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Weighted Boundedness for Toeplitz Type Operators Associated to Singular Integral Operator with Non-Smooth Kernel

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Abstract. In this paper, the weighted boundedness of the Toeplitz type operator associated to some singular integral operator with non-smooth kernel on Lebesgue spaces are obtained. To do this, some weighted sharp maximal function inequalities for the operator are proved.

1. Introduction

As the development of singular integral operators(see [8][19]), their commutators have been well studied. In [3][18], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators(also see [11][17]). In [1][12], the boundedness for the commutators generated by the singular integral operators and the weighted *BMO* and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained (also see [9]). In [12][13][15], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by *BMO* and Lipschitz functions are obtained. In [5][16], some singular integral operators with non-smooth kernel are introduced, and the boundedness for the operators and their commutators are obtained (see [4][6][14][20]). Motivated by these, in this paper, we will study the Toeplitz type operator related to some singular integral operator with non-smooth kernel and the weighted Lipschitz and *BMO* functions.

2. Preliminaries and Notations

In this paper, we will study some singular integral operator as following (see [5][16]).

Definition 1. A family of operators $D_t, t > 0$ is said to be an "approximation to the identity" if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y) f(y) dy$$

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for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies:

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2}\rho(|x-y|^2/t),$$

where ρ is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}\rho(r^2) = 0$$

for some $\epsilon > 0$.

Definition 2. A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{R^n} K(x, y)f(y)dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an "approximation to the identity" $\{B_t, t > 0\}$ such that TB_t has the associated kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} |K(x, y) - k_t(x, y)|dx \leq c_2 \ \text{ for all } \ y \in R^n.$$

(2) There exists an "approximation to the identity" $\{A_t, t > 0\}$ such that A_tT has the associated kernel $K_t(x, y)$ which satisfies

$$|K_t(x, y)| \leq c_4 t^{-n/2} \ \text{ if } \ |x-y| \leq c_3 t^{1/2},$$

and

$$|K(x, y) - K_t(x, y)| \leq c_4 t^{\delta/2}|x-y|^{-n-\delta} \ \text{ if } \ |x-y| \geq c_3 t^{1/2},$$

for some $\delta > 0, c_3, c_4 > 0$.

Let b be a locally integrable function on R^n and T be the singular integral operator with non-smooth kernel. The Toeplitz type operator associated to T is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1}M_bT^{k,2},$$

where $T^{k,1}$ are the singular integral operator T with non-smooth kernel or $\pm I$(the identity operator), $T^{k,2}$ are the bounded linear operators on $L^p(R^n, \omega)$ for $1 < p < \infty$ and $\omega \in A_p (1 \leq k \leq m)$, and $M_b(f) = bf$.

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operator T_b is the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [18]). In [5][16], the boundedness of the singular integral operator with non-smooth kernel are obtained. In [4][6][14][20], the boundedness of the commutator associated to the singular integral operator with non-smooth kernel are obtained. The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator T_b. As the application, we obtain the weighted L^p-norm inequality of the Toeplitz type operator T_b.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a nonnegative integrable function ω, let $\omega(Q) = \int_Q \omega(x)dx$ and $\omega_Q = |Q|^{-1}\int_Q \omega(x)dx$. For any locally integrable function f, let

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \leq p < \infty$ and the non-negative weight function ω, set

$$M_{\eta,p,\omega}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{\omega(Q)^{1-p\eta/n}} \int_Q |f(y)|^p \omega(y) dy \right)^{1/p}.$$

Given a double measure σ(that is $\sigma(Q) \leq C\sigma(Q)$ for any cube Q), set

$$M_\sigma(f)(x) = \sup_{Q \ni x} \frac{1}{\sigma(Q)} \int_Q |f(y)| d\sigma(y).$$

The sharp maximal function $M_A(f)$ associated with the "approximation to the identity" $\{A_t, t > 0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)| dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $\eta > 0$, let $M_{A,\eta}^{\#}(f) = M_A^{\#}(|f|^\eta)^{1/\eta}$.

The A_p weight is defined by (see [8])

$$A_p = \left\{ 0 < \omega \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{0 < \omega \in L^p_{loc}(R^n) : M(\omega)(x) \leq Cw(x), a.e.\}.$$

Given a non-negative weight function ω. For $1 \leq p < \infty$, the weighted Lebesgue space $L^p(R^n, \omega)$ is the space of functions f such that

$$||f||_{L^p(\omega)} = \left(\int_{R^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

Given the non-negative weight function ω. The weighted BMO space $BMO(\omega)$ is the space of functions b such that

$$||b||_{BMO(\omega)} = \sup_Q \frac{1}{\omega(Q)} \int_Q |b(y) - b_Q| dy < \infty.$$

For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(\omega)$ is the space of functions b such that

$$||b||_{Lip_\beta(\omega)} = \sup_Q \frac{1}{\omega(Q)^{\beta/n}} \left(\frac{1}{\omega(Q)} \int_Q |b(y) - b_Q|^p \omega(x)^{1-p} dy \right)^{1/p} < \infty.$$

Remark.(1). It has been known that(see [7]), for $b \in Lip_\beta(\omega)$, $\omega \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^jQ}| \leq Cj||b||_{Lip_\beta(\omega)} \omega(x) w(2^jQ)^{\beta/n}.$$

(2). Let $b \in Lip_\beta(\omega)$ and $\omega \in A_1$. By [8], we know that spaces $Lip_\beta(\omega)$ coincide and the norms $||b||_{Lip_\beta(\omega)}$ are equivalent with respect to different values $1 \leq p < \infty$.

3. Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. Let T be the singular integral operator as **Definition 2**, $1 < p < \infty$, $0 < \eta < 1$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$ and $b \in BMO(\omega)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$, $\varepsilon > 0$, $0 < \delta < 1$, $1 < q < p$ and $p' < r < \min(p' + \varepsilon, p'(1 + \delta))$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M_{A,\eta}^{\#}(T_b(f))(\tilde{x}) \leq C||b||_{BMO(\omega)} \sum_{k=1}^m \left([M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega T^{k,2}(f)|^q)(\tilde{x})]^{1/q} \right).$$

Theorem 2. Let T be the singular integral operator as **Definition 2**, $1 < s < \infty$, $0 < \eta < 1$, $0 < \beta < 1$, $\omega \in A_1$ and $b \in Lip_\beta(\omega)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^{\#}_{A,\eta}(T_b(f))(\tilde{x}) \le C\|b\|_{Lip_\beta(\omega)}\omega(\tilde{x})\sum_{k=1}^{m} M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}).$$

Theorem 3. Let T be the singular integral operator as **Definition 2**, $1 < p < \infty$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$ and $b \in BMO(\omega)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then T^b is bounded from $L^p(R^n, \mu)$ to $L^p(R^n, \nu)$.

Theorem 4. Let T be the singular integral operator as **Definition 2**, $\omega \in A_1$, $0 < \beta < 1$, $b \in Lip_\beta(\omega)$, $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then T^b is bounded from $L^p(R^n, \omega)$ to $L^q(R^n, \omega^{1-q})$.

Corollary 1. Let $[b, T](f) = bT(f) - T(bf)$ be the commutator generated by the singular integral operator T as **Definition 23** and b. Then Theorems 1-4 hold for $[b, T]$.

To prove the theorems, we need the following lemmas.

Lemma 1.(see [8, p.485]) Let $0 < p < q < \infty$ and for any function $f \ge 0$. We define that, for $1/r = 1/p - 1/q$

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p}\|f\|_{WL^q}.$$

Lemma 2.(see [5][16]) Let T be the singular integral operator as **Definition 2**. Then T is bounded on $L^p(R^n, \omega)$ for $\omega \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.

Lemma 3.([5][16]) Let $\{A_t, t > 0\}$ be an "approximation to the identity". For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that

$$|\{x \in R^n : M(f)(x) > D\lambda, M^{\#}_A(f)(x) \le \gamma\lambda\}| \le C\gamma|\{x \in R^n : M(f)(x) > \lambda\}|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n. Thus, for $f \in L^p(R^n), 1 < p < \infty, 0 < \eta < \infty$ and $\omega \in A_p$,

$$\|M_\eta(f)\|_{L^p(\omega)} \le C\|M^{\#}_{A,\eta}(f)\|_{L^p(\omega)}.$$

Lemma 4.(see [1]) Let $b \in BMO(\omega)$. Then

$$|b_Q - b_{2^jQ}| \le Cj\|b\|_{BMO(\omega)}\omega_{Q_j},$$

where $\omega_{Q_j} = \max_{1\le i\le j} |2^iQ|^{-1}\int_{2^iQ}\omega(x)dx$.

Lemma 5.(see [1]) Let $\omega \in A_p$, $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that $\omega^{-r/p} \in A_r$ for any $p' \le r \le p' + \varepsilon$.

Lemma 6.(see [1]) Let $b \in BMO(\omega)$, $\omega = (\mu\nu^{-1})^{1/p}$, $\mu, \nu \in A_p$ and $p > 1$. Then there exists $\varepsilon > 0$ such that for $p' \le r \le p' + \varepsilon$,

$$\int_Q |b(x) - b_Q|^r \mu(x)^{-r/p}dx \le C\|b\|^r_{BMO(\omega)}\int_Q \nu(x)^{-r/p}dx.$$

Lemma 7.(see [1]) Let $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$, $1 < p < \infty$. Then there exists $1 < q < p$ such that

$$\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-q'}\nu(x)^{-q'/q}dx\right)^{1/q'} \le C.$$

Lemma 8.(see [1]) Let $\omega \in A_p$, $1 < p < \infty$. Then there exists $0 < \delta < 1$ such that $\omega^{1-r'/p} \in A_{p/r'}(d\mu)$ for any $p' < r < p'(1 + \delta)$, where $d\mu = \omega^{r'/p}dx$.

Lemma 9.(see [2][8]) Let $0 < \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $\omega \in A_1$. Then

$$\|M_{\eta,s,\omega}(f)\|_{L^q(\omega)} \le C\|f\|_{L^p(\omega)}.$$

4. Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \left|T_b(f)(x) - A_{t_Q}(T_b(f))(x)\right|^\eta dx\right)^{1/\eta}\\ \le\ & C\|b\|_{BMO(\omega)}\sum_{k=1}^m \left([M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega T^{k,2}(f)|^q)(\tilde{x})]^{1/q}\right),\end{aligned}$$

where $t_Q = (l(Q))^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{x} \in Q$. We write

$$T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = U_1(x) + U_2(x)$$

and

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \left|T_b(f)(x) - A_{t_Q}(T_b(f))(x)\right|^\eta dx\right)^{1/\eta}\\ \le\ & \left(\frac{C}{|Q|}\int_Q |U_1(x)|^\eta dx\right)^{1/\eta} + \left(\frac{C}{|Q|}\int_Q |A_{t_Q}(U_1)(x)|^\eta dx\right)^{1/\eta} + \left(\frac{C}{|Q|}\int_Q |U_2(x) - A_{t_Q}(U_2)(x)|^\eta dx\right)^{1/\eta}\\ =\ & I_1 + I_2 + I_3.\end{aligned}$$

For I_1, we know $\nu^{-r/p} \in A_r$ by Lemma 5, thus

$$\left(\frac{1}{|Q|}\int_Q \nu(x)^{-r/p}dx\right)^{1/r} \le C\left(\frac{1}{|Q|}\int_Q \nu(x)^{r'/p}dx\right)^{-1/r'},$$

then, by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogoro's inequality (see Lemma 1), we obtain, by Lemma 6,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta}\\ =\ & \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\frac{\|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}}\\ \le\ & \frac{C}{|Q|}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1}\\ \le\ & \frac{C}{|Q|}\int_{R^n} |M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|dx\\ =\ & \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}|\mu(x)^{-1/p}|T^{k,2}(f)(x)|\omega(x)\nu(x)^{1/p}dx\\ \le\ & C\left(\frac{1}{|Q|}\int_{2Q} |b(x) - b_{2Q}|^r\mu(x)^{-r/p}dx\right)^{1/r}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)|^{r'}\omega(x)^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\ \le\ & C\|b\|_{BMO(\omega)}\left(\frac{1}{|2Q|}\int_{2Q} \nu(x)^{-r/p}dx\right)^{1/r}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)\omega(x)|^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\ \le\ & C\|b\|_{BMO(\omega)}\left(\frac{1}{|2Q|}\int_{2Q} \nu(x)^{r'/p}dx\right)^{-1/r'}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)\omega(x)|^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\ \le\ & C\|b\|_{BMO(\omega)}\left(\frac{1}{\nu(2Q)^{r'/p}}\int_{2Q} |T^{k,2}(f)(x)\omega(x)|^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\ \le\ & C\|b\|_{BMO(\omega)}[M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'},\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\le C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{j,2}(f)(x)|^{\eta}dx\right)^{1/\eta}\\
&\le C\|b\|_{BMO(\omega)}\sum_{k=1}^{m}[M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'}.
\end{aligned}$$

For I_2, taking $r > r_1 > 1$ and $r > s > 1$ such that $1/r_1 + 1/r_1{}' = 1$ and $sr_1{}' = r'$. Now by the condition on h_{t_Q} and notice for $x \in Q$, $y \in 2^{j+1}Q\backslash 2^jQ$, then $h_{t_Q}(x,y) \le Ct_Q^{-n/2}\rho(2^{2(j-1)})$, we obtain, similar to the proof of I_1,

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q |A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f))(x)|^{\eta}dx\right]^{1/\eta}\\
\le\ & C\left[\frac{1}{|Q|}\int_Q\left(\int_{R^n} h_{t_Q}(x,y)|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|dy\right)^{\eta}dx\right]^{1/\eta}\\
\le\ & C\left[\frac{1}{|Q|}\int_Q\left(\int_{2Q} h_{t_Q}(x,y)|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|dy\right)^{\eta}dx\right]^{1/\eta}\\
+\ & C\left[\frac{1}{|Q|}\int_Q\left(\int_{(2Q)^c} h_{t_Q}(x,y)|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|dy\right)^{\eta}dx\right]^{1/\eta}\\
\le\ & C\int_{2Q} t_Q^{-n/2}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|dy\\
+\ & C\sum_{j=1}^{\infty} t_Q^{-n/2}\rho(2^{2(j-1)})\int_{2^{j+1}Q\backslash 2^jQ}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|dy\\
\le\ & C\left(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
+\ & C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
\le\ & C\left(\frac{1}{|Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
+\ & C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
\le\ & C\left(\frac{1}{|Q|}\int_{2Q}|b(y)-b_{2Q}|^s\mu(y)^{-s/p}|T^{k,2}(f)(y)|^s\omega(y)^s\nu(y)^{s/p}dy\right)^{1/s}\\
+\ & C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^jl(Q))^{-n/s}(l(Q))^{-n/s}\\
\times\ & \left(\frac{1}{|Q|}\int_{2Q}|b(y)-b_{2Q}|^s\mu(y)^{-s/p}|T^{k,2}(f)(y)|^s\omega(y)^s\nu(y)^{s/p}dy\right)^{1/s}
\end{aligned}$$

$$\begin{aligned}
&\le C\left(\frac{1}{|Q|}\int_{2Q}|b(y)-b_{2Q}|^{sr_1}\mu(y)^{-sr_1/p}dy\right)^{1/sr_1}\left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(y)|^{sr_1'}\omega(y)^{sr_1'}\nu(y)^{r_1's/p}dy\right)^{1/sr_1'}\\
&+ C\sum_{j=1}^{\infty}2^{jn(1-1/s)}\rho(2^{2(j-1)})\left(\frac{1}{|Q|}\int_{2Q}|b(y)-b_{2Q}|^{sr_1}\mu(y)^{-sr_1/p}dy\right)^{1/sr_1}\\
&\times \left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(y)|^{sr_1'}\omega(y)^{sr_1'}\nu(y)^{sr_1'/p}dy\right)^{1/sr_1'}\\
&\le C\|b\|_{BMO(\omega)}\left(\frac{1}{|2Q|}\int_{2Q}\nu(y)^{-sr_1/p}dy\right)^{1/sr_1}\left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(y)\omega(y)|^{sr_1'}\nu(y)^{sr_1'/p}dy\right)^{1/sr_1'}\\
&+ C\sum_{j=1}^{\infty}2^{jn(1-1/s)}\rho(2^{2(j-1)})\|b\|_{BMO(\omega)}\left(\frac{1}{|Q|}\int_{2Q}\nu(y)^{-sr_1/p}dy\right)^{1/sr_1}\\
&\times \left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(y)\omega(y)|^{sr_1'}\nu(y)^{sr_1'/p}dy\right)^{1/sr_1'}\\
&\le C\|b\|_{BMO(\omega)}\left(\frac{1}{|2Q|}\int_{2Q}\nu(y)^{sr_1'/p}dy\right)^{-1/sr_1'}\left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(y)\omega(y)|^{sr_1'}\nu(y)^{sr_1'/p}dy\right)^{1/sr_1'}\\
&+ C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}2^{jn(1-1/s)}\rho(2^{2(j-1)})\left(\frac{1}{|Q|}\int_{2Q}\nu(y)^{sr_1'/p}dy\right)^{-1/sr_1'}\\
&\times \left(\frac{1}{|Q|}\int_{2Q}|T^{k,2}(f)(y)\omega(y)|^{sr_1'}\nu(y)^{sr_1'/p}dy\right)^{1/sr_1'}\\
&\le C\|b\|_{BMO(\omega)}\left(\frac{1}{\nu(2Q)^{sr_1'/p}}\int_{2Q}|T^{k,2}(f)(y)\omega(y)|^{sr_1'}\nu(y)^{sr_1'/p}dy\right)^{1/sr_1'}\\
&\times \sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/s)}\\
&\le C\|b\|_{BMO(\omega)}[M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'},
\end{aligned}$$

thus

$$\begin{aligned}
I_2 &\le C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_Q|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{j,2}(f)(x)|^{\eta}dx\right)^{1/\eta}\\
&\le C\|b\|_{BMO(\omega)}\sum_{k=1}^{m}[M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'}.
\end{aligned}$$

For I_3, we get, for $x\in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f))(x)|\\
&\le \int_{(2Q)^c}|b(y)-b_{2Q}||K(x-y)-K_{t_Q}(x-y)||T^{k,2}(f)(y)|dy\\
&\le C\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}\frac{l(Q)^{\delta}}{|y-x_0|^{n+\delta}}|b(y)-b_{2Q}||T^{k,2}(f)(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}||T^{k,2}(f)(y)|dy\\
&\quad +C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{k+1}d)^{n+\delta}}|b_{2^{j+1}Q}-b_{2Q}|\int_{2^{j+1}Q}|T^{k,2}(f)(y)|dy\\
&= I_3^{(1)}+I_3^{(2)}.
\end{aligned}
$$

For $I_3^{(1)}$, by using the same argument as I_1, we get

$$
\begin{aligned}
I_3^{(1)} &\leq C\sum_{j=1}^{\infty}2^{-\delta j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}|^{r}\mu(y)^{-r/p}dy\right)^{1/r}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^{r'}\omega(y)^{r'}\nu(y)^{r'/p}dy\right)^{1/r'}\\
&\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}2^{-\delta j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\nu(y)^{-r/p}dy\right)^{1/r}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)\omega(y)|^{r'}\nu(y)^{r'/p}dy\right)^{1/r'}\\
&\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}2^{-\delta j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\nu(y)^{r'/p}dy\right)^{-1/r'}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)\omega(y)|^{r'}\nu(y)^{r'/p}dy\right)^{1/r'}\\
&\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}2^{-\delta j}\left(\frac{1}{\nu(2^{j+1}Q)^{r'/p}}\int_{2^{j+1}Q}|T^{k,2}(f)(y)\omega(y)|^{r'}\nu(y)^{r'/p}dy\right)^{1/r'}\\
&\leq C\|b\|_{BMO(\omega)}[M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'}.
\end{aligned}
$$

For $I_3^{(2)}$, by Lemmas 4 and 7, we get

$$
\begin{aligned}
I_3^{(2)} &\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}j\omega_{Q_j}2^{-\delta j}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|dy\\
&\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}j2^{-\delta j}\omega_{Q_j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|\omega(y)T^{k,2}(f)(y)|^{q}\nu(y)dy\right)^{1/q}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}\\
&\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}j2^{-\delta j}\omega_{2^jQ}(\nu_{2^jQ})^{1/q}\left(\frac{1}{\nu(2^{j+1}Q)}\int_{2^{j+1}Q}|\omega(y)T^{k,2}(f)(y)|^{q}\nu(y)dy\right)^{1/q}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}
\end{aligned}
$$

$$\leq \quad C\|b\|_{BMO(\omega)}[M_\nu(|\omega T^{k,2}(f)|^q)(\tilde{x})]^{1/q}\sum_{j=1}^{\infty} j2^{-\delta j}$$
$$\times\omega_{2^jQ}(\nu_{2^jQ})^{1/q}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}$$
$$\leq \quad C\|b\|_{BMO(\omega)}[M_\nu(|\omega T^{k,2}(f)|^q)(\tilde{x})]^{1/q}\sum_{j=1}^{\infty} j2^{-\delta j}$$
$$\leq \quad C\|b\|_{BMO(\omega)}[M_\nu(|\omega T^{k,2}(f)|^q)(\tilde{x})]^{1/q}.$$

Thus

$$I_3 \leq \frac{C}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f))(x)|dx$$
$$\leq C\|b\|_{BMO(\omega)}\sum_{k=1}^{m}\left([M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega T^{k,2}(f)|^q)(\tilde{x})]^{1/q}\right).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q\left|T_b(f)(x) - A_{t_Q}(T_b(f))(x)\right|^\eta dx\right)^{1/\eta} \leq C\|b\|_{Lip_\beta(\omega)}\omega(\tilde{x})\sum_{k=1}^{m}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}),$$

where $t_Q = (l(Q))^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = V_1(x) + V_2(x)$$

and

$$\left(\frac{1}{|Q|}\int_Q\left|T_b(f)(x) - A_{t_Q}(T_b(f))(x)\right|^\eta dx\right)^{1/\eta}$$
$$\leq \left(\frac{C}{|Q|}\int_Q|V_1(x)|^\eta dx\right)^{1/\eta} + \left(\frac{C}{|Q|}\int_Q|A_{t_Q}(V_1)(x)|^\eta dx\right)^{1/\eta} + \left(\frac{C}{|Q|}\int_Q|V_2(x) - A_{t_Q}(V_2)(x)|^\eta dx\right)^{1/\eta}$$
$$= I_4 + I_5 + I_6.$$

For I_4, by Lemmas 2 and 1, we obtain

$$\left(\frac{1}{|Q|}\int_Q|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta}$$
$$\leq \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\frac{\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}}$$
$$\leq \frac{C}{|Q|}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1}$$

$$
\begin{aligned}
&\le \frac{C}{|Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|dx \\
&\le \frac{C}{|Q|}\int_{2Q}|b(x)-b_{2Q}|\omega(x)^{-1/s}|T^{k,2}(f)(x)|\omega(x)^{1/s}dx \\
&\le \frac{C}{|Q|}\left(\int_{2Q}|b(x)-b_{2Q}|^{s'}\omega(x)^{1-s'}dx\right)^{1/s'}\left(\int_{2Q}|T^{k,2}(f)(x)|^{s}\omega(x)dx\right)^{1/s} \\
&\le \frac{C}{|Q|}||b||_{Lip_\beta(\omega)}\omega(2Q)^{1/s'+\beta/n}\omega(2Q)^{1/s-\beta/n}\left(\frac{1}{\omega(Q)^{1-s\beta/n}}\int_Q|f(y)|^s\omega(y)dy\right)^{1/s} \\
&\le C||b||_{Lip_\beta(\omega)}\frac{\omega(2Q)}{|2Q|}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}) \\
&\le C||b||_{Lip_\beta(\omega)}\omega(\tilde{x})M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_4 &\le C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_Q|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta} \\
&\le C||b||_{Lip_\beta(\omega)}\omega(\tilde{x})\sum_{k=1}^{m}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

For I_5, noting that $\omega\in A_1$, ω satisfies the reverse of Hölder's inequality:

$$
\left(\frac{1}{|Q|}\int_Q\omega(x)^{p_0}dx\right)^{1/p_0}\le\frac{C}{|Q|}\int_Q\omega(x)dx
$$

for all cube Q and some $1<p_0<\infty$ (see [9]). Choose $q>1$ such that $r=(p_0-1)/q+1<s$ and let $p>1$ with $r/s+1/p+1/q=1$. By using the same argument as in the proof of I_2, we obtain

$$
\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q|A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f))(x)|^\eta dx\right]^{1/\eta} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^r dy\right)^{1/r} \\
&+ C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^r dy\right)^{1/r} \\
&\le C|Q|^{-1/r}\left(\int_{2Q}|b(y)-b_{2Q}|^r\omega(y)^{1/p-r}|T^{k,2}(f)(y)|^r\omega(y)^{r/s}\omega(y)^{r-r/s-1/p}dy\right)^{1/r} \\
&+ C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^jl(Q))^{-n/r} \\
&\times\left(\int_{2Q}|b(y)-b_{2Q}|^r\omega(y)^{1/p-r}|T^{k,2}(f)(y)|^r\omega(y)^{r/s}\omega(y)^{r-r/s-1/p}dy\right)^{1/r}
\end{aligned}
$$

$$\begin{aligned}
&\leq C|Q|^{-1/r}\left(\int_{2Q}|b(y)-b_{2Q}|^{pr}\omega(y)^{1-pr}dy\right)^{1/pr}\\
&\times \left(\int_{2Q}|T^{k,2}(f)(y)|^{s}\omega(y)dy\right)^{1/s}\left(\int_{2Q}\omega(y)^{(r-r/s-1/p)q}dy\right)^{1/qr}\\
&+ C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^{j}l(Q))^{-n/r}\left(\int_{2Q}|b(y)-b_{2Q}|^{pr}\omega(y)^{1-pr}dy\right)^{1/pr}\\
&\times \left(\int_{2Q}|T^{k,2}(f)(y)|^{s}\omega(y)dy\right)^{1/s}\left(\int_{2Q}\omega(y)^{(r-r/s-1/p)q}dy\right)^{1/qr}\\
&\leq C|Q|^{-1/r}||b||_{Lip_{\beta}(\omega)}\omega(Q)^{\beta/n+1/pr}\omega(Q)^{1/s-\beta/n}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&\times |Q|^{1/qr}\left(\frac{1}{|Q|}\int_{2Q}\omega(y)^{p_0}dy\right)^{1/qr}\\
&+ C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^{j}l(Q))^{-n/r}||b||_{Lip_{\beta}(\omega)}\omega(Q)^{\beta/n+1/pr}\omega(Q)^{1/s-\beta/n}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&\times |Q|^{1/qr}\left(\frac{1}{|Q|}\int_{2Q}\omega(y)^{p_0}dy\right)^{1/qr}\\
&\leq C||b||_{Lip_{\beta}(\omega)}|Q|^{-1/r}\omega(2Q)^{1/pr}\omega(2Q)^{1/s}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&\times |Q|^{1/qr}\left(\frac{1}{|Q|}\int_{2Q}\omega(y)dy\right)^{p_0/qr}\\
&+ C||b||_{Lip_{\beta}(\omega)}\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^{j}l(Q))^{-n/r}\omega(2Q)^{1/pr}\omega(2Q)^{1/s}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&\times |Q|^{1/qr}\left(\frac{1}{|Q|}\int_{2Q}\omega(y)dy\right)^{p_0/qr}\\
&\leq C||b||_{Lip_{\beta}(\omega)}\frac{\omega(2Q)}{|2Q|}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/r)}\\
&\leq C||b||_{Lip_{\beta}(\omega)}\omega(\tilde{x})M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_5 &\leq C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f))(x)|^{\eta}dx\right)^{1/\eta}\\
&\leq C||b||_{Lip_{\beta}(\omega)}\omega(\tilde{x})\sum_{k=1}^{m}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For I_6, notice $w\in A_1\subset A_s$, we get, for $x\in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f))(x)|\\
&\leq \int_{(2Q)^c}|b(y)-b_{2Q}||K(x-y)-K_{t_Q}(x-y)||T^{k,2}(f)(y)|dy\\
&\leq C\sum_{j=1}^{\infty}\int_{2^{j}d\leq|y-x_0|<2^{j+1}d}\frac{l(Q)^{\delta}}{|y-x_0|^{n+\delta}}|b(y)-b_{2Q}||T^{k,2}(f)(y)|dy
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}+b_{2^{j+1}Q}-b_{2Q}|\omega(y)^{-1/s}|T^{k,2}(f)(y)|\omega(y)^{1/s}dy\\
&\le C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\left(\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}|^{s'}\omega(y)^{1-s'}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^{s}\omega(y)dy\right)^{1/s}\\
&+ C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+1}}|b_{2^{j+1}Q}-b_{2Q}|\left(\int_{2^{j+1}Q}w(y)^{-1/(s-1)}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^{s}\omega(y)dy\right)^{1/s}\\
&\le C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}||b||_{Lip_\beta(\omega)}\omega(2^{j+1}Q)^{1/s'+\beta/n}\omega(2^{j+1}Q)^{1/s-\beta/n}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&+ C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}||b||_{Lip_\beta(\omega)}\omega(\tilde{x})j\omega(2^{j+1}Q)^{\beta/n}\omega(2^{j+1}Q)^{1/s-\beta/n}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&\times \frac{|2^{j+1}Q|}{\omega(2^{j+1}Q)^{1/s}}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\omega(y)dy\right)^{1/s}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\omega(y)^{-1/(s-1)}dy\right)^{(s-1)/s}\\
&\le C||b||_{Lip_\beta(\omega)}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}Q)}{|2^{j+1}Q|}2^{-\delta j}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&+ C||b||_{Lip_\beta(\omega)}\sum_{j=1}^{\infty}\omega(\tilde{x})j2^{-\delta j}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x})\\
&\le C||b||_{Lip_\beta(\omega)}\omega(\tilde{x})M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_3 &\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f))(x)|dx\\
&\le C||b||_{Lip_\beta(\omega)}\omega(\tilde{x})\sum_{k=1}^{m}M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1.

Proof of Theorem 3. Notice $\nu^{r'/p}\in A_{r'+1-r'/p}\subset A_p$ and $\nu(x)dx\in A_{p/r'}(\nu(x)^{r'/p}dx)$ by Lemma 8, thus, by Theorem 1 and Lemma 3, we have

$$\begin{aligned}
&\int_{R^n}|T_b(f)(x)|^p\nu(x)dx\le\int_{R^n}|M_\eta(T_b(f))(x)|^p\nu(x)dx\le C\int_{R^n}|M^{\#}_{A,\eta}(T_b(f))(x)|^p\nu(x)dx\\
&\le C||b||_{BMO(\omega)}\sum_{k=1}^{m}\int_{R^n}\left([M_{\nu^{r'/p}}(|\omega T^{k,2}(f)|^{r'})(x)]^{p/r'}+[M_\nu(|\omega T^{k,2}(f)|^{q})(x)]^{p/q}\right)\nu(x)dx\\
&\le C||b||_{BMO(\omega)}\sum_{k=1}^{m}\int_{R^n}|\omega(x)T^{k,2}(f)(x)|^p\nu(x)dx\\
&= C||b||_{BMO(\omega)}\sum_{k=1}^{m}\int_{R^n}|T^{k,2}(f)(x)|^p\mu(x)dx\\
&\le C||b||_{BMO(\omega)}\int_{R^n}|f(x)|^p\mu(x)dx.
\end{aligned}$$

This completes the proof.

Proof of Theorem 4. Choose $1 < s < p$ in Theorem 2 and notice $w^{1-q} \in A_\infty$, then we have, by Lemmas 3 and 9,

$$\begin{aligned}
&\|T_b(f)\|_{L^q(\omega^{1-q})} \le \|M_\eta(T_b(f))\|_{L^q(\omega^{1-q})} \le C\|M^{\#}_{A,\eta}(T_b(f))\|_{L^q(\omega^{1-q})} \\
\le\ & C\|b\|_{Lip_\beta(\omega)} \sum_{k=1}^{m} \|\omega M_{\beta,s,\omega}(T^{k,2}(f))\|_{L^q(\omega^{1-q})} \\
=\ & C\|b\|_{Lip_\beta(\omega)} \sum_{k=1}^{m} \|M_{\beta,s,\omega}(T^{k,2}(f))\|_{L^q(\omega)} \\
\le\ & C\|b\|_{Lip_\beta(\omega)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p(\omega)} \\
\le\ & C\|b\|_{Lip_\beta(\omega)} \|f\|_{L^p(\omega)}.
\end{aligned}$$

This completes the proof.

5. Applications

In this section we shall apply Theorems 1-4 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [5][16]). Given $0 \le \theta < \pi$. Define

$$S_\theta = \{z \in C : |\arg(z)| \le \theta\} \bigcup \{0\}$$

and its interior by S^0_θ. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|\|(\eta I - L)^{-1}\| \le C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S^0_\mu) = \{f : S^0_\theta \to C : \ f \ \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S^0_\mu\}$. Set

$$\Psi(S^0_\mu) = \left\{ g \in H_\infty(S^0_\mu) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \le c\frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

If L is of type θ and $g \in H_\infty(S^0_\mu)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \ge 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S^0_\mu)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1 + z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$\|g(L)\| \le N\|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S^0_\mu)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [16] and Theorems 1-4, we get

Corollary 2. Assume the following conditions are satisfied:

(i) The holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$ is represented by the kernels $a_z(x, y)$ which satisfy, for all $\nu > \theta$, an upper bound

$$|a_z(x, y)| \le c_\nu h_{|z|}(x, y)$$

for $x, y \in R^n$, and $0 \le |\arg(z)| < \pi/2 - \theta$, where $h_t(x, y) = Ct^{-n/2}s(|x - y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

(ii) The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S^0_\mu)$, the operator $g(L)$ satisfies

$$\|g(L)(f)\|_{L^2} \le c_\nu \|g\|_{L^\infty} \|f\|_{L^2}.$$

Let $g(L)_b$ be the Toeplitz type operator associated to $g(L)$. Then Theorems 1-4 hold for $g(L)_b$.

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