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# On the Hyperbolicity of Edge-Chordal and Path-Chordal Graphs

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**Abstract.** If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in $X$. The space $X$ is $\delta$-*hyperbolic* (in the Gromov sense) if any side of $T$ is contained in a $\delta$-neighborhood of the union of the other two sides, for every geodesic triangle $T$ in $X$. An important problem in the study of hyperbolic graphs is to relate the hyperbolicity with some classical properties in graph theory. In this paper we find a very close connection between hyperbolicity and chordality: we extend the classical definition of chordality in two ways, edge-chordality and path-chordality, in order to relate this property with Gromov hyperbolicity. In fact, we prove that every edge-chordal graph is hyperbolic and that every hyperbolic graph is path-chordal. Furthermore, we prove that every path-chordal cubic graph with small path-chordality constant is hyperbolic.

## 1. Introduction

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [13]), where its practical importance was discussed. This theory was mainly applied to the study of automatic groups (see [18]), which appear in computational science. Another important application of these spaces is secure transmission of information on the internet. In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [15, 16]). It has been shown in [5] that the Internet topology can be accurately embedded into an hyperbolic space. The hyperbolicity has also been applied in the field of random networks. For example, it was shown in [28, 29] that several types of small-world networks and networks with given expected degrees are not hyperbolic in some sense. It was proved in [31] the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a graph related to it; hence, it is useful to know hyperbolicity criteria for graphs from a geometrical viewpoint. In recent years, the study of mathematical properties of Gromov hyperbolic spaces has become a topic of increasing interest in graph theory and its applications; see, for instance [1–3, 6–8, 10, 11, 14, 17, 19–26, 30–32].

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For any metric space $(X, d)$, we say that $\gamma : [a, b] \to X$ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where $L$ denotes length. We say that $X$ is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining $x$ and $y$; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

In order to consider a graph $G$ as a geodesic metric space, we must identify any edge $uv \in E(G)$ with the real interval $[0, l]$ (if $l := L(uv)$); hence, if we consider the edge $uv$ as a graph with just one edge, then it is isometric to $[0, l]$. Therefore, any point in the interior of any edge will be consider as a point of $G$. A connected graph $G$ is naturally equipped with a distance defined on its points, induced by taking shortest paths in $G$. Then, we see $G$ as a metric graph. Throughout this paper we consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges); we allow edges of arbitrary lengths.

If $X$ is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that $x_1, x_2$ and $x_3$ are the vertices of $T$; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that $T$ is *$\delta$-thin* if any side of $T$ is contained in the $\delta$-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of $T$, i.e. $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space $X$ is *$\delta$-hyperbolic* (or satisfies the *Rips condition* with constant $\delta$) if every geodesic triangle in $X$ is $\delta$-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e. $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that $X$ is *hyperbolic* if $X$ is $\delta$-hyperbolic for some $\delta \geq 0$. If we have a triangle with two identical vertices, we call it a "bigon". Obviously, every bigon in a $\delta$-hyperbolic space is $\delta$-thin.

Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, for a general graph or a general geodesic metric space, deciding whether or not a space is hyperbolic is usually very difficult. Therefore, an important problem in the study of hyperbolic graphs is to relate the hyperbolicity with some classical properties in graph theory. Following the ideas given in [6, 32], we find some very close connections between hyperbolicity and different generalizations of chordality: we extend the classical definition of chordality in two ways, edge-chordality and path-chordality, in order to relate this property with Gromov hyperbolicity. In fact, we prove that every edge-chordal graph is hyperbolic (see Theorem 2.4) and that every hyperbolic graph is path-chordal (see Theorem 2.6). We also give examples showing that the converses of Theorems 2.4 and 2.6 do not hold, i.e., hyperbolicity does not imply edge-chordality and path-chordality does not imply hyperbolicity. However, we prove that every path-chordal cubic graph with small path-chordality constant is hyperbolic (see Corollary 3.5 and Theorem 3.12); Example 3.14 shows that path-chordal cubic graphs with large path-chordality constant are not necessarily hyperbolic.

## 2. Edge-Chordal and Path-Chordal Graphs

As usual, by *cycle* in a graph we mean a simple closed curve, *i.e.*, a path with different vertices, except for the last one, which is equal to the first vertex. A *shortcut* of a cycle $C$ in a graph $G$ is a path $\sigma$ joining two vertices $p, q \in C$ such that $L(\sigma) < d_C(p, q)$. An *edge-shortcut* of a cycle $C$ is an edge in $G$ which is a shortcut of $C$. Given two constants $k, m \geq 0$, we say that a graph $G$ is *$(k, m)$-edge-chordal* if for any cycle $C$ in $G$ with length $L(C) \geq k$ there exists an edge-shortcut $e$ with length $L(e) \leq m$. The graph $G$ is *edge-chordal* if there exist constants $k, m \geq 0$ such that $G$ is $(k, m)$-edge-chordal. We say that a graph $G$ is *$k$-path-chordal* if for any cycle $C$ in $G$ with $L(C) \geq k$ there exists a shortcut $\sigma$ of $C$ such that $L(\sigma) \leq k/2$.

**Remark 2.1.** *Every $(k, m)$-edge-chordal graph is* $\max\{k, 2m\}$*-path-chordal.*

According to the above definition, a graph (with edges of length 1) is said to be chordal if it is $(4, 1)$-edge-chordal. It was proved in [6] that chordal graphs are hyperbolic. Other generalization of chordality for graphs with edges of length 1 was introduced in [32], namely, a graph is said to be $k$-chordal if it does not contain any induced $n$-cycle for $n > k$. It is clear that chordal graphs are 3-chordal. It was proved in

[32, Theorem 2] that $k$-chordal graphs are also hyperbolic. Our concept of edge-chordality generalizes the $k$-chordality; in fact, $k$-chordal graphs are $(k+1,1)$-edge-chordal.

The following lemma will be used throughout the paper, it generalizes [6, Lemma 2.2] and uses the same ideas in its proof.

**Lemma 2.2.** *Given a $(k,m)$-edge-chordal graph $G$, a cycle $C$ in $G$ with length $L(C) \geq k$ and a geodesic $[ab] \subset C$ with $L([ab]) \geq k/2$, there exist two vertices $v \in V(G) \cap ([ab] \setminus \{a,b\})$ and $w \in V(G) \cap (C \setminus [ab])$ with $e = vw \in E(G)$, $L(e) < d_C(v,w)$ and $L(e) \leq m$.*

*Proof.* Since $[ab]$ is a geodesic contained in $C$, we have $L(C \setminus [ab]) \geq L([ab])$ and $L(C) \geq 2L([ab])$.

Assume first that $L(C) = 2L([ab])$. In this case $L(C \setminus [ab]) = L([ab])$ and then $(C \setminus [ab]) \cup \{a,b\}$ is also a geodesic joining $a$ and $b$. Since $L(C) \geq k$ and $G$ is a $(k,m)$-edge-chordal graph, there exists an edge $e = xy$ with $x,y \in V(G) \cap C$ such that $L(e) < d_C(x,y)$ and $L(e) \leq m$. It is not possible for $e$ to join two vertices of $[ab]$, since $[ab]$ is a geodesic. Similarly, it is not possible for $e$ to join two vertices of $(C \setminus [ab]) \cup \{a,b\}$, since $(C \setminus [ab]) \cup \{a,b\}$ is also a geodesic. Therefore, the conclusion of the lemma holds in this case.

Assume now that $L(C) > 2L([ab])$. Since $L(C) \geq k$ and $G$ is a $(k,m)$-edge-chordal graph, there exists an edge $e = xy$ with $x,y \in V(G) \cap C$ such that $L(e) < d_C(x,y)$ and $L(e) \leq m$. It is not possible for $e$ to join two vertices of $[ab]$, since $[ab]$ is a geodesic. If either $x$ or $y$ belongs to $[ab] \setminus \{a,b\}$, then the conclusion of the lemma also holds in this case. If $x,y \notin [ab] \setminus \{a,b\}$, then we consider the cycle $C_1$ obtained by pasting $e$ with the connected component of $C \setminus \{x,y\}$ which contains $[ab] \setminus \{a,b\}$. It is clear that $L(C_1) < L(C)$, $[ab] \subset C_1$ and $V(G) \cap C_1 \subseteq V(G) \cap C$.

Now we can apply the previous argument to $C_1$. If we do not obtain the conclusion of the Lemma, then we obtain a new cycle $C_2$ with $L(C_2) < L(C_1) < L(C)$, $[ab] \subset C_2$ and $V(G) \cap C_2 \subseteq V(G) \cap C$. Iterating this process we obtain either the conclusion of the Lemma or a sequence of cycles $C_1, C_2, \dots, C_j, \dots$ with $[ab] \subset C_j$, $V(G) \cap C_j \subseteq V(G) \cap C$ for every $j \geq 1$, and

$$L(C_j) < \cdots < L(C_2) < L(C_1) < L(C)\,. \tag{1}$$

Since $G$ is locally finite and we have (1), this process must stop in some finite step (in a finite number of steps) by compactness; therefore, the conclusion of the Lemma holds. □

In the next results we will show that edge-chordality implies hyperbolicity and that hyperbolicity implies path-chordality. To prove them we will need the following Lemma, which can be found in [27].

**Lemma 2.3.** *In any graph $G$ we have*

$$\delta(G) = \sup\Big\{\delta(T) :\ T \textit{ is a geodesic triangle that is a cycle}\Big\}.$$

**Theorem 2.4.** *If $G$ is a $(k,m)$-edge-chordal graph, then it is $(m+k/4)$-hyperbolic.*

*Proof.* Let us consider any fixed geodesic triangle $T = \{x,y,z\}$ in $G$. By Lemma 2.3, in order to compute $\delta(G)$, we can assume that $T$ is a cycle. If $\delta(T) \leq k/4$, then $\delta(T) \leq m + k/4$. If $\delta(T) > k/4$, without loss of generality, let us show that for every $p \in [xy]$ with $k/4 < d(p,[xz] \cup [yz]) \leq \delta(T)$, we have $d(p,[xz] \cup [yz]) \leq \frac{k}{4} + m$. In such a case, it is clear that $L([xy]) \geq k/2$ and $L(T) \geq k$. Let us consider $a,b \in [xy]$, with $a \neq b$ and $d(a,p) = d(b,p) = k/4$; then $p \in [ab] \subset [xy]$ and $L([ab]) = k/2$. Lemma 2.2 gives that there exist two vertices $v \in V(G) \cap ([ab] \setminus \{a,b\})$ and $w \in V(G) \cap (T \setminus [ab])$ with $e = vw \in E(G)$, $L(e) < d_T(v,w)$ and $L(e) \leq m$. Note that $w \notin [xy]$ since $L(vw) < d_T(v,w)$. Therefore, $d(p,[xz] \cup [yz]) \leq d(p,w) \leq d(p,v) + d(v,w) \leq \frac{k}{4} + m$. Since this is satisfied for every geodesic triangle $T$ in $G$, $G$ is $(m+k/4)$-hyperbolic. □

The following example shows that the converse of Theorem 2.4 does not hold, i.e., hyperbolicity does not imply edge-chordality. Let $P_3$ be the path graph with (adjacent) vertices $v_1, v_2, v_3$, and $G$ the Cartesian product graph $G = \mathbb{Z} \square P_3$ with $L(e) = 1$ for every $e \in E(G)$. One can check that $G$ is $\frac{5}{2}$-hyperbolic and 5-path-chordal, but it is not edge-chordal, since for every natural number $r \geq 2$ the geodesic squares with vertices $(0,v_1), (r,v_1), (r,v_3), (0,v_3)$ do not have edge-shortcuts.

Theorem 2.6 below is one of the main results in this paper, since it is a kind of converse of Theorem 2.4. In order to prove it, we need the following technical result (see [12, Theorem 16]).

**Lemma 2.5.** *Let us consider constants $\delta \geq 0, r > 0$, a $\delta$-hyperbolic geodesic metric space $X$ and a finite sequence $\{x_j\}_{0\leq j\leq n}$ in $X$ with $d_X(x_{j-1}, x_{j+1}) \geq \max\{d_X(x_{j-1}, x_j), d_X(x_j, x_{j+1})\} + 18\delta + r$ for every $0 < j < n$. Then $d_X(x_0, x_n) \geq rn$.*

**Theorem 2.6.** *Every $\delta$-hyperbolic graph is $90\delta$-path-chordal.*

*Proof.* Seeking for a contradiction, assume that $G$ is a $\delta$-hyperbolic graph which is not $90\delta$-path-chordal. Then there exists a cycle $C$ in $G$ with $L(C) \geq 90\delta$ without shortcuts $\sigma$ satisfying $L(\sigma) \leq 45\delta$. Consequently, any subcurve $g$ of $C$ with $L(g) \leq 45\delta$ is a geodesic in $G$. Let us define an integer $n$ and a positive number $\ell$ by $n := \left\lceil \frac{2L(C)}{45\delta} \right\rceil$ and $\ell := \frac{L(C)}{n}$. Since $\frac{2L(C)}{45\delta} \leq \left\lceil \frac{2L(C)}{45\delta} \right\rceil < \frac{2L(C)}{45\delta} + 1$, we deduce that $18\delta < \ell \leq \frac{45\delta}{2}$. Let us take a finite sequence $\{x_j\}_{0\leq j\leq n}$ in $C$ such that $x_0 = x_n$, $d_G(x_j, x_{j+1}) = d_C(x_j, x_{j+1}) = \ell$ for every $0 \leq j < n$, and $d_G(x_{j-1}, x_{j+1}) = 2\, d_C(x_j, x_{j+1}) = 2\ell$ for every $0 < j < n$.

If $r := \ell - 18\delta$, then $2\ell = \ell + 18\delta + r$, and

$$d_G(x_{j-1}, x_{j+1}) = \max\{d_G(x_{j-1}, x_j), d_G(x_j, x_{j+1})\} + 18\delta + r,$$

for every $0 < j < n$. In consequence, Lemma 2.5 gives $0 = d_G(x_0, x_n) \geq rn > 0$, which is a contradiction. Hence, $G$ is $90\delta$-path-chordal. □

The following example shows that the converse of Theorem 2.6 does not hold, i.e., path-chordality does not imply hyperbolicity. First of all, we assume that $0 \in \mathbb{N}$. Let $\sum_{n=0}^{\infty} a_n$ be a fixed convergent series of positive real numbers such that $a_0 = 1$ and $\sum_{n=0}^{\infty} a_n = S < \infty$. Now, Let us consider $\{S_n\}_{n=0}^{\infty}$ the sequence of partial sums. Let $G$ be the Cartesian product graph $G = \mathbb{N}\square\mathbb{N}$ with $L\big((p,q)(p+1,q)\big) = S_{p+q} = L\big((p,q)(p,q+1)\big)$. Note that $G$, although it is not hyperbolic, it is a path chordal graph, since each cycle $C$ of $G$ with $L(C) > 4S$ has a vertex $v = (p+1, q+1) \in C$ such that $(p+1,q)v, (p,q+1)v \in E(G)$ are contained in $C$ (i.e., $v$ is an upper-right vertex of $C$); then $C$ has a shortcut $\sigma \subseteq (p+1,q)(p,q) \cup (p,q)(p,q+1)$, since $S_{p+q} < S_{p+q+1}$.

## 3. Chordality in Cubic Graphs

Cubic graph (graphs in which all vertices have degree three) are very interesting in many situations. They are also very important in the study of Gromov hyperbolicity, because the study of the hyperbolicity of graphs can be reduced to the study of the hyperbolicity of cubic graphs ([4]). For more information about the hyperbolicity in cubic graphs see [19, 25].

A proper shortcut of a cycle $C$ is a geodesic which is a shortcut $\sigma$ joining two vertices $p, q \in C \cap V(G)$ such that $\sigma \cap C = \{p, q\}$. Note that for any cycle $C$ in a $k$-path-chordal graph $G$ satisfying $L(C) \geq k$ there exists a proper shortcut with length at most $k/2$. Therefore, we may replace proper shortcut by shortcut in the definition of path-chordal graph. Along this section we just consider (finite or infinite) cubic graphs with edges of length 1. In such a case, every edge-shortcut is a proper shortcut.

**Theorem 3.1.** *Let $G$ be a cubic graph. Then $G$ is 4-path-chordal if and only if it is chordal.*

*Proof.* If $G$ is a chordal graph, it is clear that it is 4-path-chordal.

Assume now that $G$ is a 4-path-chordal graph. Seeking for a contradiction, assume that there exists a cycle $C$ in $G$ with $L(C) \geq 4$ and such that $C$ has no shortcut with length 1. Since $L(C) \geq 4$ and $G$ is 4-path-chordal, the set $V_C := \{(u,v) |\, u, v \in V(G) \cap C$ and $[uv]$ is a shortcut of $C$ with length 2$\}$ is non-empty. Let $(x, y) \in V_C$ with $d_C(x,y) = \min\{d_C(u,v) |\, (u,v) \in V_C\}$. Let $g_1$ be a path joining $x$ and $y$ contained in $C$ such that $L(g_1) = d_C(x,y)$. Define $C_1 := g_1 \cup [xy]$; then $L(C_1) \geq 2L([xy]) \geq 4$ and there exists a proper shortcut $\rho = [zw]$ of $C_1$. Since it is not possible to have $\{z,w\} \subset [xy]$ or $\{z,w\} \subset g_1$, without loss of generality we can assume that $z \in g_1 \setminus \{x,y\}$ and $w \in [xy] \setminus \{x,y\}$; since $L([xy]) = 2$, then $w$ is the midpoint of $[xy]$.

Note that we have either $L(\rho) = 1$ or $L(\rho) = 2$.

If $L(\rho) = 1$, then $d_C(z,x) \leq 2$ and $d_C(z,y) \leq 2$, since $xw \cup wz$ and $yw \cup wz$ are not shortcuts of $C$. We prove now that $d_C(z,x) = d_C(z,y) = 1$. Otherwise, by symmetry, we can assume that $d_C(z,x) = 2$; then the cycle

$C_2 = xw \cup \rho \cup zx$ has length 4 and there exists a shortcut of $C_2$; but since $x, z, w$ have "full degree", there is just one vertex in $C_2$ that can be an endpoint of the shortcut. This is a contradiction and we conclude that $d_C(z, x) = d_C(z, y) = 1$. But, in that case $d_C(x, y) = 2 = L([xy])$ and, consequently, $[xy]$ is not a shortcut of $C$, which is also a contradiction.

If $L(\rho) = 2$, then we have a shortcut for each of the two cycles $[xw] \cup [wz] \cup [zx]$ and $[yw] \cup [wz] \cup [zy]$. Since $x, y, w, z$ have already degree 3, the midpoint $v$ of $\rho$ must be the endpoint of the two shortcuts, which is a contradiction because it has degree 3. $\square$

**Lemma 3.2.** *Let $G$ be a 4-path-chordal cubic graph and let $C$ be any cycle in $G$ with two different shortcuts with length 1. Then, $G$ is isomorphic to the complete graph $K_4$.*

*Proof.* By Theorem 3.1 any cycle of $G$ with length greater than 3 has an edge-shortcut. Let $\sigma_1 := xx'$ and $\sigma_2 := yy'$ be two different edge-shortcuts of $C$. Let $g$ (respectively, $g'$) be a subcurve of $C$ joining $x$ and $y$ (respectively, $x'$ and $y'$) such that $g \cap g' = \emptyset$; then $C_1 := \sigma_1 \cup g' \cup \sigma_2 \cup g$ is a cycle with $L(C_1) \geq 4$. The cycle $C$ can be oriented either by: (1) $x \to y \to y' \to x'$, or (2) $x \to y \to x' \to y'$.

Assume that $C$ is oriented by (1). Then $C_1$ has an edge-shortcut $e_1$ joining $g \setminus \{x, y\}$ and $g' \setminus \{x', y'\}$. Let $C_2$ be a cycle obtained by joining $e_1$ with a path contained in $C_1$. Proceeding this way, we obtain a finite sequence of cycles $C_1, C_2, \dots, C_k$ such that $L(C) > L(C_1) > L(C_2) > \cdots > L(C_k) = 4$ and the four vertices of $C_k$ have full degree; then there is no shortcut for $C_k$, which is a contradiction.

Assume now that $C$ is oriented by (2). Let $\gamma_1, \gamma_2$ be two curves with $\gamma_1 \cup \gamma_2 = C$ and $\gamma_1 \cap \gamma_2 = \{x, x'\}$. If $\max\{L(\gamma_1), L(\gamma_2)\} > 2$, then without loss of generality we can assume that $L(\gamma_1) > 2$; hence, $\gamma_1 \cup xx'$ is a cycle with $L(\gamma_1 \cup xx') \geq 4$ and there is an edge-shortcut $e_1$ for $\gamma_1 \cup xx'$; since $x$ and $x'$ have full degree, $xx' \cap e_1 = \emptyset$; consequently, $xx'$ and $e_1$ are two edge-shortcuts of $C$ in the case (1), and we have proved that this is a contradiction. Therefore, $\max\{L(\gamma_1), L(\gamma_2)\} \leq 2$; we conclude that $L(\gamma_1) = L(\gamma_2) = 2$, and then $G$ is isomorphic to $K_4$. $\square$

**Theorem 3.3.** *If $G$ is a 4-path-chordal cubic graph, then $G$ does not have cycles with length greater than 4.*

*Proof.* We assume that there exists a cycle $C$ such that $L(C) \geq 5$, so there exists an edge-shortcut $\sigma$ of $C$ and, consequently, there exists a cycle $C_1$ such that $L(C_1) \geq 4$ and it contains two vertices of degree 3. Therefore, $C_1$ must have an edge-shortcut $\sigma_1$ which will be a second edge-shortcut for $C$. By Lemma 3.2 we get a contradiction. $\square$

Let us define the *circumference* $c(G)$ of a graph $G$ as the supremum of the lengths of its cycles if $G$ is not a tree; we define $c(G) = 0$ for every tree $G$. The following result appears in [8].

**Lemma 3.4.** *For any graph $G$, $\delta(G) \leq \frac{1}{4}\, c(G)$.*

Theorem 3.3 and Lemma 3.4 have the following consequence.

**Corollary 3.5.** *If $G$ is a 4-path-chordal cubic graph, then $G$ is 1-hyperbolic.*

**Theorem 3.6.** *If $G$ is a 4-path-chordal cubic graph, then $\delta(G) = c(G)/4$.*

In order to prove Theorem 3.6 above, that provides a simple explicit formula for the hyperbolicity constant of every 4-path-chordal cubic graph, we need the following result (see [8]). A subgraph $G'$ of $G$ is said *isometric* if $d_{G'}(x, y) = d_G(x, y)$ for every $x, y \in G'$.

**Lemma 3.7.** *If $G$ is any graph, then*

$$\delta(G) \geq \frac{1}{4} \sup\{L(g) : g \text{ is an isometric cycle in } G\}.$$

*Proof.* [Proof of Theorem 3.6] By Proposition 3.4, $\delta(G) \leq c(G)/4$. Let us prove the converse inequality. By Theorem 3.3 we have $c(G) \leq 4$. If $c(G) \leq 3$, then $\delta(G) \geq c(G)/4$ by Lemma 3.7. Assume now that $c(G) = 4$ and consider a cycle $g$ with length 4. Let $x, y$ be midpoints of edges in $g$ with $d(x, y) = 2$ and paths $g_1, g_2$ with $g_1 \cup g_2 = g$ and $g_1 \cap g_2 = \{x, y\}$. Then $\{g_1, g_2\}$ is a geodesic bigon in $G$. If $p$ is the midpoint of $g_1$, then $\delta(G) \geq d(p, g_2) = d(p, \{x, y\}) = 1 = c(G)/4$. □

**Proposition 3.8.** *If $G$ is a 5-path-chordal cubic graph, then there are no proper shortcuts with length 2 for any cycle in $G$.*

*Proof.* We prove the Lemma by complete induction. It is clear that on every cycle in $G$ with length 5 the proper shortcuts have length 1. Now, we assume that any cycle in $G$ with length at most $k$ does not have proper shortcuts with length 2. Consider a cycle $C$ in $G$ with $k + 1$ vertices. Seeking for a contradiction, assume that $C$ has a proper shortcut $\sigma := [xy]$ with length 2, and let $v$ be the midpoint of $\sigma$. Let $g_1, g_2$ be two paths in $G$ joining $x$ and $y$ such that $C = g_1 \cup g_2$ and $g_1 \cap g_2 = \{x, y\}$. Consider the cycles $C_1 := g_1 \cup \sigma$ and $C_2 := g_2 \cup \sigma$. Since $L(C_1) \geq 5$, there exists a proper shortcut $\rho_1$ of $C_1$ joining two vertices $u, v$ in $C_1$. Note that $u, v$ are different from $x, y$ because $G$ is a cubic graph. If $u$ and $v$ belong to $g_1$, then denote by $g_1'$ the path joining $u, v$ contained in $C$ and which contains $g_2$; the cycle $\rho_1 \cup g_1'$ verifies $L(\rho_1 \cup g_1') \leq k$ and has the proper shortcut $\sigma$ with length 2, which is a contradiction. Hence, $\rho_1 = vz$ with $z \in g_1 \setminus \{x, y\}$. In a similar way, there should exist another shortcut $vw$ with $w \in g_2 \setminus \{x, y\}$, which is a contradiction because $\deg(v) = 3$. We conclude that $C$ does not have proper shortcuts with length 2. □

**Corollary 3.9.** *Every 5-path-chordal cubic graph is $(5, 1)$-edge-chordal.*

By Corollary 3.9 any 5-path-chordal cubic graph $G$ is $(5, 1)$-edge-chordal, and Theorem 2.4 gives that $\delta(G) \leq 9/4$. However, Theorem 3.12 below improves this inequality.

We need the following result.

**Lemma 3.10.** *Let $C$ be a cycle in a 5-path-chordal cubic graph $G$ and $[xy]$ a geodesic contained in $C$. If there are two edge-shortcuts $\rho_1 := xu$, $\rho_2 := yv$ of $C$ and there is no other edge-shortcut of $C$ starting in $[xy]$, then $xy, uv \in E(G)$.*

*Furthermore, the cycle obtained by joining the shortcuts $\rho_1$ and $\rho_2$ with paths contained in $C$ has length 4.*

*Proof.* Denote by $\gamma$ the path contained in $C$ which joins $u$ and $v$ such that $x, y \notin \gamma$. Denote by $C_1$ the cycle $C_1 := \rho_1 \cup [xy] \cup \rho_2 \cup \gamma$. Notice that is suffices to prove that $L(C_1) = 4$. Let us assume that $L(C_1) > 4$ and let us take an edge-shortcut $\sigma := u_1v_1$ of $C_1$ joining two vertices of $\gamma$ such that $d_\gamma(u_1, v_1)$ is maximum. Without loss of generality we can suppose that $\gamma$ can be oriented by $u \to u_1 \to v_1 \to v$. Since $G$ is a cubic graph, we have $\rho_i \cap \sigma = \emptyset$ for $i \in \{1, 2\}$; then we have that the cycle $C_2 := \rho_1 \cup [uu_1] \cup \sigma \cup [v_1v] \cup \rho_2 \cup [xy]$ has length greater than 5; since $C_2$ does not have edge-shortcuts, we obtain the contradiction. Therefore, we conclude that $L(C_1) = 4$ and $xy, uv \in E(G)$. □

The following result (see [2, Theorem 2.7]) will be used to prove our next theorem. Let us denote by $J(G)$ the set of vertices and midpoints of the edges of $G$, and by $\mathbb{T}(G)$ the set of geodesic triangles $T = \{x, y, z\}$ in $G$ that are cycles with $x, y, z \in J(G)$.

**Lemma 3.11.** *For any graph $G$ with edges of length 1 we have $\delta(G) = \sup\{\delta(T) | T \in \mathbb{T}(G)\}$. Furthermore, if $G$ is hyperbolic, there exists a geodesic triangle $T \in \mathbb{T}(G)$ with $\delta(T) = \delta(G)$.*

**Theorem 3.12.** *If $G$ is a 5-path-chordal cubic graph, then $\delta(G) \leq 3/2$. Moreover, this upper bound is sharp.*

*Proof.* Fix a geodesic triangle $T = \{x, y, z\}$ in $G$. By Lemma 3.11, in order to study $\delta(G)$ we can assume that $T$ is a cycle with $x, y, z \in J(G)$. If $L(T) \leq 6$, then the three geodesic sides of $T$ have length at most 3 and, consequently, $\delta(T) \leq 3/2$. Assume now that $L(T) \geq 7$. By Corollary 3.9 there exists an edge-shortcut of $T$. By symmetry, it suffices to prove that for every $p \in [xy]$ we have $d_G(p, [yz] \cup [zx]) \leq 3/2$.

Assume first that there is no edge-shortcut of $T$ starting in $[xy]$. Since $G$ is $(5,1)$-edge-chordal, by Lemma 2.2 we have that $L([xy]) \le 2$; therefore, we have for every $p \in [xy]$,

$$d_G(p, [yz] \cup [zx]) \le d_G(p, \{x, y\}) \le 1.$$

Assume now that there is an edge-shortcut of $T$ joining $[xy]$ and $[xz]$, but there is no edge-shortcut joining $[xy]$ and $[yz]$. Let $\sigma_1$ be an edge-shortcut of $T$ joining $P_1$ and $Q_1$, where $P_1 \in [xy]$, $Q_1 \in [xz]$ and $P_1$ is the closest vertex to $x$ with an edge-shortcut. Consider the cycle $C := [xP_1] \cup \sigma_1 \cup [Q_1 x]$. Then, $C$ does not have edge-shortcuts; therefore, $L(C) \le 4$ and $L([xP_1]) + L([xQ_1]) \le 3$. Hence, since $L([xP_1]) \le 1 + L([xQ_1])$, we have $L([xP_1]) \le 2$, $L([xP_1]) + L([P_1 Q_1]) \le 3$ and we obtain $d_G(p, [yz] \cup [zx]) \le d_G(p, [zx]) \le d_G(p, \{x, Q_1\}) \le 3/2$ for every $p \in [xP_1]$. Let $n$ be the exact number of edge-shortcuts of $T$ joining $[xy]$ and $[xz]$. Let $P_1, \dots, P_n \in [xy]$, $Q_1, \dots, Q_n \in [xz]$ with $P_i Q_i \in E(G)$ for $1 \le i \le n$ and $L([xP_i]) < L([xP_{i+1}])$ for $1 \le i < n$. Hence, by Lemma 3.10 we have that $P_i P_{i+1}, Q_i Q_{i+1} \in E(G)$ for every $1 \le i < n$; thus, for every $p \in [P_i P_{i+1}]$, we obtain $d_G(p, [yz] \cup [zx]) \le d_G(p, [zx]) \le d_G(p, \{Q_i, Q_{i+1}\}) \le 3/2$.

Furthermore, since there is no edge-shortcut of $T$ from $[P_n y]$, by Lemma 2.2 we have that $L([P_n y]) \le 2$; therefore, for every $p \in [P_n y]$ we have $d_G(p, [yz] \cup [zx]) \le d_G(p, \{Q_n, y\}) \le 3/2$. Hence, we obtain

$$d_G(p, [yz] \cup [zx]) \le 3/2, \qquad \text{for every } p \in [xy].$$

Finally, assume that there are shortcuts of $T$ joining $[xy]$ with $[xz]$, and $[xy]$ with $[yz]$. Let $m$ be the exact number of edge-shortcuts of $T$ joining $[xy]$ and $[yz]$. Let $P_1, \dots, P_n \in [xy]$ as above, and let $R_1, \dots, R_m \in [xy]$, $S_1, \dots, S_m \in [yz]$ with $R_i S_i \in E(G)$ for $1 \le i \le m$ and $L([yR_i]) < L([yR_{i+1}])$ for $1 \le i < m$. Let $1 \le k \le m$ with $[P_n R_k] \cap \{R_1, \dots, R_m\} = R_k$; by Lemma 3.10 we have that $[P_n R_k]$ is an edge. So, a similar argument to the one in the previous case gives

$$d_G(p, [yz] \cup [zx]) \le 3/2, \qquad \text{for every } p \in [xy].$$

This upper bound for the hyperbolicity constant of 5-path-chordal graphs is sharp because it is attained by the Cartesian product graph $\mathbb{Z}_2 \square \mathbb{Z}$. $\square$

The Example 3.14 below shows that the converse of Theorem 2.6 does not hold even for cubic graphs, i.e., path-chordality does not imply hyperbolicity in cubic graphs. In order to obtain the non-hyperbolicity of the graph in that example we will need the following result (see [9, Theorem 2.7]).

**Theorem 3.13.** *Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. If $\inf_n A(F_n) > 0$, then $G$ is not hyperbolic.*

**Example 3.14.** *Consider a graph $G$ which is the 1-skeleton of the semiregular tessellation of the Euclidean plane obtained by octagons and squares, see Figure 1.*

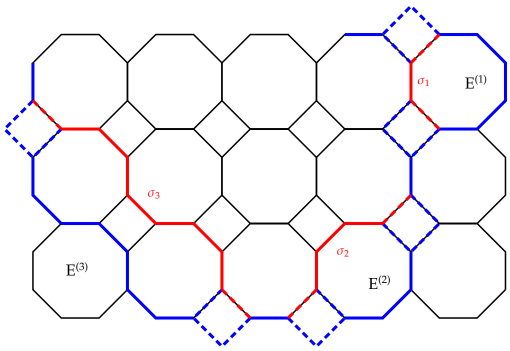


Figure 1: Semiregular tessellation of $\mathbb{R}^2$ whose 1-skeleton is a cubic 18-path-chordal graph.

Clearly, $G$ is a cubic graph; we show now that $G$ is a 18-path-chordal graph. Let us consider a cycle $C$ in $G$ with length greater than 17. Let $R_C$ be the compact region in $\mathbb{R}^2$ whose boundary is $C$. We pay attention to the relative position of the octagons contained in $R_C$. Notice that at least one of the following situations holds:

(1) there is an octagon $E$ in $R_C$ intersecting $C$ such that either ($a$) neither of the two octagons which are horizontal neighbors of $E$ are contained in $R_C$ or ($b$) neither of the two octagons which are vertical neighbors of $E$ are contained in $R_C$ or ($c$) both of the above, simultaneously (see $E^{(1)}$ in Figure 1),

(2) there are three octagons in $R_C$ intersecting $C$ to form a "right angle" (i.e., there is a octagon $E$ in $R_C$ intersecting $C$, the "corner", such that one of the octagons which are horizontal neighbors of $E$ is contained in $R_C$ and the other one is not contained in $R_C$, and one of the octagons which are vertical neighbors of $E$ is contained in $R_C$ and the other one is not contained in $R_C$, see $E^{(2)}$ in Figure 1),

(3) there are four octagons in $R_C$ intersecting $C$ to form a "right angle without the corner" (see $E^{(3)}$ in Figure 1).

Note that each octagon in the "corners" of $R_C$ satisfy (1), (2) or (3).

If (1) holds, then $\sigma_1 := E \setminus C$ is a shortcut of $C$ with length at most 3. If (2) holds, then $C$ has a shortcut $\sigma_2$ of length at most 5 (delimiting the octagon at the corner). If (3) holds, then $C$ has a shortcut $\sigma_3$ of length at most 9 (delimiting the two octagons closest to the corner). This proves that $G$ is a 18-path-chordal graph. Finally, by Theorem 3.13 we have that $G$ is not hyperbolic.

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