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An Extension of the Generalized Hurwitz-Lerch Zeta Function of Two Variables

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Abstract. The main object of this paper is to introduce a new extension of the generalized Hurwitz-Lerch Zeta functions of two variables. We then systematically investigate such its several interesting properties and related formulas as (for example) various integral representations, which provide certain new and known extensions of earlier corresponding results, a summation formula and Mellin-Barnes type contour integral representations. We also consider some important special cases.

1. Introduction, Definitions and Preliminaries

The Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ is defined by (see, *e.g.*, [14, p. 121] and [15, p. 194]):

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \qquad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\right). \tag{1}$$

Various properties and the special cases of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$, one may refer to [4, Chapter I] (see also [14, 15]). Certain generalizations of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ have been investigated by many authors (see, *e.g.*, [1, 2, 7, 8, 16, 18, 20]). Recently, Srivastava [13, p. 1487, Eq. (1.14)] studied and investigated more generalized form of Hurwitz-Lerch Zeta function:

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-\frac{b}{t^\lambda}}\, {}_p\Psi_q^* \left[\begin{matrix} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{matrix}\ ze^{-t} \right] \mathrm{d}t \tag{2}$$

$$\left(\min\{\Re(s),\ \Re(a)\}>0,\ \Re(b) \geqq 0;\ \lambda \geqq 0\right).$$

Various special cases of (2) can be found in [13]. Motivated mainly by various extensions of the Hurwitz-Lerch Zeta function, we introduce a new extension of the generalized Hurwitz-Lerch Zeta function of two variables. We then systematically investigate such its several interesting properties as (for example) various integral representations which provide new and known extensions of the corresponding results earlier given by various authors. We also derive a summation formula and Mellin-Barnes type contour integral representations. We also consider some important special cases of the main results.

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2. Extended Hurwitz-Lerch Zeta Function of Two Variables

Here, we introduce a new extension of the generalized Hurwitz-Lerch Zeta functions of two variables defined in the following form:

$$\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!} \frac{z^m t^n}{(m+n+a)^s} \tag{3}$$

$$\Big(\alpha,\ \beta,\ \beta' \in \mathbb{C};\ a,\ \gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s,\ z,\ t \in \mathbb{C} \text{ when } |z|<1 \text{ and } |t|<1;\ \text{and}$$

$$\Re(s+\gamma-\alpha-\beta-\beta')>0 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

Remark 1. Among many other things, the following special or limiting cases of the extended Hurwitz-Lerch Zeta function $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ in (3) are considered in our present investigation.

Case 1. The special case $\alpha=\gamma$ in (3) reduces to the generalized Hurwitz-Lerch Zeta function of Pathan and Daman [11]:

$$\Phi^*_{\mu,\lambda}(z,t,s,a) := \sum_{m,k=0}^{\infty} \frac{(\mu)_m(\lambda)_k}{m!k!} \frac{z^m t^k}{(m+k+a)^s} \tag{4}$$

$$\Big(\mu,\lambda \in \mathbb{C};\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1 \text{ and } |t|<1; \Re(s-\mu-\lambda)>0 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

Case 2. The following limiting case of the extended Hurwitz-Lerch Zeta function $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ in (3) will be used:

$$\Phi^*_{\alpha,\beta;\gamma}(z,t,s,a) = \lim_{\beta'\to\infty}\left\{\Phi_{\alpha,\beta,\beta';\gamma}\left(z,t/\beta',s,a\right)\right\} = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!} \frac{z^m t^n}{(m+n+a)^s} \tag{5}$$

$$\Big(\alpha,\beta \in \mathbb{C};\ a,\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-; s,z,t \in \mathbb{C} \text{ when } |z|<1 \text{ and } |t|<1;\ \text{and } \Re(s+\gamma-\alpha-\beta)>0 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

Case 3. The following limiting case of the extended Hurwitz-Lerch Zeta function $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ in (3) will also be used:

$$\Phi^*_{\beta,\beta';\gamma}(z,t,s,a) = \lim_{\alpha\to\infty}\left\{\Phi_{\alpha,\beta,\beta';\gamma}\left(z/\alpha,t/\alpha,s,a\right)\right\} = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!} \frac{z^m t^n}{(m+n+a)^s} \tag{6}$$

$$\Big(\beta,\beta' \in \mathbb{C};\ a,\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-; s,z,t \in \mathbb{C} \text{ when } |z|<1 \text{ and } |t|<1;\ \text{and } \Re(s+\gamma-\beta-\beta')>0 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

Case 4. Another limiting case of the extended Hurwitz-Lerch Zeta function $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ in (3) is given as follows:

$$\Phi_{\beta;\gamma}(z,t,s,a) = \lim_{\min\{|\alpha|,|\beta'|\}\to\infty}\left\{\Phi_{\alpha,\beta,\beta';\gamma}\left(z/\alpha,t/(\alpha\beta'),s,a\right)\right\} := \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}m!n!} \frac{z^m t^n}{(m+n+a)^s} \tag{7}$$

$$\Big(\beta \in \mathbb{C}; a,\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-; s,z,t \in \mathbb{C} \text{ when } |z|<1 \text{ and } |t|<1;\ \text{and } \Re(s+\gamma-\beta)>0 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

3. Integral Representations of $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$

Here we begin by recalling the Appell hypergeometric function F_1 of two variables (see, *e.g.*, [17, p. 22]): For $\alpha,\ \beta, \ldots,\ \beta' \in \mathbb{C}$ and $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$,

$$F_1[\alpha,\beta,\beta';\gamma;x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\, {}_2F_1\left[\begin{matrix} \alpha+m, \beta'; \\ \gamma+m; \end{matrix}\ y\right]\frac{x^m}{m!} \tag{8}$$

$$\Big(\max\{\Re(x), \Re(y)\} < 1 \quad \text{and} \quad \Re(\alpha) > 0\Big).$$

The following confluent forms of the Appell hypergeometric function F_1 or Humbert functions are recalled (see, *e.g.*, [17, p. 25 *et seq.*]):

$$\Phi_1[\alpha,\beta;\gamma;x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad \Big(|x| < 1,\ |y| < \infty\Big), \tag{9}$$

$$\Phi_2[\beta,\beta';\gamma;x,y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad \Big(|x| < \infty,\ |y| < \infty\Big) \tag{10}$$

and

$$\Phi_3[\beta;\gamma;x,y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad \Big(|x| < \infty,\ |y| < \infty\Big). \tag{11}$$

Theorem 1. *The following integral representation for* $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ *in* (3) *holds true:*

$$\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a) := \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}e^{-ax}\, F_1(\alpha,\beta,\beta';\gamma;ze^{-x},te^{-x})\,\mathrm{d}x \tag{12}$$

$$\Big(\min\{\Re(s),\ \Re(a)\} > 0 \ \ \textit{when}\ |z| \leqq 1\,(z \neq 1),\ |t| \leqq 1\,(t \neq 1);\ \Re(s) > 1 \ \ \textit{when}\ z = 1,\ t = 1\Big).$$

Proof. Using the following Eulerian integral:

$$\frac{1}{(m+n+a)^s} := \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-(m+n+a)t}\,\mathrm{d}t \quad \left(\min\{\Re(s),\ \Re(a)\} > 0;\ m,\ n \in \mathbb{N}_0\right) \tag{13}$$

in (3) and then interchanging the order of summation and integration, which is valid under the given conditions, and which upon using (8), we are led to the desired integral representation (12). □

Similarly as Theorem 1, if we use the Eulerian integral (13) in the relationships (5), (6) and (7), use (9), (10) and (11), respectively, we obtain the following integral representations asserted by Corollary 1 below.

Corollary 2. *Each of the following integral representations for*

$$\Phi^*_{\alpha,\beta;\gamma}(z,t,s,a),\ \ \Phi^*_{\beta,\beta';\gamma}(z,t,s,a) \ \ \textit{and} \ \ \Phi_{\beta;\gamma}(z,t,s,a)$$

in (5), (6) *and* (7) *holds true:*

$$\Phi^*_{\alpha,\beta;\gamma}(z,t,s,a) := \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}e^{-ax}\, \Phi_1[\alpha,\beta;\gamma;ze^{-x},te^{-x}]\,\mathrm{d}x, \tag{14}$$

$$\Phi^*_{\beta,\beta';\gamma}(z,t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}\, \Phi_2[\beta,\beta';\gamma;ze^{-x},te^{-x}]\,\mathrm{d}x \tag{15}$$

and

$$\Phi_{\beta;\gamma}(z,t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}\, \Phi_3[\beta;\gamma;ze^{-x},te^{-x}]\,\mathrm{d}x \tag{16}$$

$$\Big(\min\{\Re(s),\ \Re(a)\} > 0 \ \ \textit{when}\ |z| \leqq 1\,(z \neq 1),\ |t| \leqq 1\,(t \neq 1);\Re(s) > 1 \ \ \textit{when}\ z = 1,\ t = 1\Big).$$

Remark 2. It is interesting to see that the case $\gamma = \alpha$ in the integral representation (12) and using the reduction formula for the Appell function F_1 (see, *e.g.*, [19, p. 54]):

$$F_1[\alpha,\beta,\beta';\alpha;x,y] = (1-x)^{-\beta}\,(1-y)^{-\beta'} \tag{17}$$

yields the known integral representation given by Pathan and Daman [11] (see also [3, p. 252, Eq.(7)]):

$$\Phi_{\alpha,\beta,\beta';\alpha}(z,t,s,a) = \Phi^*_{\beta,\beta'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-ze^{-x})^{\beta}(1-te^{-x})^{\beta'}}\ \mathrm{d}x \tag{18}$$

$$\Big(\Re(a) > 0;\ \Re(s) > 0 \text{ when } |z| \leqq 1\,(z \neq 1),\ |t| \leqq 1\,(t \neq 1);\Re(s) > 1 \text{ when } z = 1,\ t = 1\Big).$$

Further, if we put $t = 0$ in (18), we again obtain the known integral representation due to Goyal and Laddha [6, p. 100, Eq. (1.6)]:

$$\Phi^*_{\beta}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-ze^{-x})^{\beta}}\,\mathrm{d}x \qquad \Big(\Re(a) > 0;\ \Re(s) > 0 \text{ when } |z| \leqq 1\,(z \neq 1);\ \Re(s) > 1 \text{ when } z = 1\Big). \tag{19}$$

The special case $\beta = 1$ of the above result (19) is precisely the well-known integral representation in (1) (see, *e.g.*, [15, p. 194, Eq. 2.5(4)]):

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}}\,\mathrm{d}t \qquad \Big(\Re(a) > 0;\ \Re(s) > 0 \text{ when } |z| \leqq 1\,(z \neq 1);\ \Re(s) > 1 \text{ when } z = 1\Big). \tag{20}$$

Theorem 3. *Each of the following integral representations for* $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ *in* (3) *holds true:*

$$\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^\infty \frac{w^{\alpha-1}}{(1+w)^{\gamma}} \Phi^*_{\beta;\beta'}\left(\frac{zw}{1+w}, \frac{tw}{1+w}, s, a\right)\mathrm{d}w \tag{21}$$

$$\Big(\Re(\gamma) > \Re(\alpha) > 0\Big)$$

and

$$\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a) = \frac{\Gamma(\gamma)}{\Gamma(s)\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}w^{\alpha-1}}{(1+w)^{\gamma}} \left(1-\frac{zwe^{-x}}{1+w}\right)^{-\beta} \left(1-\frac{twe^{-x}}{1+w}\right)^{-\beta'} \mathrm{d}x\,\mathrm{d}w \tag{22}$$

$$\Big(\Re(\gamma) > \Re(\alpha) > 0;\ \min\{\Re(s),\ \Re(a)\} > 0\Big).$$

Proof. Setting $a = \alpha + m + n$ and $b = \gamma + m + n$ in the Eulerian Beta function formula (see, *e.g.*, [4, p. 9, Eq. 1.5 (2)]; see also [15, p. 8, Eq. (45)]):

$$B(a, b-a) = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} = \int_0^\infty \frac{y^{a-1}}{(1+y)^{b}}\,\mathrm{d}y \qquad (\Re(b) > \Re(a) > 0), \tag{23}$$

we find that

$$\frac{(\alpha)_{m+n}}{(\gamma)_{m+n}} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^\infty \frac{w^{\alpha+m+n-1}}{(1+w)^{\gamma+m+n}}\, \mathrm{d}w \tag{24}$$

$$(\Re(\gamma) > \Re(\alpha) > 0;\ m,\ n \in \mathbb{N}_0),$$

which, by appealing to the definition (4), immediately yields the first assertion (21). Moreover, by (12) and (24), we also obtain

$$\begin{aligned}\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a) &:= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{m,n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{(ze^{-x})^m}{m!} \frac{(te^{-x})^n}{n!}\, \mathrm{d}x \\ &= \frac{\Gamma(\gamma)}{\Gamma(s)\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}w^{\alpha-1}}{(1+w)^\gamma} \sum_{m=0}^\infty \frac{(\beta)_m}{m!} \left(\frac{ze^{-x}w}{1+w}\right)^m \sum_{n=0}^\infty \frac{(\beta')_n}{n!} \left(\frac{te^{-x}w}{1+w}\right)^n \mathrm{d}x\, \mathrm{d}w,\end{aligned}$$

which leads us to the second assertion (22). □

Theorem 4. *The following summation formula for* $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ *in* (3) *holds true:*

$$\sum_{k=0}^\infty \frac{(s)_k}{k!} \Phi_{\alpha,\beta,\beta';\gamma}(z,t,s+k,a)\, x^k = \Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a-x) \qquad (|x| < |a|;\ s \neq 1). \tag{25}$$

Proof. Using (3) in right-hand side of the assertion (25), we have

$$\begin{aligned}\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a-x) &= \sum_{m,n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!(m+n+a-x)^s} \\ &= \sum_{m,n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!(m+n+a)^s} \left(1 - \frac{x}{m+n+a}\right)^{-s} \\ &= \sum_{m,n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!(m+n+a)^s} \left\{\sum_{k=0}^\infty \frac{(s)_k}{k!} \frac{x^k}{(m+n+a)^k}\right\} \\ &= \sum_{k=0}^\infty \frac{(s)_k}{k!} \left\{\sum_{m,n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!(m+n+a)^{s+k}}\right\} x^k,\end{aligned}$$

which, upon using (3), leads to the desired formula in (25). □

4. Mellin-Barnes Contour Integral Representation for $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$

Here we present Mellin-Barnes contour integral representation of the function $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ given in Theorem 5.

Theorem 5. *The following contour integral representation for* $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$ *in* (3) *holds true:*

$$\begin{aligned}\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')} \\ &\times \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_z} \int_{\mathcal{L}_t} \frac{\Gamma(-\xi)\Gamma(-\zeta)\Gamma(\alpha+\xi+\zeta)\Gamma(\beta+\xi)\Gamma(\beta'+\zeta)\{\Gamma(\xi+\zeta+a)\}^s}{\Gamma(\gamma+\xi+\zeta)\{\Gamma(\xi+\zeta+a+1)\}^s} \\ &\times (-z)^\xi (-t)^\zeta\, \mathrm{d}\xi\, \mathrm{d}\zeta\end{aligned} \tag{26}$$

$$(|\arg(-z)| < \pi,\ |\arg(-t)| < \pi),$$

where it is assumed that the poles of the integrand in (26) *are simple ; the contours of the integration are described so that the poles of* $\Gamma(-\xi)\,\Gamma(-\zeta)$ *can be separated from those of* $\Gamma(\beta+\xi)$ *and* $\Gamma(\beta'+\zeta)$ *with indentations, if necessary.*

Proof. If we calculate the integral (26) as sum of the residues at the simple poles of $\Gamma(-\xi)$ at the points $\xi = m$ $(m \in \mathbb{N}_0)$ and of the $\Gamma(-\zeta)$ at the points $\zeta = n$ $(n \in \mathbb{N}_0)$, respectively, we immediately obtain the following series expansion:

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')} \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m+n)\Gamma(\beta+m)\Gamma(\beta'+n)}{\Gamma(\gamma+m+n)m!n!} \frac{z^m t^n}{(m+n+a)^s}, \tag{27}$$

which is just the $\Phi_{\alpha,\beta,\beta';\gamma}(z,t,s,a)$. This completes the proof. □

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