Studying the Krull Dimension of Finite Lattices Under the Prism of Matrices

T. Dube\textsuperscript{a}, D.N. Georgiou\textsuperscript{b}, A.C. Megaritis\textsuperscript{c}, F. Sereti\textsuperscript{b}

\textsuperscript{a}Department of Mathematical Sciences, University of South Africa, P.O. Box 392, Pretoria, 0003, South Africa
\textsuperscript{b}University of Patras, Department of Mathematics, 265 00 Patras, Greece
\textsuperscript{c}Technological Educational Institute of Western Greece, Department of Accounting and Finance, 302 00 Messolonghi, Greece

Abstract. The Krull dimension of a finite lattice \((X, \preceq)\) is equal to the height of the poset of join prime elements of \(X\) minus 1. To every partially ordered set we assign an order-matrix, and we use these order-matrices to characterize the join prime elements of finite lattices. In addition, we present a reduction algorithm for the computation of the height of a finite poset. The algorithm is based on the concept of the incidence matrix. Our main objective, ultimately, is to use these processes to calculate the Krull dimension of any given finite lattice.

Introduction

The main goal in this paper is to use methods of matrix theory to calculate the Krull dimension of any finite lattice. This requires developing certain algorithms, starting with one for calculating the height of a finite partially ordered set using matrix theory. The matrices that come to play are what are called order-matrices.

We then characterize join prime elements of a finite lattice in terms of its order-matrix (Theorem 2.5). This theorem enables us to develop an algorithm for computing the set of join prime elements. The algorithm terminates in eight steps. This is the contents of Section 2.

In Section 3 we consider the height of a finite poset, and in this case the matrices that form the basic tools are the incidence matrices. The main step here in developing the algorithm for calculating the height of a finite poset with \(n\) elements is Corollary 3.4 which, for any positive integer \(k \leq n\), characterizes when the height of the poset is \(k\), in terms of columns of the incidence matrix of the poset.

The results outlined above are used in Section 4 to calculate the Krull dimension of a finite lattice. We end the paper with a few questions associated with this topic.

1. Preliminaries

In this section we recall some definitions and notations (see, for example, [1–4]) that are needed in the sequel.
Let $X$ be a partially ordered set and $M \subseteq L$. The set of all upper bounds of $M$ is denoted by $\text{UB}(M)$. A supremum (respectively, infimum) of $M$, if it exists, is denoted by $\lor M$ (respectively, $\land M$) and often referred to as the join (respectively, the meet) of $M$. We will also use the symbols $x \lor y$ for $\lor\{x, y\}$ and $x \land y$ for $\land\{x, y\}$. Henceforth, we shall write, as usual, “poset” for “partially ordered set”.

The height of a finite poset $X$, denoted by $h(X)$, is the maximum of cardinalities of maximal chains in $X$. When we say a lattice is finite, we shall mean that its underlying set is a finite set. A finite lattice has a least element and a greatest element, which are denoted by 0 and 1, respectively.

Let $(X, \leq)$ be a finite lattice.

(1) A non-empty subset $F$ of $X$ is called a filter if $F$ has the following properties:

(i) $F \neq X$.

(ii) If $x \in F$ and $x \leq y$, then $y \in F$.

(iii) If $x, y \in F$, then $x \land y \in F$.

Thus, by a filter we mean a proper filter. The set of all filters of $X$ is denoted by $\mathcal{F}(X)$. For every $x \in X$, the subset $\uparrow x = \{y \in X : x \leq y\}$ of $X$ is called the principal filter generated by $x$.

(2) A filter $F$ is called prime if for every $x, y \in X$ with $x \lor y \in F$, we have $x \in F$ or $y \in F$. The set of all prime filters of $X$ is denoted by $\mathcal{PF}(X)$.

(3) If $\mathcal{PF}(X) \neq \emptyset$, then the Krull dimension (see, for example, [5] and [6]) of $(X, \leq)$ is defined as follows:

$$\text{Kdim}(X) = \max\{k : \text{there exist prime filters } F_0 \subset F_1 \subset \cdots \subset F_k\}.$$ 

It is known that a finite lattice $(X, \leq)$ has Krull dimension zero if and only if it is a Boolean algebra (see for example [7]).

(4) An element $x$ of $(X, \leq)$ is said to be join prime if it is nonzero and the inequality $x \leq a \lor b$ implies $x \leq a$ or $x \leq b$, for all $a, b \in X$. For distributive lattices, these are exactly the join-irreducible elements.

Let $(X, \leq)$ be a finite poset, where $X = \{x_1, \ldots, x_n\}$. The $n \times n$ matrix $A^\leq_x = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } x_i < x_j \\ -2, & \text{if } x_j < x_i \\ 0, & \text{if } x_i \parallel x_j \end{cases}$$

is called the order-matrix of $X$. For example, let $(X, \leq)$ be the poset represented by the diagram of Figure 1.

![Figure 1: The poset $(X, \leq)$](image-url)
The order-matrix of $X$ is the following $5 \times 5$ matrix:

$$A^5_X = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ -2 & 1 & 2 & 2 & 2 \\ -2 & -2 & 1 & 0 & 2 \\ -2 & -2 & 0 & 1 & 2 \\ -2 & -2 & -2 & -2 & 1 \end{pmatrix}.$$  

In what follows, we denote by $r_i(A^5_X), \ldots, r_n(A^5_X)$ the $n$ rows of the matrix $A^5_X$. Also, if $\alpha$ is an entry in the matrix $A^5_X$, then, for the sake of simplicity, we write $\alpha \in A^5_X$.

2. Join Prime Elements and Order-Matrices

Throughout this section, $(X, \leq)$ denotes a finite lattice, where $X = \{x_1, \ldots, x_n\}$, and $A^5_X = (a_{ij})$ denotes the $n \times n$ order-matrix of $X$. We write $|S|$ for the cardinality of a set $S$.

**Proposition 2.1.** Let $i_1, i_2$ be two distinct elements of $\{1, 2, \ldots, n\}$ such that

$$r_{i_1}(A^5_X) + r_{i_2}(A^5_X) = \left( \begin{array}{cccc} a_{i_1}^{i_1i_2} & a_{i_2}^{i_1i_2} & \cdots & a_{n}^{i_1i_2} \end{array} \right).$$

The following statements are true:

1. The set $\{i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1\}$ is one of the following sets: $\emptyset, \{i_1, i_2\}$.

2. If $i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1$, then $x_{i_1} \parallel x_{i_2}$.

3. If $i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1 \neq \emptyset$ if and only if $x_{i_1} \parallel x_{i_2}$.

**Proof.** (1) We have $a_{i_1i_1} = a_{i_2i_2} = 1$. Let $i \in \{1, \ldots, n\}$ such that $a_{i}^{i_1i_2} = 1$. Then, $i = i_1$ or $i = i_2$.

(a) Let $a_{i}^{i_1i_2} = 1$.

In this case $a_{i_1i_1} = 0$ and, therefore, $a_{i_2i_2} = 0$ which means that $a_{i_2}^{i_1i_2} = 1$. Thus, $\{i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1\} = \{i_1, i_2\}$.

(b) Let $a_{i}^{i_1i_2} = 1$.

Similar to (a), $a_{i_1i_1} = 0$ and, therefore, $a_{i_2i_2} = 0$ which means that $a_{i_1}^{i_1i_2} = 1$. Thus, $\{i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1\} = \{i_1, i_2\}$.

(2) If $\{i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1\} = \{i_1, i_2\}$, then $a_{i_1}^{i_1i_2} = a_{i_2}^{i_1i_2} = 1$. Therefore, $a_{i_1i_1} = a_{i_2i_2} = 0$ which means that $x_{i_1} \parallel x_{i_2}$.

Conversely, let $x_{i_1} \parallel x_{i_2}$. Then, $a_{i_1i_1} = a_{i_2i_2} = 0$. Moreover, $a_{i_1i_2} = a_{i_2i_1} = 1$. Therefore, $a_{i_1}^{i_1i_2} = a_{i_2}^{i_1i_2} = 1$ which means that $\{i \in \{1, \ldots, n\} : a_{i}^{i_1i_2} = 1\} = \{i_1, i_2\}$.

(3) This statement is a consequence of statements (1) and (2). □

**Proposition 2.2.** Let $i_0, i_1, i_2$ be three distinct elements of $\{1, 2, \ldots, n\}$ such that $x_{i_1} \parallel x_{i_2}$ and

$$r_{i_0}(A^5_X) + r_{i_1}(A^5_X) + r_{i_2}(A^5_X) = \left( \begin{array}{cccc} a_{i_0}^{i_0i_1i_2} & a_{i_1}^{i_0i_1i_2} & \cdots & a_{i_2}^{i_0i_1i_2} \end{array} \right).$$

The following statements are true:

1. The set $\{i \in \{1, \ldots, n\} : a_{i}^{i_0i_1i_2} = 1\}$ is one of the following sets: $\emptyset, \{i_1\}, \{i_2\}, \{i_0, i_1, i_2\}$. 


(2) If $|\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\}| \neq 3$ and
\[3 \notin (-1) \cdot r_{s_i}(A^\infty_\chi) + r_{i_i}(A^\infty_\chi) + r_{s_i}(A^\infty_\chi),\]
then $\alpha_{ij} = 2$ or $\alpha_{ij} = 2$.

Proof. (1) Since $x_{i_1} \parallel x_{i_2}$, we have $\alpha_{ij} = \alpha_{i_{ij}} = 0$. Also, we have $\alpha_{ij} = \alpha_{ij} = 1$. Let $i \in [1,\ldots,n]$ such that $\alpha_i^{j_{i_{ij}}} = 1$. Then, $i = i_0$ or $i = i_1$ or $i = i_2$.

(a) Let $\alpha_i^{j_{i_{ij}}} = 1$.

In this case $\alpha_{ij} = \alpha_{ij} = 0$ and, therefore, $\alpha_{ij} = \alpha_{ij} = 0$ which means that $\alpha_i^{j_{i_{ij}}} = \alpha_i^{j_{i_{ij}}} = 1$. Thus, $\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\} = \{i_0, i_1, i_2\}$.

(b) Let $\alpha_i^{j_{i_{ij}}} \neq 1$.

In this case $\alpha_{ij} = \alpha_{ij} \neq 0$ or $\alpha_{ij} \neq 0$.

(i) If $\alpha_{ij} \neq 0$, then $\alpha_{ij} \neq 0$ and, therefore, $\alpha_i^{j_{i_{ij}}} \neq 1$. Thus, the set $\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\} = \{i_0, i_1, i_2\}$.

(ii) If $\alpha_{ij} \neq 0$, then $\alpha_{ij} \neq 0$ and, therefore, $\alpha_i^{j_{i_{ij}}} \neq 1$. Thus, the set $\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\} = \{i\}$.

(2) By statement (1) we have the following cases:

(a) $\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\} = \{i_1\}$

In this case $\alpha_i^{j_{i_{ij}}} \neq 1$, $\alpha_{ij} = 0$, and $\alpha_{ij} \in \{-2,2\}$. By (2.1) we have $\alpha_{ij} = 2$.

(b) $\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\} = \{i_2\}$

In this case $\alpha_i^{j_{i_{ij}}} \neq 1$, $\alpha_{ij} = 0$, and $\alpha_{ij} \in \{-2,2\}$. By (2.1) we have $\alpha_{ij} = 2$.

(c) $\{i \in [1,\ldots,n] : \alpha_i^{j_{i_{ij}}} = 1\} = \emptyset$

In this case $\alpha_{ij} = \alpha_{ij} \in \{-2,2\}$. If $\alpha_{ij} = -2$ or $\alpha_{ij} = -2$, then $3 \notin (-1) \cdot r_{i}(A^\infty_\chi) + r_{i_i}(A^\infty_\chi) + r_{i}(A^\infty_\chi)$, which is a contradiction. Thus, $\alpha_{ij} = \alpha_{ij} = 2$. \(\square\)

Proposition 2.3. Let $j, i_1, i_2$ be three distinct elements of $[1,2,\ldots,n]$ such that $x_{i_1} \parallel x_{i_2}$. The following statements are true:

(1) $x_j \in UB(x_{i_1}, x_{i_2})$ if and only if
\[5 \in r_{i_1}(A^\infty_\chi) + r_{i_2}(A^\infty_\chi) + r_{i_3}(A^\infty_\chi).\]

(2) $x_j \in UB(x_{i_1}, x_{i_2})$. Then, $x_j \neq x_{i_1} \lor x_{i_2}$ if and only if
\[6 \in r_{i_1}(A^\infty_\chi) + r_{i_2}(A^\infty_\chi) + (-1) \cdot r_{i_3}(A^\infty_\chi).\]

Proof. (1) Let $x_j \in UB(x_{i_1}, x_{i_2})$. Then, $\alpha_{ij} = \alpha_{ij} = 2$. Moreover, $\alpha_{ij} = 1$. Therefore, the condition (2.2) is satisfied.

Conversely, we suppose that the condition (2.2) is satisfied. Since $\alpha_{ij} = \alpha_{ij} = \alpha_{ij} = 1$ and $\alpha_{ij} = \alpha_{ij} = 0$, we have $\alpha_{ij} = \alpha_{ij} = 2$. Therefore, $x_j \in UB(x_{i_1}, x_{i_1})$.

(2) We suppose that $x_j \neq x_{i_1} \lor x_{i_2}$. Then, there exists $k \in [1,\ldots,n]$ such that $x_{i_1} < x_k$, $x_{i_2} < x_k$, and $x_k < x_j$. Therefore, $\alpha_{ik} = \alpha_{ik} = 2$ and $\alpha_{ik} = 2$. \(\square\)
Proposition 2.4. Let $i_0, i_1, i_2$ be three distinct elements of $\{1, 2, \ldots, n\}$ such that $x_{i_0} \parallel x_{i_2}$. Then, $x_{i_0} < x_{i_1} \lor x_{i_2}$ if and only if

$$-6 \not\in r_{i_0}(\alpha_X) + (-1) \cdot r_{i_1}(\alpha_X) + (-1) \cdot r_{i_2}(\alpha_X)$$

(2.4)

and

$$4 \not\in 5 \cdot r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + r_{i_2}(\alpha_X).$$

(2.5)

Proof. Let $x_{i_0} < x_{i_1} \lor x_{i_2}$. We suppose that the condition (2.4) is not satisfied. Then, there exists $k \in \{1, 2, \ldots, n\}$ such that $\alpha_{i_0 k} = -2, \alpha_{i_1 k} = 2, \alpha_{i_2 k} = 2$. Thus, $x_{i_1} \lor x_{i_2} \leq x_k < x_{i_0}$, which is a contradiction.

Now, we suppose that the condition (2.5) is not satisfied. Then, there exists $k$ such that $\alpha_{i_0 k} = 0, \alpha_{i_1 k} = 2, \alpha_{i_2 k} = 2$. Therefore, $x_{i_1} < x_k$ and $x_{i_2} < x_k$ which means that $x_k \in H\mathcal{B}(x_{i_1}, x_{i_2})$. Hence, $x_{i_1} \lor x_{i_2} \leq x_k$. By assumption, $x_{i_0} < x_{i_1} \lor x_{i_2} \leq x_k$, which is a contradiction since $x_{i_0} \parallel x_k$.

Conversely, we suppose that the conditions (2.4) and (2.5) are satisfied. We prove that $x_{i_0} < x_{i_1} \lor x_{i_2}$. Let $x_j = x_{i_1} \lor x_{i_2}$. Then, $x_{i_0} < x_j$ and $x_{i_2} < x_j$. Therefore, $\alpha_{i_0 j} = \alpha_{i_2 j} = 2$. It suffices to prove that $x_{i_0} < x_j$. We have the following cases:

(a) Let $x_{i_0} \parallel x_j$. In this case $\alpha_{i_0 j} = 0$ and, therefore, $4 \not\in r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + r_{i_2}(\alpha_X)$, which is a contradiction.

(b) Let $x_j < x_{i_0}$. In this case $\alpha_{i_0 j} = -2$. Hence, $-6 \not\in r_{i_0}(\alpha_X) + (-1) \cdot r_{i_1}(\alpha_X) + (-1) \cdot r_{i_2}(\alpha_X)$, which is a contradiction.

Thus, $x_{i_0} < x_j$. □

Theorem 2.5. Let $i_0 \in \{1, 2, \ldots, n\}$. The element $x_{i_0}$ is join prime if and only if $0 \not\in r_{i_0}(\alpha_X)$ or $-2 \not\in r_{i_0}(\alpha_X)$ and for every $i_1, i_2 \in \{1, \ldots, n\} \setminus \{i_0\}$, with $x_{i_1} \parallel x_{i_2}$, the following conditions are satisfied:

(C1) If

$$5 \in r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + r_{i_2}(\alpha_X),$$

(2.6)

then

$$6 \in r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + (-1) \cdot r_{i_2}(\alpha_X).$$

(2.7)

(C2) If

$$-6 \not\in r_{i_0}(\alpha_X) + (-1) \cdot r_{i_1}(\alpha_X) + (-1) \cdot r_{i_2}(\alpha_X),$$

(2.8)

$$4 \not\in 5 \cdot r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + r_{i_2}(\alpha_X),$$

(2.9)

then

$$\left|\{i \in \{1, \ldots, n\} : \alpha_{i_0 i_1} = 1\right| \neq 3$$

(2.10)

and

$$3 \not\in (-1) \cdot r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + r_{i_2}(\alpha_X).$$

(2.11)

Proof. Let $x_{i_0} \in X$. We suppose that the conditions of the theorem are satisfied. We shall prove that the element $x_{i_0}$ is join prime. Since $0 \not\in r_{i_0}(\alpha_X)$ or $-2 \not\in r_{i_0}(\alpha_X)$, the element $x_{i_0}$ of $X$ is nonzero. Let $x_{i_1}, x_{i_2} \in X$, where $i_1, i_2 \in \{1, 2, \ldots, n\} \setminus \{i_0\}$, such that $x_{i_0} \parallel x_{i_1}$ and $x_{i_0} \leq x_{i_1} \lor x_{i_2}$. It suffices to prove that $x_{i_0} \leq x_{i_1} \lor x_{i_2} \leq x_{i_2}$. First we observe that $x_{i_0} < x_{i_1} \lor x_{i_2}$. Indeed, if $x_{i_0} = x_{i_1} \lor x_{i_2}$, then by Proposition 2.3 (1), the equation (2.6) is satisfied. Thus, $6 \in r_{i_0}(\alpha_X) + r_{i_1}(\alpha_X) + (-1) r_{i_2}(\alpha_X)$ or equivalently, by Proposition 2.3 (2), $x_{i_0} \not\in x_{i_1} \lor x_{i_2},$
which is a contradiction. Thus, $x_{i_0} < x_{1_1} \lor x_{2_1}$. Since $x_{i_1} \parallel x_{i_2}$, by Proposition 2.4, (2.8) and (2.9) are satisfied. Therefore, by assumption, (2.10) and (2.11) are satisfied. Thus, by Proposition 2.2 (2), $\alpha_{i_{01}} = 2$ or $\alpha_{i_{02}} = 2$. Equivalently, $x_{i_0} < x_{1_1}$ or $x_{i_0} < x_{2_1}$. Thus, $x_{i_0}$ is join prime element of $X$.

Conversely, let $x_{i_0}$ be a join prime element of $X$. We shall prove the conditions of the theorem. Since the element $x_{i_0}$ of $X$ is nonzero, we have $0 \in r_0(A_X^{\alpha})$ or $-2 \in r_0(A_X^{\alpha})$. We consider $i_1, i_2 \in \{1, \ldots, n\} \setminus \{i_0\}$ with $x_{i_0} \parallel x_{i_1}$. We suppose that $\alpha_{i_1} = 2$ or $\alpha_{i_1} = 2$. By Proposition 2.3 (2), $x_{i_0} = x_{i_1} \lor x_{i_2}$, which is a contradiction since $x_{i_0}$ is join prime. We suppose that the conditions (2.8) and (2.9) are satisfied. Then, by Proposition 2.4, $x_{i_0} < x_{1_1} \lor x_{2_1}$. We prove the equations (2.10) and (2.11). We suppose that $\alpha_{i_0} = 0$ or $\alpha_{i_0} = 0$. Moreover, since $x_{i_0} < x_{i_1} \lor x_{i_2}$ and $x_{i_0}$ is a join prime element of $X$, we have $\alpha_{i_0} = 2$ or $\alpha_{i_0} = 2$. Therefore, we have the following cases:

1. $\alpha_{i_0} = -2$ or $\alpha_{i_0} = -2$

   In this case $x_{i_0} < x_{i_1}$, which is a contradiction since $x_{i_0} \parallel x_{i_1}$.

2. $\alpha_{i_0} = 2$ or $\alpha_{i_0} = -2$

   In this case $x_{i_0} < x_{i_1}$, which is a contradiction since $x_{i_0} \parallel x_{i_1}$.

Let $X = \{x_1, \ldots, x_n\}$ be a finite lattice. Using Theorem 2.5 we can compute the set of join prime elements of $X$ in the following manner.

**Algorithm 2.6.** Our intended algorithm consists of the following steps:

**Step 1:** Find the $n$ rows $r_1(A_X^{\alpha}), r_2(A_X^{\alpha}), \ldots, r_n(A_X^{\alpha})$ of the order-matrix $A_X^{\alpha}$ of $X$.

**Step 2:** Set $k = 0$ and $j = 1$.

**Step 3:** If $0 \in r_j(A_X^{\alpha})$ or $-2 \in r_j(A_X^{\alpha})$, then go to Step 4. Otherwise, go to Step 7.

**Step 4:** Find the $1 \times n$ matrices

\[
\begin{align*}
r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + r_{i_3}(A_X^{\alpha}) \\
r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + (-1) \cdot r_{i_3}(A_X^{\alpha}) \\
r_{i_1}(A_X^{\alpha}) + (-1) \cdot r_{i_2}(A_X^{\alpha}) + (-1) \cdot r_{i_3}(A_X^{\alpha}) \\
5 \cdot r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + r_{i_3}(A_X^{\alpha}) \\
(-1) \cdot r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + r_{i_3}(A_X^{\alpha}),
\end{align*}
\]

and the set

\[
\{i \in \{1, \ldots, n\} : a_{i_{02}}^{\alpha_{i_{02}}} = 1\}
\]

for each $i_1, i_2 \in \{1, 2, \ldots, n\} \setminus \{j\}$ with $\alpha_{i_1} = 0$.

**Step 5:** Check the following:

- If $5 \in r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + r_{i_3}(A_X^{\alpha})$,

  then $6 \in r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + (-1) \cdot r_{i_3}(A_X^{\alpha})$. 

**Step 6:** Check the following:

- If $5 \in r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + r_{i_3}(A_X^{\alpha})$,

  then $6 \in r_{i_1}(A_X^{\alpha}) + r_{i_2}(A_X^{\alpha}) + (-1) \cdot r_{i_3}(A_X^{\alpha})$. 

**Step 7:** Stop. 

T. Dube et al. / Filomat 31:10 (2017), 2901–2915
• If 

\[ -6 \not\in r_i(A_X^+) + (-1) \cdot r_i(A_X^+X) + (-1) \cdot r_i(A_X^+) \]

\[ 4 \not\in 5 \cdot r_i(A_X^+) + r_i(A_X^+X) + r_i(A_X^+) \]

then

\[ |\{ i \in \{1, \ldots, n\} : a_{ji}^{\text{join}} = 1 \}| \neq 3 \]

and

\[ 3 \not\in (-1) \cdot r_i(A_X^+) + r_i(A_X^+X) + r_i(A_X^+) \]

If the above are true, then put \( x_i = x_j \) and go to Step 6. Otherwise, go to Step 7.

**Step 6:** Put \( k \leftarrow k + 1 \) (meaning the number \( k + 1 \) replaces the old \( k \)) and go to Step 7.

**Step 7:** If \( j < n \), then put \( j \leftarrow j + 1 \) (meaning the number \( j + 1 \) replaces the old \( j \)) and go to Step 3. Otherwise, go to Step 8.

**Step 8:** Create the set \( \{x_0, x_1, \ldots, x_n\} \).

**Remark 2.7.** Let \( m \) be the number of 2-element antichains. Algorithm 2.6 iterates at most \( n \cdot \binom{m}{2} \) times to calculate 5 matrices and the set mentioned above.

**Example 2.8.** Let \((X, \leq)\) be the lattice represented by the diagram of Figure 2.

![Figure 2: The lattice \((X, \leq)\)](image)

The order-matrix of \( X \) is the following \( 8 \times 8 \) matrix:

\[
A_X^+ = \begin{bmatrix}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
-2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 \\
-2 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \\
-2 & -2 & -2 & 1 & 2 & 2 & 2 & 2 \\
-2 & -2 & -2 & -2 & 1 & 2 & 2 & 2 \\
-2 & -2 & -2 & -2 & -2 & 1 & 2 & 2 \\
-2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 \\
\end{bmatrix}
\]

Using Theorem 2.5, we show that \( x_4 \) is not join prime, but \( x_5 \) is. Later on in Example 4.6 we shall exhibit all join primes of this lattice. Consider the element \( x_5 \). For the elements \( x_6 \) and \( x_7 \), for which \( x_6 \parallel x_7 \), we have

\[ 5 \not\in r_6(A_X^+X) + r_7(A_X^+X) + r_5(A_X^+) = \begin{bmatrix}
-6 & -6 & -6 & -6 & -6 & 3 & 3 & 6 \\
\end{bmatrix}. \]
Moreover,
\[-6 \not\in r_5(A_\chi) + (-1) \cdot r_6(A_\chi) + (-1) \cdot r_7(A_\chi) = \begin{pmatrix} 2 & 2 & 2 & 5 & 1 & 1 & -2 \\ \end{pmatrix},
\]
\[4 \not\in 5 \cdot r_5(A_\chi) + r_6(A_\chi) + r_7(A_\chi) = \begin{pmatrix} -14 & -14 & -14 & -14 & 1 & 11 & 11 & 14 \\ \end{pmatrix},\]
\[\{ i \in \{ 1, 2, \ldots, 8 \} : a_{ij}^{567} = 1 \} = \emptyset\]
and
\[3 \not\in (-1) \cdot r_5(A_\chi) + r_6(A_\chi) + r_7(A_\chi) = \begin{pmatrix} -2 & -2 & -2 & -2 & -5 & -1 & -1 & 2 \\ \end{pmatrix}.
\]

In a similar way we see that for the elements $x_2$ and $x_3$, for which $x_2 \parallel x_3$, the conditions of the theorem are satisfied. It follows directly that the element $x_5$ is join prime.

The element $x_4$ is not join prime since
\[5 \in r_2(A_\chi) + r_3(A_\chi) + r_4(A_\chi) = \begin{pmatrix} -6 & -1 & 5 & 6 & 6 & 6 \\ \end{pmatrix},\]
and
\[6 \not\in r_2(A_\chi) + r_3(A_\chi) + (-1) \cdot r_4(A_\chi) = \begin{pmatrix} -2 & 3 & 3 & 2 & 2 & 2 \\ \end{pmatrix}.
\]

3. An Algorithm for Calculating the Height of a Finite Poset

We start by reminding the reader how the incidence matrix is defined.

**Definition 3.1.** (see [1]) Let $(X, \leq)$ be a poset, where $X = \{ x_1, \ldots, x_n \}$. The $n \times n$ matrix $T^{\leq}_X = (t_{ij})$, where
\[t_{ij} = \begin{cases} 1, & \text{if } x_i \leq x_j \\ 0, & \text{otherwise} \end{cases}
\]
is called the incidence matrix of $X$.

**Notation 3.2.** (1) The $n$ columns of the incidence matrix $T^{\leq}_X$ are denoted by $C_1(T^{\leq}_X), \ldots, C_n(T^{\leq}_X)$. Let
\[C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and} \quad C' = \begin{pmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{pmatrix}
\]
be two $n \times 1$ matrices. Then, by $C + C'$ we denote the $n \times 1$ matrix
\[C + C' = \begin{pmatrix} c_1 + c'_1 \\ c_2 + c'_2 \\ \vdots \\ c_n + c'_n \end{pmatrix}.
\]
Also, we write $C \leq C'$ if only if $c_i \leq c'_i$ for each $i = 1, \ldots, n$.

(2) We denote by $1_i$, and $2_i$, where $i \in \{ 1, \ldots, n \}$, respectively, the $n \times 1$ matrices
\[\begin{pmatrix} a^1_{i1} \\ a^1_{i2} \\ \vdots \\ a^1_{in} \\ a^2_{i1} \\ \vdots \\ a^n_{i1} \end{pmatrix}, \quad a^i_{ji} = \begin{cases} 1, & \text{if } i = i_j \\ 0, & \text{otherwise} \end{cases}
\]
and
\[
\begin{pmatrix}
\rho^1_{i_1} \\
\rho^2_{i_1} \\
\vdots \\
\rho^m_{i_1}
\end{pmatrix}, \quad \rho^i_{i_1} = \begin{cases} 
2, & \text{if } i = i_1 \\
0, & \text{otherwise}.
\end{cases}
\]

Also, we denote by \(2_{i_1i_2...i_m}\), where \(i_1, i_2, \ldots, i_m\) are distinct elements of \([1, \ldots, n]\) and \(m \leq n\), the \(n \times 1\) matrix
\[
\begin{pmatrix}
y^1_{i_1i_2...i_m} \\
y^2_{i_1i_2...i_m} \\
\vdots \\
y^m_{i_1i_2...i_m}
\end{pmatrix}, \quad y^i_{i_1i_2...i_m} = \begin{cases} 
2, & \text{if } i \in \{i_1, i_2, \ldots, i_m\} \\
0, & \text{otherwise}.
\end{cases}
\]

(3) Let \(T_X^e = (t_{ij})\) be the incidence matrix of \(X\). We denote by \(C_{ijl}(T_X^e)\), where \(i_1, i_2\) are distinct elements of \([1, \ldots, n]\), the \(n \times 1\) matrix
\[
\begin{pmatrix}
c^1_{i_1i_2} \\
c^2_{i_1i_2} \\
\vdots \\
c^m_{i_1i_2}
\end{pmatrix}, \quad c^i_{i_1i_2} = \begin{cases} 
t_{i_1i_2}, & \text{if } i = i_1 \\
1, & \text{if } i = i_2 \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem 3.3.** Let \((X, \leq)\) be a poset, \(X = \{x_1, \ldots, x_n\}\) and let \(i_1, i_2, \ldots, i_k\), where \(k \leq n\), be distinct elements of \([1, \ldots, n]\). Then,
\[
1_{i_1} + C_{i_1i_2}(T_X^e) + C_{i_2i_3}(T_X^e) + \ldots + C_{i_{k-1}i_k}(T_X^e) \geq 2_{i_1i_2...i_k} \quad (3.1)
\]
if and only if
\[
x_{i_1} > x_{i_2} > x_{i_3} > \ldots > x_{i_k}. \quad (3.2)
\]

**Proof.** Let the condition (3.1) be satisfied. We prove that \(x_{i_1} > x_{i_2} > x_{i_3} > \ldots > x_{i_k}\). We suppose that there exists \(a \in \{1, 2, \ldots, k-1\}\) such that \(x_{i_{a+1}} \neq x_{i_a}\) or equivalently \(t_{i_{a+1}i_a} = 0\). By the definition of the matrix \(C_{i_{a+1}i_a}(T_X^e)\), we have \(c^i_{i_{a+1}i_a} = t_{i_{a+1}i_a} = 0\).

Let \(a = 1\). Then, \(c^1_{i_1i_2} = 0\). Since the elements \(i_1, i_2, \ldots, i_k\) of \([1, \ldots, n]\) are distinct, by the definition of the matrix \(C_{i_1i_2}(T_X^e)\) we have \(c^i_{i_1i_2} = 0\), for each \(i \in \{2, 3, \ldots, k-1\}\). Thus,
\[
a_{i_1} + c^i_{i_1i_2} + c^i_{i_1i_3} + \ldots + c^i_{i_1i_k} = 1 + 0 + 0 + \ldots + 0 = 1 < 2 = y^1_{i_1i_2...i_k}
\]
and, therefore, \(1_{i_1} + C_{i_1i_2}(T_X^e) + C_{i_2i_3}(T_X^e) + \ldots + C_{i_{k-1}i_k}(T_X^e) \not\geq 2_{i_1i_2...i_k}\), which is a contradiction.

Let \(a > 1\). By the definition of the matrix \(C_{i_{a-1}i_a}(T_X^e)\), \(c^i_{i_{a-1}i_a} = 1\). Since the elements \(i_1, i_2, \ldots, i_k\) of \([1, \ldots, n]\) are distinct, by the definition of the matrix \(C_{i_{a-1}i_a}(T_X^e)\) we have \(c^i_{i_{a-1}i_a} = 0\), for each \(i \in \{1, 2, \ldots, k-1\} \setminus \{a-1, a\}\). Hence,
\[
a^i_{i_1} + c^i_{i_1i_2} + \ldots + c^i_{i_1i_{a-1}} + c^i_{i_1i_{a+1}} + \ldots + c^i_{i_1i_k} = 0 + 0 + \ldots + 1 + 0 + \ldots + 0 = 1 < 2 = y^i_{i_1i_2...i_k}
\]
and, therefore, \(1_{i_1} + C_{i_1i_2}(T_X^e) + C_{i_2i_3}(T_X^e) + \ldots + C_{i_{k-1}i_k}(T_X^e) \not\geq 2_{i_1i_2...i_k}\), which is a contradiction. Thus, in each case the condition (3.2) is satisfied.

Conversely, let the condition (3.2) be satisfied. We prove that
\[
1_{i_1} + C_{i_1i_2}(T_X^e) + C_{i_2i_3}(T_X^e) + \ldots + C_{i_{k-1}i_k}(T_X^e) \geq 2_{i_1i_2...i_k},
\]
We suppose that \( 1_i + C_{i,\infty}(T_X^\alpha) + C_{i,\infty}(T_X^\beta) + \ldots + C_{i,\infty}(T_X^{k-1}) \geq 2_{\infty,\ldots,\infty}. \) There exists \( a \in \{1, 2, \ldots, k-1\} \) such that \( \alpha_{i,a} + \beta_{i,a} + \ldots + \gamma_{i,a} \in \{0, 1\}. \)

Let \( a = 1. \) Then, \( \alpha_{i,1} = 1. \) Hence, \( \alpha_{i,1} + \beta_{i,1} + \ldots + \gamma_{i,1} = 1 \) and \( \gamma_{i,1} = 0. \) Therefore, by the definition of the matrix \( C_{i,\infty}(T_X^\alpha) \), we have \( t_{i,\infty} = 0 \) or equivalently \( x_i \not< x_n, \) which is a contradiction.

Let \( a > 1. \) By the definition of the matrix \( C_{i,\infty}(T_X^\alpha), \) \( \gamma_{i,a} = 1. \) So we have

\[
\alpha_{i,a} + \beta_{i,a} + \ldots + \gamma_{i,a} + \ldots + \gamma_{i,a} = 1.
\]

Thus, \( \gamma_{i,a} = 0 \) and, therefore, by the definition of the matrix \( C_{i,\infty}(T_X^\alpha) \), we have \( t_{i,\infty} = 0 \) or equivalently \( x_i \not< x_n, \) which is a contradiction. Thus, the condition (3.1) is satisfied. □

**Corollary 3.4.** Let \( (X, \leq) \) be a poset, where \( X = \{x_1, \ldots, x_n\}. \) Then, \( h(X) = k, \) \( 1 \leq k \leq n, \) if and only if there exist distinct elements \( i_1, i_2, \ldots, i_k \) of \( \{1, \ldots, n\} \) such that

1. \( 1_i + C_{i,\infty}(T_X^\alpha) + \ldots + C_{i,\infty}(T_X^{k-1}) \geq 2_{i,\infty,\ldots,\infty}. \)
2. \( 1_i + C_{i,\infty}(T_X^\alpha) + \ldots + C_{i,\infty}(T_X^{k-1}) \not\geq 2_{i,\infty,\ldots,\infty} \) for every distinct elements \( i_1, i_2, \ldots, i_k \) of \( \{1, \ldots, n\}. \)

**Proof.** Let \( h(X) = k, \) where \( 1 \leq k \leq n. \) Then, there exist distinct elements \( i_1, i_2, \ldots, i_k \) of \( \{1, \ldots, n\} \) such that \( x_{i_1} < x_{i_2} < \ldots < x_{i_k}. \) By Theorem 3.3 we have

\[
1_i + C_{i,\infty}(T_X^\alpha) + \ldots + C_{i,\infty}(T_X^{k-1}) \geq 2_{i,\infty,\ldots,\infty}.
\]

Also, any subset \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}\} \) of \( X \) is not a chain or equivalently, by Theorem 3.3,

\[
1_1 + C_{1,\infty}(T_X^\alpha) + \ldots + C_{1,\infty}(T_X^{k-1}) \not\geq 2_{1,\infty,\ldots,\infty}.
\]

Conversely, suppose that there exist distinct elements \( i_1, i_2, \ldots, i_k \) of \( \{1, \ldots, n\} \) such that

\[
1_i + C_{i,\infty}(T_X^\alpha) + \ldots + C_{i,\infty}(T_X^{k-1}) \geq 2_{i,\infty,\ldots,\infty}.
\]

and

\[
1_i + C_{i,\infty}(T_X^\alpha) + \ldots + C_{i,\infty}(T_X^{k-1}) \not\geq 2_{i,\infty,\ldots,\infty}
\]

for every distinct elements \( i_1, i_2, \ldots, i_k \) of \( \{1, \ldots, n\}. \) Then, by Theorem 3.3, we have \( x_{i_1} < x_{i_2} < \ldots < x_{i_k}. \)

Also, any subset \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}\} \) of \( X \) is not a chain. Thus, \( h(X) = k. \) □

Using Corollary 3.4 we give an algorithm for computing the height of a finite poset.

**Algorithm 3.5.** Let \( (X, \leq) \) be a poset, where \( X = \{x_1, \ldots, x_n\}. \) Our intended algorithm consists of the following \( n-1 \) steps:

**Step 1.**
Find the sums \( 1_i + C_{i,\infty}(T_X^\alpha) \) for each distinct elements \( i_1, i_2 \) of \( \{1, \ldots, n\}. \) If there exist distinct elements \( i_1, i_2 \) of \( \{1, \ldots, n\} \) such that \( 1_i + C_{i,\infty}(T_X^\alpha) \geq 2_i, \) then go to the Step 2. Otherwise print \( h(X) = 1. \)

**Step 2.**
Find the sums \( 1_i + C_{i,\infty}(T_X^\alpha) + C_{i,\infty}(T_X^\beta) \) for each distinct elements \( i_1, i_2, i_3 \) of \( \{1, \ldots, n\} \) with \( 1_i + C_{i,\infty}(T_X^\alpha) \geq 2_i. \)
If there exist distinct elements \( i_1, i_2, i_3 \) of \( \{1, \ldots, n\} \) such that \( 1_i + C_{i,\infty}(T_X^\alpha) + C_{i,\infty}(T_X^\beta) \geq 2_i, \) then go to the Step 3. Otherwise print \( h(X) = 2. \)

...........
...........
...........
Step $n - 2$. Find the sums $1_{i_1} + C_{i_1}(T_X^1) + \ldots + C_{i_{n-1}}(T_X^{n-1})$ for each distinct elements $i_1, i_2, \ldots, i_{n-1}$ of $\{1, \ldots, n\}$ with 

$$1_{i_1} + C_{i_1}(T_X^1) + \ldots + C_{i_{n-1}}(T_X^{n-1}) \geq 2_{i_{n-1}}.$$ 

If there exist distinct elements $i_1^{-2}, i_2^{-2}, \ldots, i_{n-1}^{-2}$ of $\{1, \ldots, n\}$ such that 

$$1_{i_1^{-2}} + C_{i_1^{-2}}(T_X^1) + \ldots + C_{i_{n-1}^{-2}}(T_X^{n-1}) \geq 2_{i_{n-1}^{-2}},$$

then go to the Step $n - 1$. Otherwise print $h(X) = n - 2$.

Step $n - 1$. Find the sums $1_{i_1} + C_{i_1}(T_X^1) + \ldots + C_{i_{n-1}}(T_X^{n-1})$ for each distinct elements $i_1, i_2, \ldots, i_n$ of $\{1, \ldots, n\}$ with 

$$1_{i_1} + C_{i_1}(T_X^1) + \ldots + C_{i_{n-1}}(T_X^{n-1}) \geq 2_{i_{n-1}}.$$ 

If there exist distinct elements $i_1^{-1}, i_2^{-1}, \ldots, i_{n-1}^{-1}$ of $\{1, \ldots, n\}$ such that 

$$1_{i_1^{-1}} + C_{i_1^{-1}}(T_X^1) + \ldots + C_{i_{n-1}^{-1}}(T_X^{n-1}) \geq 2_{i_{n-1}^{-1}},$$

then print $h(X) = n$. Otherwise print $h(X) = n - 1$.

**Proposition 3.6.** An upper bound on the number of iterations of the Algorithm 3.5 is the number 

$$n(n - 1)(1 + (n - 2) + (n - 2)(n - 3) + \ldots + (n - 2)!).$$

**Proof.** We observe that the number of iterations Algorithm 3.5 performs in Steps 1, 2, 3, $\ldots$, $n - 1$ is 

$$n(n - 1), n(n - 1)(n - 2), n(n - 1)(n - 2)(n - 3), \ldots, n!,$$

respectively. Thus, the number of iterations the algorithm performs is 

$$n(n - 1) + n(n - 1)(n - 2) + n(n - 1)(n - 2)(n - 3) + \ldots + n! = n(n - 1)(1 + (n - 2) + (n - 2)(n - 3) + \ldots + (n - 2)!).$$

**Example 3.7.** Let $(X, \leq)$ be the poset represented by the diagram of Figure 3.

![Figure 3: The poset $(X, \leq)$](image)

The incidence matrix of $X$ is 

$$T_X^\leq = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

In order to find $h(X)$ we follow the following steps (see Algorithm 3.5):
Step 1. Find the sums $1_i + C_{iij}(T^c_X)$ for every $i_1, i_2 \in \{1, 2, \ldots, 5\}$.

We observe that there exist $5, 3 \in \{1, 2, \ldots, 5\}$ such that

$$1_5 + C_{35}(T^c_X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} = 2_5.$$ 

Thus, $h(X) > 1$.

Step 2. Find the sums $1_i + C_{ijk}(T^c_X) + C_{ijk}(T^c_X)$ for every $i_1, i_2, i_3 \in \{1, 2, \ldots, 5\}$ with $1_i + C_{ijk} \geq 2_i$.

We observe that there exist $5, 3, 2 \in \{1, 2, \ldots, 5\}$ such that

$$1_5 + C_{35}(T^c_X) + C_{25}(T^c_X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = 2_{53}.$$ 

Thus, $h(X) > 2$.

Step 3. Find the sums $1_i + C_{ijl}(T^c_X) + C_{ijm}(T^c_X) + C_{ijm}(T^c_X)$ for every $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, 5\}$ with $1_i + C_{ijl} \geq 2_{ij}$.

We have only the following cases:

$$1_4 + C_{34}(T^c_X) + C_{25}(T^c_X) + C_{12}(T^c_X) = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \not\geq \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} = 2_{432},$$

$$1_5 + C_{35}(T^c_X) + C_{23}(T^c_X) + C_{12}(T^c_X) = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \not\geq \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} = 2_{532}.$$ 

Thus, $h(X) = 3$.

4. Krull Dimension of Finite Lattices through the Order Matrices

Throughout this section $(X, \leq)$ denotes a lattice, where $X = \{x_1, \ldots, x_n\}$ and $A^c_X = (a_{ij})$ denotes the $n \times n$ order-matrix of $X$.

Proposition 4.1. The following statements are true:

1. For every $i \in \{1, \ldots, n\}$ we have $\uparrow x_i = \{x_j: a_{ij} \in \{1, 2\}\}$.

2. $\mathcal{F}(X) = \{x_j: a_{ij} \in \{1, 2\}: i = 1, 2, \ldots, n\} \setminus \{X\}$. 
(3) \( \uparrow x_i \subset \uparrow x_i \) if and only if \( x_i < x_i \).

(4) \( \uparrow x_i \subset \uparrow x_i \) if and only if \( \alpha_{ini} = -2 \).

**Proof.** (1) Follows from the definitions of the set \( \uparrow x_i \) and the order-matrix.

(2) Since \( X \) is finite, we have \( \mathcal{F}(X) = \{ \uparrow x_i : i = 1, 2, \ldots, n \} \setminus \{X\} \). Therefore, by statement (1) we have that \( \mathcal{F}(X) \) is the desired set.

(3) Let \( \uparrow x_i \subset \uparrow x_i \). Then, \( x_i \in \uparrow x_i \subset \uparrow x_i \) and, therefore, \( x_i < x_i \). Conversely, let \( x_i < x_i \). We prove that \( \uparrow x_i \subset \uparrow x_i \). Indeed, let \( x_i \in \uparrow x_i \). Then, \( x_i \subset \uparrow x_i \) and, therefore, \( x_i < x_i \). Thus, \( x_i \in \uparrow x_i \).

(4) Follows from the definition of the order-matrix and the statement (3). □

**Proposition 4.2.** Let \( x_i, x_i, \ldots, x_i \in X \). The following statements are equivalent:

1. \( \uparrow x_i \subset \uparrow x_i \subset \cdots \subset \uparrow x_i \).
2. \( x_i > x_i > \cdots > x_i \).
3. \( \alpha_{iin} = \alpha_{ini} = \cdots = \alpha_{ini} = -2 \).

**Proof.** The proof is a straightforward verification from Proposition 4.1 (3) and (4). Indeed, for \( l = 1, 2, \ldots, k \) we have \( \uparrow x_i \subset \uparrow x_i \) and the order-matrix. □

**Proposition 4.3.** Let \( x_i, x_i, \ldots, x_i \in X \). Then, \( \uparrow x_i \subset \uparrow x_i \) if and only if the following condition is satisfied:

\[ 1 \notin r_a(A^x_i) - r_b(A^x_i). \]

**Proof.** Let \( \uparrow x_i \subset \uparrow x_i \). We suppose that \( 1 \in r_a(A^x_i) - r_b(A^x_i) \). We have the following cases:

1. \( \alpha_{ini} = 1 \) and \( \alpha_{ini} = 2 \).
2. \( \alpha_{ini} = 1 \) and \( \alpha_{ini} = 0 \).

Therefore, in each case \( \alpha_{ini} \neq -2 \), which contradicts Proposition 4.1 (4). Thus, \( 1 \notin r_a(A^x_i) - r_b(A^x_i) \).

Conversely, we suppose that the condition of the proposition is satisfied. We prove that \( \uparrow x_i \subset \uparrow x_i \). By Proposition 4.1 (4) it suffices to prove that \( \alpha_{ini} = -2 \). We have the following cases:

1. Let \( \alpha_{ini} = 0 \).
2. Let \( \alpha_{ini} = 2 \).

Since \( \alpha_{ini} = 1 \) and \( \alpha_{ini} = 0 \), we have \( 1 \notin r_a(A^x_i) - r_b(A^x_i) \), which is a contradiction.

**Theorem 4.4.** Let \( x_i, x_i, \ldots, x_i \in X \). Then, \( \uparrow x_i \subset \uparrow x_i \subset \cdots \subset \uparrow x_i \) if and only if the following condition is satisfied:

\[ 1 \notin r_a(A^x_i) - r_b(A^x_i) \text{ for } l = 1, 2, \ldots, k. \]

**Proof.** The proof is a straightforward verification from Proposition 4.3. Indeed, for \( l = 1, 2, \ldots, k \) we have \( \uparrow x_i \subset \uparrow x_i \) if and only if \( 1 \notin r_a(A^x_i) - r_b(A^x_i) \). □

**Proposition 4.5.** If \( \mathcal{P} \mathcal{F}(X) \neq \emptyset \), then the following statements hold:

1. \( \mathcal{P} \mathcal{F}(X) = \{ \uparrow x : x \text{ is join prime} \} \).
2. \( \mathcal{P} \mathcal{F}(X) = \{ x_j : a_i \in [1, 2], x_i \text{ is join prime} \} \).
3. \( \text{Kdim}(X) = \max \{ k : \uparrow x_i \subset \cdots \subset \uparrow x_i, x_i, \ldots, x_i \text{ are join primes} \} \).
4. \( \text{Kdim}(X) = \max \{ k : x_i > \cdots > x_i, x_i, \ldots, x_i \text{ are join primes} \} \).
5. \( \text{Kdim}(X) = h(Y) - 1 \), where \( Y \) is the poset of join primes elements of \( X \).
Proof. (1) Let \( F \in \mathcal{P}F(X) \). Since \( X \) is finite, \( \land F \in X \). We observe that \( x \) is join prime. Let \( a, b \in X \) such that \( x \leq a \lor b \). Then, \( a \lor b \in F \). Hence, \( a \in F \) or \( b \in F \) and, therefore, \( x \leq a \) or \( x \leq b \). Thus, \( x \) is join prime.

Conversely, let \( x \) be a join prime element of \( X \). We set \( F = \uparrow x \) and prove that \( F \in \mathcal{P}F(X) \). Let \( a, b \in X \) such that \( a \lor b \in F \). Then, \( x \leq a \lor b \). Hence, \( x \leq a \) or \( x \leq b \) and, therefore, \( a \in F \) or \( b \in F \). Thus, \( F \) is prime.

(2) It follows from Proposition 4.1 (1) and from the (1) of this proposition.

(3) It is known that

\[
\text{Kdim}(X) = \max\{k : \text{there exist prime filters } F_0 \subset F_1 \subset \cdots \subset F_k\}.
\]

Therefore, the result follows from the (1).

(4) Follows from Proposition 4.2 and the statement (3).

(5) Follows from the definition of the height and the statement (4). □

Example 4.6. Consider the finite lattice \((X, \leq)\) which is represented in the Example 2.8. By Algorithm 2.6 we find that the poset of join primes elements of \( X \) is \( Y = \{x_2, x_3, x_5, x_6, x_7\} \) and represented by the diagram of Figure 4.

![Figure 4: The poset \((Y, \leq)\)](image)

By Example 3.7 we have \( h(Y) = 3 \). Hence, by Proposition 4.5 (5),

\[
\text{Kdim}(X) = h(Y) - 1 = 2.
\]

5. Questions

An element \( x \) of a finite lattice \((X, \leq)\) is said to be meet prime (see, for example, [4]) if \( x \neq 1 \) and the inequality \( a \land b \leq x \) implies \( a \leq x \) or \( b \leq x \), for all \( a, b \in X \).

Question 1. Are there some characterizations of the meet-prime elements of a finite lattice, using order matrices?

Question 2. We know that for an arbitrary finite lattice \((X, \leq)\) with \( \mathcal{P}F(X) \neq \emptyset \) the Krull dimension can be defined, also, as follows:

\[
\text{Kdim}(X) = \max\{k : \text{there exist prime ideals } P_0 \subset P_1 \subset \cdots \subset P_k\}.
\]

Are there some characterizations of the prime ideals and, therefore, of the Krull dimension, using matrices?

Acknowledgements. The authors would like to thank the referees for their careful reading of this paper and helpful comments.
References