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Common Fixed Point Theorems for Hybrid Pair of Mappings

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Abstract. In this paper we define the concept of conditionally commuting mappings and establish some common fixed point theorems for hybrid pair of mappings satisfying a nonexpansive type condition.

1. Introduction

The literature of fixed point theory contains numerous fixed point theorems for the pair of single valued mappings in metric and Banach spaces. There exists many situations when both mappings under examinations are not single valued. Therefore, fixed/coincidence point theorems for hybrid pair of maps are also worth investigating. Nadler [4] started the study of fixed points for multivalued mappings. Later several mathematicians were attracted towards this field because of its applicability in diverse disciplines of mathematics, statistics, engineering and economics. Singh and Kulshrestha [10] gave the concept of coincidence points for single valued and multivalued mappings. This result was generalized by several mathematicians taking different types of hybrid contraction conditions.

Meanwhile, the concept of commutativity and its weaker versions like compatibility, weak compatibility, R-weak commutativity etc. have been extended for the multivalued mappings as well as for the pair of a singlevalued and multivalued mappings (see [6], [8], [9]). The concept of conditionally commuting mappings was introduced by Pant et al. [5]. Following this concept, we extend and generalize the concept of conditionally commuting for a pair of singlevalued and multivalued mappings by introducing following definition.

Definition 1.1. *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to 2^X$. Then the pair (f, T) is said to be conditionally commuting if they commute on the subset of the set of coincidence points whenever the set of their coincidences is nonempty.*

It is remarkable that the pair (f, T) does not necessarily commute at all the coincidence points. It is illustrated by following example.

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Example 1.2. *Let $X = [2, 20]$, define f and T as follows:*

$$f(x) = \begin{cases} 4 & \text{if } 2 < x \leq 5 \\ 2 & \text{if } x \geq 5 \end{cases}$$

and

$$T(x) = \begin{cases} \{2, 4\} & \text{if } x = 2 \\ \{x, 2x\} & \text{if } 2 < x < 8 \\ \{\frac{x}{4}, \frac{x}{2}\} & \text{if } x \geq 8. \end{cases}$$

Here, $f(2) = 2 \in T(2), f(4) = 4 \in T(4)$ and $f(8) = 2 \in T(8)$ i.e., $2, 4$and 8 are coincidence points and f and T. Further, $fT(2) = \{2, 4\} = Tf(2)$, $fT(8) = \{2, 4\} = Tf(8)$ and $fT(4) = \{4, 8\} \neq Tf(4)$.
Since f and T commute at subset $\{2, 8\}$ of the set of coincidence points $\{2, 4, 8\}$. Thus f and T are conditionally commuting mappings.

Let (X, d) be a metric space and H denotes the Hausdorff metric on $C(X)$ (resp.$CB(X), CL(X)$) induced by metric d, where $C(X)$ (resp.$CB(X), CL(X)$) is the collection of all nonempty compact (resp. closed and bounded, closed) subsets of X.

An orbit of multivalued mapping T at a point x_0 in X is a sequence $\{x_n : x_n \in Tx_{n-1}\}$. A space X is T-orbitally complete if every Cauchy sequence of the form $\{x_n : x_n \in Tx_{n-1}\}$ converges in X. If for a point x_0 in X, there exists a sequence $\{x_n\} \subset X$ such that $fx_{n+1} \in Tx_n, n = 0, 1, 2, \ldots$, then $O_f = \{fx_n : n = 0, 1, 2, \ldots\}$ is an orbit of (T, f) at x_0. A space X is called (T, f)-orbitally complete if every Cauchy sequence of the form $\{fx_n : fx_{n+1} \in Tx_n\}$ converges in X (see [1], [4], [7], [10]).

Lemma 1.3. *[4]: Let A, B be in $CB(X)$, then for any $\epsilon > 0$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \leq H(a, b) + \epsilon$. If A, B are in $C(X)$, then one can choose $b \in B$ such that $d(a, b) \leq H(A, B)$.*

2. Main Results

Theorem 2.1. *Let (X, d) be a metric space. $T : X \to C(X), f : X \to X$ such that $T(X) \subset 2^{f(X)}$ where $2^{f(X)}$ is set of all subset of $f(X)$. Suppose T and f satisfy the condition*

$$H(Tx, Ty) \leq a \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[M(x, y) + m(x, y)]\right\} + c\,[M(x, y) + hm(x, y)] \quad (1)$$

where $M(x, y) = \max\{d(fx, Ty), d(fy, Tx)\}$ and $m(x, y) = \min\{d(fx, Ty), d(fy, Tx)\}$ with $a, c \geq 0$ such that $a+2c = 1$ and $h < 1$, then there exists a coincidence point $z \in X$ if either of the following condition is satisfied.

i. X is (T, f)-orbitally complete and f is surjective.
ii. $f(X)$ is (T, f)-orbitally complete.

Further, if pair (f, T) is conditionally commuting, then f and T have a common fixed point.

Proof. Let $x_0 \in X$, since $T(X) \subset 2^{f(X)}$ then there exists a point $x_1 \in X$ such that $y_1 = fx_1 \in Tx_0$ and in general $y_{n+1} = fx_{n+1} \in Tx_n$, for each $n = 0, 1, 2, \ldots$.
By (1) and based on the lemma (1.3),

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq H(Tx_n, Tx_{n+1}) \\ &\leq a \max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1}) + m(x_n, x_{n+1})]\} \\ &\quad +c[M(x_n, x_{n+1}) + hm(x_n, x_{n+1})]. \end{aligned} \quad (2)$$

Since,

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(fx_n, Tx_{n+1}), d(fx_{n+1}, Tx_n)\} = d(fx_n, Tx_{n+1}) \\ &\leq d(y_n, y_{n+2}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) \end{aligned}$$

and $m(x_n, x_{n+1}) = \min\{d(fx_n, Tx_{n+1}), d(fx_{n+1}, Tx_n)\} = d(fx_{n+1}, Tx_n) = 0$

therefore by (2)

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq a \max\left\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \frac{1}{2}[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})]\right\} \\ &\quad + c\,[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})]. \end{aligned}$$

If $d(y_n, y_{n+1}) < d(y_{n+1}, y_{n+2})$ for some n, then we have

$$d(y_{n+1}, y_{n+2}) < a\,d(y_{n+1}, y_{n+2}) + 2c\,d(y_{n+1}, y_{n+2}) = (a + 2c)d(y_{n+1}, y_{n+2}) = d(y_{n+1}, y_{n+2})$$

a contradiction. Thus

$$d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) \qquad \forall n \in \mathbb{N}. \tag{3}$$

Again, for $n = 1, 2, 3, \ldots$

$$\begin{aligned} d(y_n, y_{n+1}) &\leq H(Tx_{n-1}, Tx_n) \\ &\leq a \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-1}, x_n) + m(x_{n-1}, x_n)]\} \\ &\quad + c\,[M(x_{n-1}, x_n) + hm(x_{n-1}, x_n)] \end{aligned} \tag{4}$$

By similar arguments as above, we get

$$d(y_n, y_{n+1}) \leq a \max\left\{d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_{n+1}, y_{n+2}]\right\} + c\,d(y_{n-1}, y_{n+1}) \tag{5}$$

Again using (1), we have

$$\begin{aligned} d(y_{n-1}, y_{n+1}) &\leq H(Tx_{n-2}, Tx_n) \\ &\leq a \max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-2}, x_n) + m(x_{n-2}, x_n)]\} \\ &\quad + c\,[M(x_{n-2}, x_n) + hm(x_{n-2}, x_n)]. \end{aligned} \tag{6}$$

From(3), we get

$$\begin{aligned} M(x_{n-2}, x_n) + m(x_{n-2}, x_n) &\leq d(y_{n-2}, y_{n+1}) + d(y_n, y_{n-1}) \\ &\leq d(y_{n-2}, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + d(y_n, y_{n+1}) \\ &\leq 4d(y_{n-2}, y_{n-1}) \end{aligned} \tag{7}$$

$$\text{and } M(x_{n-2}, x_n) + hm(x_{n-2}, x_n) \leq d(y_{n-2}, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + hd(y_n, y_{n-1}) \tag{8}$$

By (3), (6), (7) and (8), we get

$$\begin{aligned} d(y_{n-1}, y_{n+1}) &\leq 2a\,d(y_{n-2}, y_{n-1}) + c[d(y_{n-2}, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + hd(y_n, y_{n-1})] \\ &\leq 2a\,d(y_{n-2}, y_{n-1}) + c(3 + h)d(y_{n-2}, y_{n-1}) \\ &\leq (2 - c(1 - h))d(y_{n-2}, y_{n-1}). \end{aligned} \tag{9}$$

From (3), (5) and (9),

$$\begin{aligned} d(y_n, y_{n+1}) &\leq ad(y_{n-1}, y_{n-2}) + c[\{2 - c(1 - h)\}]d(y_{n-2}, y_{n-1}) \\ &\leq (a + 2c)d(y_{n-2}, y_{n-1}) - c^2(1 - h)d(y_{n-2}, y_{n-1}) \\ &= (1 - c^2(1 - h))d(y_{n-2}, y_{n-1}) \\ &\leq (1 - c^2(1 - h))^2 d(y_{n-4}, y_{n-3}) \\ &\quad \ldots \\ &\leq (1 - c^2(1 - h))^{\frac{n}{2}} d(y_0, y_1). \end{aligned}$$

Hence $\{y_n\}$ is a Cuachy sequence in X. Thus from (i) or (ii)

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n \to p \in X.$$

If f is surjective then there exists $z \in X$ such that $p = fz$, this is obviously true in case (ii) as well.
To show that z is coincidence point of f and T, we have

$$\begin{aligned}
d(fz,Tz) &\le d(fz, fx_{n+1}) + d(fx_{n+1}, Tz)\\
&\le d(fz, fx_{n+1}) + H(Tx_n, Tz)\\
&\le d(fz, fx_{n+1}) + a \max\left\{d(fx_n, fz), d(fx_n, Tx_n), d(fz,Tz), \frac{1}{2}[M(x_n,z) + m(x_n,z)]\right\}\\
&\quad + c[M(x_n,z) + hm(x_n,z)] \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} M(x_n,z) &= \lim_{n\to\infty} \max\{d(fx_n,Tz), d(fz,Tx_n)\} = d(fz,Tz)\\
\lim_{n\to\infty} m(x_n,z) &= \lim_{n\to\infty} \min\{d(fx_n,Tz), d(fz,Tx_n)\} = 0\\
\Rightarrow\quad d(fz,Tz) &\le a\,d(fz,Tz) + c\,d(fz,Tz) = (a+c)d(fz,Tz).
\end{aligned}$$

Since $a + c < 1$, hence $fz \in Tz$, i.e., z is coincidence point of f and T.
Now, taking (f, T) a conditionally commuting pair, there exists two cases:
Case 1: f and T commute at z, i.e., $fTz = Tfz$, so $p = fz \in Tz$ implies that $fp \in fTz = Tfz = Tp$.
Now,

$$\begin{aligned}
d(fp,p) &\le H(Tp,Tz)\\
&\le a \max\left\{d(fz, fp), d(fp,Tp), d(fz,Tz), \frac{1}{2}[M(p,z) + m(p,z)]\right\} + c[M(p,z) + hm(p,z)].
\end{aligned}$$

We have

$$\begin{aligned}
M(p,z) &= \max\{d(fp,Tz), d(fz,Tp)\}\\
&\le \max\{d(fp,fz) + d(fz,Tz), d(fz,fp) + d(fp,Tp)\} = d(fp,fz)\\
\text{and } m(p,z) &= \min\{d(fp,Tz), d(fz,Tp)\}\\
&\le \min\{d(fp,fz) + d(fz,Tz), d(fz,fp) + d(fp,Tp)\} = d(fp,fz).\\
\text{Therefore,}\quad & d(fp,p) \le a\,d(fp,fz) + c[d(fp,fz) + hd(fp,fz)] = (a+c+ch)d(fp,fz).
\end{aligned}$$

Since $a + c + ch < 1$, hence $p = fp \in Tp$, i.e., p is a common fixed point of f and T.
Case 2: If f and T do not commute at z, then by definition of conditionally commuting mappings there exists at least one coincidence point at which f and T commute, i.e., there exists $y \in X$ such that $p = fy \in Ty$ and $fTy = Tfy$. By similar calculations as in case 1, it can be easily proved that $d(fp, p) = 0$. Hence $p = fy$ is a common fixed point of f and T. □

Replacing $C(X)$ by $CL(X)$ in Theroem (2.1), we get following result.

Theorem 2.2. *Let (X,d) be a metric space. $T : X \to CL(X)$, $f : X \to X$ satisfying all conditions of Theorem (2.1) with $a, c \ge 0$ and $a + 2c < 1$. Then f and T have common fixed point $z \in X$.*

Proof. Let $x_0 \in X$. Choose a point $x_1 \in X$ such that $y_1 = fx_1 \in Tx_0$. In general for each n choose $y_{n+1} = fx_{n+1} \in Tx_n$. Since $Tx_n \in CL(X)$, using lemma (1.3),

$$d(y_{n+1}, y_{n+2}) \le \lambda H(Tx_n, Tx_{n+1}), \qquad \text{where } \lambda > 1 \text{ and } \lambda(a+2c) < 1.$$

From (1)

$$\begin{aligned}
d(y_{n+1}, y_{n+2}) &\le \lambda H(Tx_n, Tx_{n+1})\\
&\le \lambda(a+2c)d(y_n, y_{n+1})\\
&\le \{\lambda(a+2c)\}^n d(y_0, y_1).
\end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in X, therefore $\lim_{n\to\infty} y_n \to p \in X$. Since f is surjective then there exists $z \in X$ such that $p = fz$, which is obviously true for case (ii) also.

To show that z is a coincidence point of f and T, we have

$$\begin{aligned} d(fz, Tz) &\leq d(fz, fx_{n+1}) + d(fx_{n+1}, Tz) \\ &\leq d(fz, fx_{n+1}) + \lambda H(Tx_n, Tz) \\ &\leq \lambda(a + c)d(fz, Tz). \end{aligned}$$

Which implies that $fz \in Tz$,i.e., z is coincidence point of f and T.
Since (f, T) is conditionally commuting. There are two cases:
Case1: f and T commute at z, then $p = fz \in Tz$ implies $fp \in Tp$.
Now,

$$d(fp, p) \leq \lambda H(Tp, Tz) \leq \lambda(a + c + ch)d(fp, p).$$

Since $\lambda(a + c + ch) < 1$, hence, $p = fp \in Tp$, i.e., p is common fixed point of f and T.
Case 2: If f and T do not commute at z then by definition of conditionally commuting mappings there exists $y \in X$ such that $p = fy \in Ty$ and $fTy = Tfy$.
By similar calculations as in case 1, we get $d(fp, p) = 0$. Hence $p = fy$ is common a fixed point of f and T. □

In Theorem (2.1) and Theorem (2.2) if we take $f = I$ an identity mapping we get the following corollaries:

Corollary 2.3. *Let (X, d) be a T-orbitally complete metric space. $T : X \to C(X)$ satisfying*

$$H(Tx, Ty) \quad \leq \quad a \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[M(x, y) + m(x, y)]\right\} + c\,[M(x, y) + hm(x, y)]$$

where $M(x, y) = \max\{d(x, Ty), d(y, Tx)\}$ and $m(x, y) = \min\{d(x, Ty), d(y, Tx)\}$ with $a, c \geq 0$ and $a + 2c = 1$, then T has a fixed poinrt.

Corollary 2.4. *Let (X, d) be a T-orbitally complete metric space and $T : X \to CL(X)$ satisfying all conditions of Corollary (2.3) with $a, c \geq 0$ and $a + 2c < 1$. Then T has a fixed point.*

Remark 2.5. *In Corollary (2.4) if we take T a single valued mapping, then we get the result of Ciric [3]. Further for $h = 1$ we get the result of Ciric [2].*

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