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# Contributions to the Coupled Coincidence Point Problem in $b$-Metric Spaces with Applications

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**Abstract.** The aim of this paper is to present, in the context of $b$-metric spaces, coupled coincidence point theorems under some contraction type conditions. The method is based on the application of some coincidence point theorems of Ran-Reurings type in ordered $b$-metric spaces. Some applications will illustrate the theory. The approach is new even for the case of usual metric spaces.

*To the memory of Professor Lj. Ćirić (1935–2016)*

## 1. Introduction

A relevant generalization of Banach's contraction principle was given, in the framework of a complete $b$-metric space (also called, in some papers, quasi-metric spaces or metric type spaces), by S. Czerwik, see [5]. For several extensions of this result see [1], [2], [10].

On the other hand, the concepts of coupled fixed point and of coupled coincidence point, as well as the the study of coupled fixed point problems for contraction type operators were started with the seminal papers of Gnana Bhaskar-Lakshmikantham [7] and respectively Lakshmikantham-Ćirić [11]. An extension of this problem is the coupled coincidence problem for single-valued operators. For previous results on this subject see [3], [4], [9], [11], [13], [14], [18].

Let $(X, d)$ be a metric space and $T : X \times X \to X$, $g : X \to X$ be two operators. The coupled coincidence point problem for $T$ and $g$ means to find $(x, y) \in X \times X$ satisfying

$$\begin{cases} g(x) = T(x, y) \\ g(y) = T(y, x). \end{cases} \tag{1}$$

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*Keywords*. Single-valued operator, $b$-metric space, fixed point, ordered metric space, coupled coincidence point, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing (Ostrowski) property, integral equation, differential equation

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We denote by $CCP(T, g)$ the coupled coincidence point set for $T$ and $g$.

The aim of this paper is to present, in the context of $b$-metric spaces, coupled coincidence point theorems under some contraction type conditions. The method is based on the application of some coincidence point theorems of Ran-Reurings type in ordered $b$-metric spaces. Some applications will illustrate the theory. The approach is new even for the case of usual metric spaces. Our coincidence results, proved in [15], are related with some recent theorems given in J. Garcia Falset, O. Mleşniţe [6], while the coupled coincidence approach is connected to the papers V. Berinde [3], B. S. Choudhurya, A. Kundu [4] and V. Lakshmikantham, L. Ćirić [11], as well as to some other recent results in the literature, see [13], [14], [19].

## 2. Preliminary Notions and Results

Throughout this paper $\mathbb{N}$ stands for the set of natural numbers, while $\mathbb{N}^*$ for the set of natural numbers except 0. By $\mathbb{R}_+$ we will denote the set of real non-negative numbers. We will recall now the definition of a $b$-metric space.

**Definition 2.1.** (Bakhtin [1], Czerwik [5]) Let $X$ be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a $b$-metric if the usual axioms of the metric take place with the following modification of the triangle inequality axiom $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$. A pair $(X, d)$ with the above properties is called a $b$-metric space.

Some examples of $b$-metric spaces are given in [2], [5], [10] and in many other papers.

It is worth to mention that the $b$-metric structure produces some differences to the classical case of metric spaces: the $b$-metric on a nonempty set $X$ need not be continuous, open balls in such spaces need not be open sets and so on. For example, a set $Y \subset X$ is said to be closed if for any sequence $(x_n)$ in $Y$ which is convergent to some $x$, we have that $x \in Y$.

If $X$ is a nonempty set and $f, g : X \to X$ are two single-valued operators, then we denote by $C(f, g) := \{x \in X \mid f(x) = g(x)\}$ the coincidence point set for $f$ and $g$.

The following theorem is a coincidence point theorem in an ordered complete $b$-metric spaces.

**Theorem 2.2.** ([15]) *Let $(X, d)$ be a b-metric space with constant $s_1 \geq 1$, $Y$ be a nonempty set and "$\leq$" be a partial order relation on $X$ and on $Y$. Let $\rho$ be a complete b-metric on $Y$ with constant $s_2 \geq 1$ and $g, t : X \to Y$ be two operators. Suppose that the following assumptions take place:*

*(i) $g(X) \subset t(X)$;*

*(ii) there exists a comparison function $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\rho(g(x_1), g(x_2)) \leq \varphi(d(x_1, x_2)), \text{ for all } x_1, x_2 \in X \text{ with } x_1 \leq x_2;$$

*(iii) $t : X \to Y$ is increasing with respect to $\leq$ and expansive, i.e.,*

$$\rho(t(x_1), t(x_2)) \geq d(x_1, x_2), \text{ for all } x_1, x_2 \in X;$$

*(iv) $g$ has closed graph with respect to $d$ and $\rho$ and it is increasing with respect to $\leq$;*

*(v) one of the following conditions hold:*

*(v) (a) $t : X \to Y$ is continuous;*

*(v) (b) $t(X)$ is closed with respect to the b-metric $\rho$;*

*(v) (c) the b-metrics $d$ and $\rho$ are continuous;*

*(vi) there exists $x_0 \in X$ such that $t(x_0) \leq g(x_0)$;*

*(vii) for every $y, w \in Y$ there exists $z \in Y$ which is comparable to $y$ and $w$.*

*Then $C(g, t) = \{x^*\}$.*

We recall now another coincidence point theorem of Ran-Reurings type, which will be also applied in our main section.

**Theorem 2.3.** ([15]) *Let $(X, d)$ be a b-metric space with constant $\lambda \geq 1$, $Y$ be a nonempty set and "$\leq$" be a partial order relation on $Y$. Let $\rho$ be a b-metric on $Y$ with constant $s \geq 1$ and $g, t : X \to Y$ be two operators which have closed graph. Suppose that the following assumptions take place:*

*(i)* $t(X) \subset g(X)$;
*(ii)* $(t(X), \rho)$ *is a complete subset of* $Y$;
*(iii) there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\rho(t(x), t(y)) \le \varphi(\rho(g(x), g(y))), \text{ for all } x, y \in X \text{ with } g(x) \preceq g(y);$$

*(iv) there exists* $x_0 \in X$ *such that* $g(x_0) \in t(X)$ *and* $g(x_0) \preceq t(x_0)$;
*(v)* $t$ *is increasing with respect to* $g$, *i.e.,*

$$x_1, x_2 \in X \text{ and } g(x_1) \preceq g(x_2) \Rightarrow t(x_1) \preceq t(x_2).$$

*Then, there exists* $x^* \in X$ *such that* $g(x^*) = t(x^*)$ *and the sequence* $z_n$ *defined by* $g(z_{n+1}) = t(z_n)$ *(where* $n \in \mathbb{N}$ *and* $z_0 := x_0 \in X$*) converges to* $x^*$ *as* $n \to \infty$.

*If, in addition:*
*(vi) for every* $y, w \in Y$ *there exists* $z \in Y$ *which is comparable to* $y$ *and* $w$;
*(vii)* $g$ *is an injection,*
*then* $C(t, g) = \{x^*\}$ *and the sequence* $(z_n)_{n\in\mathbb{N}}$ *defined by* $g(z_{n+1}) = t(z_n)$, *starting from any point* $z_0 \in X$ *converges to a coincidence point of* $t$ *and* $g$.

## 3. Coupled Coincidence Results

Let $(Y, \preceq)$ be a partially ordered set. Then we can endow the product space $Y \times Y$ with the following partial order:

$$\text{for } (x, y), (u, v) \in Y \times Y, \text{ we have } (x, y) \preceq_P (u, v) \Leftrightarrow x \preceq u \text{ and } y \succeq v.$$

**Lemma 3.1.** *Let* $(X, d)$ *be a b-metric space with constant* $s \ge 1$. *Then, the functionals* $d_1, d_2 : (X\times X)\times(X\times X) \to \mathbb{R}_+$ *defined by*

$$d_1((x, y), (u, v)) := d(x, u) + d(y, v), \quad d_2((x, y), (u, v)) := \max\{d(x, u), d(y, v)\}$$

*are b-metric on* $X \times X$ *with the same constant* $s \ge 1$.

Our first main result concerning the coupled coincidence problem is the following.

**Theorem 3.2.** *Let* $(X, d)$ *be a b-metric space with constant* $s_1 \ge 1$, $(Y, \rho)$ *be a b-metric space with constant* $s_2 \ge 1$ *and* $T : X \times X \to Y$, $g : X \to Y$ *be two operators with closed graph. Let* $\preceq$ *be a partial order relation on* $Y$. *We suppose:*
*(i)* $T(X \times X) \subset g(X)$;
*(ii)* $(T(X \times X), \rho)$ *is complete in* $(Y, \rho)$;
*(iii) there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\rho(T(x, y), T(u, v)) + \rho(T(y, x), T(v, u)) \le \varphi(\rho(g(x), g(u)) + \rho(g(y), g(v))),$$

*for all* $(x, y), (u, v) \in X \times X$ *with* $g(x) \preceq g(u)$ *and* $g(y) \succeq g(v)$ *(or reversely);*
*(iv) there exists* $(x_0, y_0) \in X \times X$ *such that* $g(x_0), g(y_0) \in T(X \times X)$, $g(x_0) \preceq T(x_0, y_0)$ *and* $g(y_0) \succeq T(y_0, x_0)$;
*(v)* $T$ *is increasing with respect to* $g$, *in the sense that, for all* $(x, y), (u, v) \in X \times X$ *the following implication holds*

$$[g(x) \preceq g(u), g(y) \succeq g(v)] \Rightarrow [T(x, y) \preceq T(u, v) \text{ and } T(y, x) \succeq T(v, u)];$$

*(vi)* $g$ *is injective and for every* $(x, y), (u, v) \in Y \times Y$ *there exists* $(z, w) \in Y \times Y$ *such that* $x \preceq z$ *and* $y \succeq w$ *(or reversely) and* $u \preceq z$ *and* $v \succeq w$ *(or reversely).*
*Then, there exists a unique* $z^* = (x^*, y^*) \in X \times X$ *such that*

$$g(x^*) = T(x^*, y^*) \text{ and } g(y^*) = T(y^*, x^*).$$

*Moreover, the sequence* $z_n = (u_n, v_n) \in X \times X$ *defined by*

$$g(u_{n+1}) = T(u_n, v_n) \text{ and } g(v_{n+1}) = T(v_n, u_n), \; n \in \mathbb{N}$$

*starting from any element* $(u_0, v_0) \in X \times X$ *converges to* $z^*$.

*Proof.* Denote $Z := X \times X$ and $W := Y \times Y$. Define on $Z$ the $b$-metric

$$\widetilde{d}((x,y),(u,v)) := d(x,u) + d(y,v)$$

and on $W$ the $b$-metric

$$\widetilde{\rho}((x,y),(u,v)) := \rho(x,u) + \rho(y,v).$$

Obvious $(Z, \widetilde{d})$ and $(W, \widetilde{\rho})$ are a $b$-metric spaces with constant $s_1 \geq 1$ and respectively $s_2 \geq 1$. Consider the following two mappings

$$S, G : Z \to W,\ S(x,y) := (T(x,y), T(y,x)) \text{ and } G(x,y) := (g(x), g(y)).$$

We will prove that $S$ and $G$ satisfy all the assumption of Theorem 2.3. Indeed, we have:

a) by (i) we have that $S(Z) \subset G(Z)$;

b) by (ii) we have that $(S(Z), \widetilde{\rho})$ is complete in $W$;

c) by (iii) we get that

$$\widetilde{\rho}(S(x,y), S(u,v)) \leq \varphi(\widetilde{\rho}(G(x,y), G(u,v))),\ \forall\ (x,y),(u,v) \in Z$$

with $G(x,y) \leq_P G(u,v)$;

d) by (iv) it follows that $(x_0, y_0) \in Z$ such that $G(x_0, y_0) \in S(Z)$ and $G(x_0, y_0) \leq_P S(x_0, y_0)$;

e) by (v) we get that $S$ is increasing with respect to $G$, i.e.,

$$(x,y),(u,v) \in Z \text{ with } G(x,y) \leq_P G(u,v) \Rightarrow S(x,y) \leq_P S(u,v);$$

f) by (vi) it follows that $G$ is injective and for every $z, w \in W$ there exists $t \in W$ which is comparable to $z$ and $w$ with respect to $\leq_P$.

Hence the conclusion follows by Theorem 2.3. □

If the contraction condition (v) is satisfied on $X \times X$, then no monotonicity assumptions are needed. We have then the following result.

**Theorem 3.3.** *Let $(X,d)$ be a b-metric space with constant $s_1 \geq 1$, $(Y,\rho)$ be a b-metric space with constant $s_2 \geq 1$ and $T : X \times X \to Y$, $g : X \to Y$ be two operators with closed graphs. We suppose:*

*(i) $T(X \times X) \subset g(X)$;*

*(ii) $(T(X \times X), \rho)$ is complete in $(Y, \rho)$;*

*(iii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\rho(T(x,y), T(u,v)) + \rho(T(y,x), T(v,u)) \leq \varphi(\rho(g(x), g(u)) + \rho(g(y), g(v))),$$

*for all $(x,y),(u,v) \in X \times X$.*

*Then, there exists a unique $z^* = (x^*, y^*) \in X \times X$ such that*

$$g(x^*) = T(x^*, y^*) \text{ and } g(y^*) = T(y^*, x^*).$$

*Moreover, the sequence $z_n = (u_n, v_n) \in X \times X$ defined by*

$$g(u_{n+1}) = T(u_n, v_n) \text{ and } g(v_{n+1}) = T(v_n, u_n),\ n \in \mathbb{N}$$

*starting from any element $(u_0, v_0) \in X \times X$ converges to $z^*$.*

*Proof.* The proof goes in the lines of the proof of the above theorem and the conclusion follows by Theorem 3 in [12]. □

We will consider now the following system of operator equations:

$$\begin{cases} s(x) = G(x,y) \\ s(y) = G(y,x). \end{cases} \tag{2}$$

Our next result will be a consequence of Theorem 2.2.

**Theorem 3.4.** *Let $(X, d)$ be a b-metric space with constant $s_1 \geq 1$, $(Y, \rho)$ be a complete b-metric space with constant $s_2 \geq 1$ and $G : X \times X \to Y$, $s : X \to Y$ be two operators. Let $\leq$ be a partial order relation on $X$ and on $Y$. We suppose:*

*(i) $G(X \times X) \subset s(X)$;*

*(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\rho(G(x, y), G(u, v)) + \rho(G(y, x), G(v, u)) \leq \varphi(d(x, u) + d(y, v)),$$

*for all $(x, y), (u, v) \in X \times X$ with $x \leq u$ and $y \geq v$;*

*(iii) $s$ is increasing with respect to $\leq$ and expansive, i.e.,*

$$\rho(s(x_1), s(x_2)) \geq d(x_1, x_2), \text{ for all } x_1, x_2 \in X;$$

*(iv) there exists $(x_0, y_0) \in X \times X$ such that $s(x_0) \leq G(x_0, y_0)$ and $s(y_0) \geq G(y_0, x_0)$;*

*(v) $G$ has closed graph and it is mixed monotone, i.e., $G$ is increasing in the first variable and decreasing in the second one;*

*(vi) $s$ is continuous or $s(X)$ is closed or the b-metrics $d$ and $\rho$ are continuous;*

*(vii) for every $(x, y), (u, v) \in Y \times Y$ there exists $(z, w) \in Y \times Y$ such that $x \leq z$ and $y \geq w$ (or reversely) and $u \leq z$ and $v \geq w$ (or reversely).*

*Then, there exists a unique solution $z^* = (x^*, y^*) \in X \times X$ for the coupled coincidence system (2).*

*Proof.* Denote $Z := X \times X$ and $W := Y \times Y$. Define on $Z$ the $b$-metric

$$\widetilde{d}((x, y), (u, v)) := d(x, u) + d(y, v)$$

and on $W$ the $b$-metric

$$\widetilde{\rho}((x, y), (u, v)) := \rho(x, u) + \rho(y, v).$$

Consider on $Z$ and on $W$ the partial order relation $\leq_P$ defined by

$$(x, y) \leq_P (u, v) \Leftrightarrow x \leq u,\ y \geq v.$$

As before we notice that $(W, \widetilde{\rho})$ is a complete $b$-metric space with constant $s_2 \geq 1$. Consider the mappings $T, S : Z \to W$ given by

$$T(x, y) := (G(x, y), G(y, x)) \text{ and } S(x, y) := (s(x), s(y)).$$

By our assumptions we have:

(a) $T(Z) \subset S(Z)$;

(b) $\widetilde{\rho}(T(z), T(w)) \leq \varphi(\widetilde{d}(z, w))$, for all $z = (x, y)$, $w = (u, v) \in Z$ with $z \leq_P w$;

(c) $S$ is increasing and $\widetilde{\rho}(S(z), S(w)) \geq \widetilde{d}(z, w)$, for all $z = (x, y)$, $w = (u, v) \in Z$;

(d) $T$ has closed graph and it is increasing with respect to $\leq_P$;

(e) $S$ is continuous or $S(Z)$ is closed in $(W, \widetilde{\rho})$ or the $b$-metrics $\widetilde{d}$ and $\widetilde{\rho}$ are continuous;

(f) there exists $z_0 := (x_0, y_0) \in Z$ such that $S(z_0) \leq_P T(z_0)$;

(g) for all $z, \widetilde{z} \in Z$ there exists $w \in Z$ which is comparable (with respect to $\leq_P$) to $z$ and $\widetilde{z}$.

Hence our conclusion follows by Theorem 2.2. □

As before, if the contraction condition (ii) is satisfied on $X \times X$, then no monotonicity assumptions are needed and we obtain the following result.

**Theorem 3.5.** *Let $(X, d)$ be a b-metric space with constant $s_1 \geq 1$, $(Y, \rho)$ be a complete b-metric space with constant $s_2 \geq 1$ and $G : X \times X \to Y$, $s : X \to Y$ be two operators. We suppose:*

*(i) $G(X \times X) \subset s(X)$;*

*(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\rho(G(x, y), G(u, v)) + \rho(G(y, x), G(v, u)) \leq \varphi(d(x, u) + d(y, v)),$$

*for all* $(x, y), (u, v) \in X \times X$;

*(iii) s is expansive, i.e.,* $\rho(s(x_1), s(x_2)) \geq d(x_1, x_2)$, *for all* $x_1, x_2 \in X$;

*(iv) s is continuous or* $s(X)$ *is closed or the b-metrics d and* $\rho$ *are continuous.*

*Then, there exists a unique* $z^* = (x^*, y^*) \in X \times X$ *solution for the coupled coincidence system (2).*

*Proof.* The proof runs in a similar way to the proof of the above theorem and the conclusion follows by Theorem 3.12 in [6].

It is easy to see that a similar result takes place under a slight modification of the contraction condition, as follows.

**Theorem 3.6.** *Let* $(X, d)$ *be a b-metric space with constant* $s_1 \geq 1$, $(Y, \rho)$ *be a complete b-metric space with constant* $s_2 \geq 1$ *and* $G : X \times X \to Y$, $s : X \to Y$ *be two operators. We suppose:*

*(i)* $G(X \times X) \subset s(X)$;

*(ii) there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\max\{\rho(G(x, y), G(u, v)), \rho(G(y, x), G(v, u))\} \leq \varphi(\max\{(d(x, u), d(y, v))\}),$$

*for all* $(x, y), (u, v) \in X \times X$;

*(iii) s is expansive, i.e.,* $\rho(s(x_1), s(x_2)) \geq d(x_1, x_2)$, *for all* $x_1, x_2 \in X$;

*(iv) s is continuous or* $s(X)$ *is closed or the b-metrics d and* $\rho$ *are continuous;*

*Then, there exists a unique* $z^* = (x^*, y^*) \in X \times X$ *solution for the coupled coincidence system (2).*

*Proof.* The proof runs in a similar way to the proof of Theorem 5.4 working with the the $b$-metric

$$\widetilde{d}((x, y), (u, v)) := \max\{d(x, u), d(y, v)\}$$

on $Z := X \times X$ and with the $b$-metric

$$\widetilde{\rho}((x, y), (u, v)) := \max\{\rho(x, u), \rho(y, v)\}$$

on $W := Y \times Y$. The conclusion follows by Theorem 3.12 in [6].

**Remark 3.1.** *A study of various qualitative properties (data dependence, well-posedness, Ulam-Hyers stability) of the solution of the coupled coincidence problem can be realized using the corresponding results given in [15] for the case of coincidence point problem. It is an open question to obtain Ostrowski's property for the coincidence point problem via the fixed point approach. See [13] and [14] for results in the coupled fixed point problem case.*

## 4. An Application

Our application concerns the following periodic boundary value problem attached to a system of second order differential equations on $[0, 1]$:

$$\begin{cases} x''(t) = F(t, x(t), y'(t)) \\ y''(t) = F(t, y(t), x'(t)) \\ x(0) = y(0) = x(1) = y(1) = 0. \end{cases} \tag{3}$$

Let $X := \{x \in C^2[0, 1] \mid x(0) = x(1) = 0\}$, $Y := C[0, 1]$ and consider the mappings $g : X \to Y$, $T : X \times X \to Y$ defined by

$$g(x)(t) := x''(t) \text{ and } T(x, y)(t) := F(t, x(t), y'(t)).$$

Then the system (3) can be written as a coupled coincidence problem of the following form:

$$\begin{cases} g(x) = T(x, y) \\ g(y) = T(y, x). \end{cases} \tag{4}$$

We intend to apply Theorem 3.6 from the previous section.
Consider on $Y$ the usual supremum norm

$$\|x\|_C := \max_{t\in[0,1]} |x(t)|$$

and endow $X$ with the norm given by

$$\|x\|_M := \max\{\|x\|_C, \|x'\|_C, \|x''\|_C\}.$$

Then $(X, \|\cdot\|_M)$ and $(Y, \|\cdot\|_C)$ are Banach spaces.

Notice that, by [6] (see Lemma 4.22 and Lemma 4.23), we know that

$$\|x\|_C \le \|x'\|_C \le \|x''\|_C,\ \|x\|_M = \|x''\|_C$$

and $g : (X, \|\cdot\|_M) \to (Y, \|\cdot\|_C)$ is expansive and onto.

We can prove the following result.

**Theorem 4.3.** *Consider the boundary value problem (3). We suppose that* $F : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *is continuous and there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|F(t,u_1,v_1) - F(t,u_2,v_2)| \le \varphi(\max\{|u_1-u_2|, |v_1-v_2|\}),$$

$\forall\ t \in [0,1]$ *and* $\forall\ u_1,u_2,v_1,v_2 \in \mathbb{R}$.
*Then, the system (3) has a unique solution.*

*Proof.* Consider the Banach spaces $X$ and $Y$ defined below and the coupled coincidence problem (4), where the operators $g$ and $T$ are defined below. Then, $g$ is expansive and onto (see Lemma 4.23 in [6]). Additionally, $T$ satisfies a nonlinear contraction condition. Indeed, we have:

$$\begin{aligned}|T(x,y)(t) - T(u,v)(t)| &= |F(t,x(t),y'(t)) - F(t,u(t),v'(t))|\\ &\le \varphi(\max\{|x(t)-u(t)|, |y'(t)-v'(t)|\})\\ &\le \varphi(\max\{\|x-u\|_C, \|y'-v'\|_C\})\\ &\le \varphi(\max\{\|x-u\|_M, \|y-v\|_M\}).\end{aligned}$$

Thus

$$\|T(x,y) - T(u,v)\|_C \le \varphi(\max\{\|x-u\|_M, \|y-v\|_M\}).$$

Similarly, we can write

$$\begin{aligned}|T(y,x)(t) - T(v,u)(t)| &= |F(t,y(t),x'(t)) - F(t,v(t),u'(t))|\\ &\le \varphi(\max\{|y(t)-v(t)|, |x'(t)-u'(t)|\})\\ &\le \varphi(\max\{\|y-v\|_C, \|x'-u'\|_C\})\\ &\le \varphi(\max\{\|y-v\|_M, \|x-u\|_M\}).\end{aligned}$$

Hence

$$\|T(y,x) - T(v,u)\|_C \le \varphi(\max\{\|y-v\|_M, \|x-u\|_M\}).$$

As a consequence

$$\max\{\|T(x,y) - T(u,v)\|_C, \|T(y,x) - T(v,u)\|_C\} \le \varphi(\max\{\|y-v\|_M, \|x-u\|_M\}).$$

Hence, all the conditions of Theorem 3.6 are satisfied and the conclusion follows. □

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