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Common Fixed Points of $(\alpha-\psi)$- Generalized Rational Multivalued Contractions in Dislocated Quasi b-Metric Spaces and Applications

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Abstract. In this paper, the concept of $(\alpha-\psi)$-generalized rational contraction multivalued operator is introduced and then the existence of common fixed points of such mapping in complete dislocated quasi b-metric spaces is obtained. Some examples are presented to show that the results proved herein are potential generalization and extension of comparable existing results in the literature. We also study Ulam-Hyers stability of fixed point problems of $(\alpha-\psi)$-generalized rational contraction multivalued operator. We also obtain some common fixed point results for single and multivalued mappings in a complete dq b-metric space endowed with a partial order. As an application, the existence of a continuous solution of an integral equation under appropriate assumptions is obtained.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

Fixed point theory results are widely used in the economics, computer science, engineering and other related disciplines. The most remarkable result in metric fixed point theory is Banach fixed point theorem [8] . This result has been extended and generalized in different directions (see, [1, 4, 7, 31]). Recently, Klin-eam and Suanoom [18] introduced the concept of dislocated quasi b-metric spaces which generalize abstract spaces such as quasi b-metric spaces [31], b-metric-like spaces [1], b-metric spaces [7] and metric spaces.

In the sequel, the letters, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the set of all nonnegative real numbers, the set of all real numbers, the set of all natural numbers and the set of all nonnegative integer numbers, respectively.

Definition 1.1. *[18] Let X be a nonempty set and $s \geq 1$ a real number. Suppose that for any $x, y, z \in X$, the mapping $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:*

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(b$_1$) $d(x,y) = d(y,x) = 0$ implies $x = y$;

(b$_2$) $d(x,y) \leq s\,[d(x,z) + d(z,y)]$.

The pair (X,d) is called a dislocated quasi b-metric (or simply dq b-metric) space.

Klin-eam and Suanoom [18] proved fixed point theorem for cyclic contractions in dq b-metric spaces. Since then, fixed point results for various classes of single valued and multivalued operators have been proved in the framework of dq b-metric spaces (see [25] and references therein).

Remark 1.2. *If $s = 1$ in the definition 1.1, then dq b-metric space (or quasi b-metric-like space) is a dq metric space (or quasi metric-like space).*

Note that a b-metric is not necessarily continuous in each variable. However, if b-metric is continuous in one variable, then it is continuous in the other variable (see [2]).

It is obvious that b-metric spaces, quasi-b-metric spaces and dislocated b-metric spaces are dq b-metric spaces, but the converse does not hold in general.

Example 1.3. *[25, Example2.1] Let $X = \mathbb{R}^+$ and $p > 1$. Define $d : X \times X \to \mathbb{R}^+$ by*

$$d(x,y) = \left|x - y\right|^p + |x|^p \text{ for all } x, y \in X.$$

Then (X,d) is a dq b-metric space with $s = 2^p > 1$. As, $d(1,1) \neq 0$, (X,d) is not a quasi b-metric space. Also $d(0,1) \neq d(1,0)$ implies that (X,d) is not a dislocated b-metric space. It is obvious that (X,d) is neither b-metric space nor dislocated quasi metric space.

In view of the following proposition, some more examples of dq b-metric spaces can easily be constructed.

Proposition 1.4. *[25] Let X be a nonempty set such that d_q is a dq metric and d_b is a b-metric with $s > 1$. Then the function $d : X \times X \to \mathbb{R}^+$ defined by $d(x,y) = d_q(x,y) + d_b(x,y)$ is dq b-metric on X.*

Definition 1.5. *[18] Let (X,d) be a dq b-metric space. A sequence $\{x_n\}$ in (X,d) is called:*

(a) dq b-convergent if there exists some point $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

In this case x is called a dq b-limit of $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$.

(b) Cauchy sequence if

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0 = \lim_{n,m\to\infty} d(x_n, x_m).$$

The space (X,d) is called complete if every Cauchy sequence in X is dq b-convergent.

Each dq b-metric d generates a topology on X whose base is the family of open balls $\{B(x_0,r) : x_0 \in X, r > 0\}$, where $B(x_0,r) = \{x \in X : \max\{d(x_0,x), d(x,x_0)\} < r\}$.

Unless stated otherwise from now onwards, X denotes dq b-metric space equipped with dq b-metric d with $s \geq 1$ and we assume that a dq b-metric d is continuous in one variable.

We denote by $N(X)$ the space of all nonempty subsets of X, by $CL(X)$ the space of all nonempty closed subsets of X, and by $CB(X)$ the space of all nonempty closed and bounded subsets of X.

Let $S, T : X \to N(X)$. A point $x^* \in X$ is called:

(1) a fixed point of T if $x^* \in Tx^*$.

(2) a common fixed point of T and S if $x^* \in Tx^* \cap Sx^*$.

We denote by $F(T)$ the set of fixed point of T.
For $A, B \in CB(X)$ and $x \in X$, define

$$\begin{aligned} \delta(A,B) &= \sup\{d(x,B) : x \in A\}, \\ \delta(B,A) &= \sup\{d(y,A) : y \in B\}, \text{ and} \\ H(A,B) &= \max\{\delta(A,B), \delta(B,A)\}, \end{aligned}$$

where

$$d(x,B) = \inf\{d(x,y) : y \in B\}.$$

The function H is called the Hausdorff dq b-metric on $CB(X)$ induced by d. Note that, $H(A,B) \leq s(H(A,C) + H(C,B))$. Also, $H(A,B) = 0$ implies that $A = B$. Furthermore, $(CB(X), H)$ is complete if (X,d) is complete.

We need the following analogous lemmas [22] in the framework of dq b-metric spaces. For sake of completeness, we give the proofs.

Lemma 1.6. *Let A, $B \in CB(X)$. If $a \in A$, then $d(a,B) \leq H(A,B)$.*

Proof. $d(a,B) \leq \sup\{d(x,B) : x \in A\} = \delta(A,B) \leq H(A,B)$. □

Lemma 1.7. *Let (X,d) be a dq b-metric space. For $A \in CB(X)$ and $x \in X$, $d(x,A) = 0$ implies that $x \in A$.*

Proof. Let $d(x,A) = 0$. Then $x \in \overline{A} = A$ since $A \in CB(X)$. □

Lemma 1.8. *Let (X,d) be dq b-metric space. Suppose that $\{A_n\}$ is a sequence in $CB(X)$ such that $\lim_{n\to\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.*

Proof. By our assumption that d is continuous in one of the variable, we have

$$d(x,A) = \lim_{n\to\infty} d(x_n, A) \leq \lim_{n\to\infty} H(A_n, A) = 0.$$

Hence $d(x,A) = 0$. Then Lemma 1.7 implies that $x \in A$. □

Lemma 1.9. *Let (X,d) be a dq b-metric space with constant $s > 1$ and $B \in CB(X)$. Assume that there exists $x \in X$ such that $d(x,B) > 0$. Then for each $q > 1$, there exists $y = y(x) \in B$ such that $d(x,y) < qd(x,B)$.*

Proof. Assume on the contrary that there exists $q > 1$, such that for all $y \in B$, there is $d(x,y) \geq qd(x,B)$. Then, $d(x,B) = \inf\{d(x,y) : y \in B\} \geq qd(x,B)$. Hence, $q \leq 1$, which is a contradiction. □

Lemma 1.10. *Let (X,d) be a dq b-metric space, $A, B \in P(X)$. If there exists a $\lambda > 0$ such that (i) for each $a \in A$, there exists a $b \in B$ such that $d(a,b) \leq \lambda$, then $H(A,B) \leq \lambda$. (ii) for each $b \in B$, there exists an $a \in A$ such that $d(a,b) \leq \lambda$, then $H(A,B) \leq \lambda$.*

Recently, Mohammadi et al. [21] introduced the concept of α-admissiblity for a set-valued mapping different from the notion of α_*-admissible mappings in [3].

We now introduce a new concept of α-closed mappings as follows.

Definition 1.11. *Let X be a nonempty set, $\alpha : X \times X \to \mathbb{R}^+$ and $T, S : X \to N(X)$. A pair (T,S) is called α-closed if for any $x, y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies that } \alpha(u,v) \geq 1 \text{ for any } u \in Tx \text{ and } v \in Sy.$$

Following is the dq b-metric space version of the concept of α-continuity of multivalued mappings introduced in [19].

Definition 1.12. *Let (X,d) be a dq b-metric space, $\alpha : X \times X \to \mathbb{R}^+$ and $T,S : X \to CL(X)$. A pair (T,S) is an α-continuous on $(CL(X),H)$ if, for any sequence $\{x_n\}$ in X,*

$$\lim_{n\to\infty} d(x_n,x) = 0 \text{ and } \alpha(x_n,x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}_0 \text{ imply that } \lim_{n\to\infty} H(Tx_n,Sx) = 0.$$

Note that the continuity implies α-continuity for any mapping α but converse does not hold in general.
Recently, Samet [30] obtained fixed point theorems for (α,ψ)-type contraction mappings in metric spaces. For more results in this direction, we refer to [4, 30].

Definition 1.13. *[28] By Ψ, we denote the set of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which have the following properties:*

(Ψ_1) ψ is monotone nondecreasing;

(Ψ_2) $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^n(t)$ is the $n-$th iterate of ψ.

The function $\psi \in \Psi$ is known as Bianchini-Grandolfi gauge functions. For some useful properties of such functions we refer to [28] and the references cited therein.
The following result follows from Definition 1.13.

Lemma 1.14. *If $\psi \in \Psi$, then (i) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \geq 0$; (ii) $\psi(t) < t$ for all $t > 0$; and (iii) $\psi(t) = 0$ if and only if $t = 0$.*

A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if it is increasing and $\varphi^n(t)$ converges to 0 as $n \to \infty$, for all $t \geq 0$. We denote the class of the comparison function φ by Φ.

Lemma 1.15. *If $\varphi \in \Phi$, then (i) each iterate φ^n of φ ,$n \geq 1$, is also a comparison function; (ii) $\varphi(t) < t$ for all $t > 0$; and (iii) φ is continuous at 0.*

Berinde [11] introduced the concept of a (c)-comparison function as follows.

Definition 1.16. *A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a (c)-comparison function if*

(a) it is increasing;

(b) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$ for $k \geq k_0$ and any $t \geq 0$.

In order to extend some fixed point results to the class of b-metric spaces, Berinde [10] extended the concept of a (c)-comparison function to (b)-comparison function as follows.

Definition 1.17. *[10] Let $s \geq 1$ be a real number. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a (b)-comparison function if*

(c) φ is monotone increasing;

(d) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$ for $k \geq k_0$ and any $t \geq 0$.

The next Lemma is very important in the proof of our main result.

Lemma 1.18. *[10, 11] Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a (b)-comparison function. Then*

(e) the series $\sum\limits_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \geq 0$;

(f) the function $S_b = \mathbb{R}^+ \to \mathbb{R}^+$ defined by $S_b(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t)$, $t \geq 0$ is increasing and continuous at 0.

Note that any (b)-comparison function is a comparison function.

In this paper, by Ψ_b we denote by the set of (b)-comparison functions.

The aim of this paper is to introduce the notion of $(\alpha-\psi)$-generalized rational contraction multivalued mappings and then to study the necessary conditions for existence of a common fixed point of two mappings in the framework of a dq b-metric space.

Definition 1.19. *Let (X, d) be a dq b-metric space, $\alpha : X \times X \to \mathbb{R}^+$, $\psi \in \Psi_b$ and $T, S : X \to CL(X)$.*

(a) A pair (T, S) is called an $(\alpha-\psi)$-generalized rational contraction if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, the following condition holds:

$$\alpha(x, y)H(Tx, Sy) \leq \psi\left(M_{T,S}(x, y)\right), \tag{1}$$

where

$$M_{T,S}(x, y) = \max\left\{\begin{array}{c} d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}d(x, Sy), \\ \dfrac{d(y, Sy)[1 + d(x, Tx)]}{1 + d(x, y)} \end{array}\right\}.$$

(b) A pair (S, T) is called an $(\alpha-\psi)$-generalized rational contraction if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, the following condition holds:

$$\alpha(x, y)H(Sx, Ty) \leq \psi\left(M_{S,T}(x, y)\right), \tag{2}$$

where

$$M_{S,T}(x, y) = \max\left\{\begin{array}{c} d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2s}d(x, Ty), \\ \dfrac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)} \end{array}\right\}.$$

(c) A mapping T is called an $(\alpha-\psi)$-generalized rational contraction if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, the following condition holds:

$$\alpha(x, y)H(Tx, Ty) \leq \psi\left(M_{T}(x, y)\right), \tag{3}$$

where

$$M_{T,T}(x, y) = \max\left\{\begin{array}{c} d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2s}d(x, Ty), \\ \dfrac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \end{array}\right\}.$$

Remark 1.20. *(a) if $\alpha : X \times X \to \mathbb{R}^+$ is defined as $\alpha(x, y) = 1$ for all $x, y \in X$ in Definition 1.19, then the pairs (T, S), (S, T) and the mapping T are called ψ-generalized rational contraction. (b) if $\psi \in \Psi_b$ is a strictly increasing function in the Definition 1.19, then the pairs (T, S), (S, T) and the mapping T are said to be strictly $(\alpha-\psi)$-generalized rational contraction (c) if $\alpha : X \times X \to \mathbb{R}^+$ is defined as $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi \in \Psi_b$ is a strictly increasing function in the Definition 1.19, then the pairs (T, S), (S, T) and the mapping T are called a strictly ψ-generalized rational contraction on X.*

2. Common Fixed Point Results

In this section, we obtain some common fixed point results of $(\alpha - \psi)$-generalized rational contraction multivalued mappings in the framework of complete dq b-metric spaces.

We start with the following result.

Theorem 2.1. *Let (X, d) be a complete dq b-metric space and $T, S : X \to CB(X)$. Suppose that the pairs (T, S) and (S, T) are strictly $(\alpha - \psi)$-generalized rational contraction mappings such that*

(C_1) (T, S) and (S, T) are α-closed;

(C_2) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$;

(C_3) (T, S) and (S, T) are α-continuous.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.

Proof. If $M_{T,S}(x, y) = 0$ and $M_{S,T}(x, y) = 0$ for some $x, y \in X$, then the result is obvious. We assume that $M_{T,S}(x, y) > 0$ and $M_{S,T}(x, y) > 0$ for all $x, y \in X$. By hypothesis, there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Clearly, if $x_0 = x_1$ or $x_1 \in Sx_1$, then x_1 is a common fixed point of T and S. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin Sx_1$. So, $d(x_0, x_1) > 0$ and $d(x_1, Sx_1) > 0$. As (T, S) is strictly $(\alpha - \psi)$-generalized rational contraction, we have

$$\begin{aligned}
0 &< d(x_1, Sx_1) \leq \alpha(x_0, x_1)H(Tx_0, Sx_1) \leq \psi(M_{T,S}(x_0, x_1)) \\
&= \psi\left(\max\left\{\begin{array}{l} d(x_0, x_1), d(x_0, Tx_0), d(x_1, Sx_1), \frac{1}{2s}d(x_0, Sx_1), \\ \dfrac{d(x_1, Sx_1)[1 + d(x_0, Tx_0)]}{1 + d(x_0, x_1)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_0, x_1), d(x_0, x_1), d(x_1, Sx_1), \frac{1}{2s}d(x_0, Sx_1), \\ \dfrac{d(x_1, Sx_1)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\{d(x_0, x_1), d(x_1, Sx_1)\}\right). \qquad (4)
\end{aligned}$$

Indeed, $\dfrac{1}{2s}d(x_0, Sx_1) \leq \dfrac{1}{2}[d(x_0, x_1) + d(x_1, Sx_1)] \leq \max\{d(x_0, x_1), d(x_1, Sx_1)\}$. Assume that

$$\max\{d(x_0, x_1), d(x_1, Sx_1)\} = d(x_1, Sx_1).$$

Then from (4) we have

$$0 < d(x_1, Sx_1) \leq \psi(d(x_1, Sx_1)),$$

which is a contradiction to our assumption. Thus, $\max\{d(x_0, x_1), d(x_1, Sx_1)\} = d(x_0, x_1)$. Then from (4) we have

$$0 < d(x_1, Sx_1) \leq \psi(d(x_0, x_1)). \qquad (5)$$

By Lemma 1.9, there exists $x_2 \in Sx_1$ such that

$$d(x_1, x_2) < qd(x_1, Sx_1) \leq q\psi(d(x_0, x_1)) \qquad (6)$$

where $q > 1$. As ψ is increasing, from (6) we obtain that

$$0 < \psi(d(x_1, x_2)) \leq \psi(q\psi(d(x_0, x_1))). \qquad (7)$$

Put $q_1 = \dfrac{\psi(q\psi(d(x_0, x_1)))}{\psi(d(x_1, x_2))}$. Then $q_1 > 1$. Since $x_1 \in Tx_0$, $x_2 \in Sx_1$, $\alpha(x_0, x_1) \geq 1$, and (T, S) is α-closed, we have $\alpha(x_1, x_2) \geq 1$. Clearly, if $x_1 = x_2$ or $x_2 \in Tx_2$, then x_2 is a common fixed point of T and S. Now, we assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. So then, $d(x_1, x_2) > 0$ and $d(x_2, Tx_2) > 0$. As (S, T) is strictly $(\alpha - \psi)$-generalized rational contraction, we have

$$\begin{aligned}
0 &< d(x_2, Tx_2) \leq \alpha(x_1, x_2) H(Sx_1, Tx_2) \leq \psi(M_{S,T}(x_1, x_2)) \\
&= \psi\left(\max\left\{\begin{array}{l} d(x_1, x_2), d(x_1, Sx_1), d(x_2, Tx_2), \frac{1}{2s} d(x_1, Tx_2), \\ \dfrac{d(x_2, Tx_2)[1 + d(x_1, Sx_1)]}{1 + d(x_1, x_2)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), \frac{1}{2s} d(x_1, Tx_2), \\ \dfrac{d(x_2, Tx_2)[1 + d(x_1, x_2)]}{1 + d(x_1, x_2)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{d(x_1, x_2), d(x_2, Tx_2)\right\}\right) \qquad (8)
\end{aligned}$$

Indeed, $\frac{1}{2s} d(x_1, Tx_2) \leq \frac{1}{2}\left[d(x_1, x_2) + d(x_2, Tx_2)\right] \leq \max\left\{d(x_1, x_2), d(x_2, Tx_2)\right\}$. Assume that

$$\max\left\{d(x_1, x_2), d(x_2, Tx_2)\right\} = d(x_2, Tx_2).$$

Then from (8) we have

$$0 < d(x_2, Tx_2) \leq \psi\left(d(x_2, Tx_2)\right),$$

a contradiction to our assumption. Thus, $\max\left\{d(x_1, x_2), d(x_2, Tx_2)\right\} = d(x_1, x_2)$. Now from (8) we have

$$0 < d(x_2, Tx_2) \leq \psi\left(d(x_1, x_2)\right). \qquad (9)$$

For $q_1 > 1$, Lemma 1.9 gives that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < q_1 d(x_2, Tx_2) \leq q_1 \psi\left(d(x_1, x_2)\right) \leq \psi\left(q\psi\left(d(x_0, x_1)\right)\right). \qquad (10)$$

As ψ is increasing, from (10) we obtain that

$$0 < \psi\left(d(x_2, x_3)\right) < \psi^2\left(q\psi\left(d(x_0, x_1)\right)\right). \qquad (11)$$

Put $q_2 = \dfrac{\psi^2\left(q\psi\left(d(x_0, x_1)\right)\right)}{\psi\left(d(x_2, x_3)\right)}$. Then $q_2 > 1$. As $x_2 \in Sx_1$, $x_3 \in Tx_2$, $\alpha(x_1, x_2) \geq 1$, and (S, T) is α-closed, we have $\alpha(x_2, x_3) \geq 1$. Clearly, if $x_2 = x_3$ or $x_3 \in Sx_3$, then x_3 is a common fixed point of T and S. Now, we assume that $x_2 \neq x_3$ and $x_3 \notin Sx_3$. So then, $d(x_2, x_3) > 0$ and $d(x_3, Sx_3) > 0$. As (T, S) is strictly $(\alpha - \psi)$-generalized rational contraction, we have

$$\begin{aligned}
0 &< d(x_3, Sx_3) \leq \alpha(x_2, x_3) H(Tx_2, Sx_3) \leq \psi(M_{T,S}(x_2, x_3)) \\
&= \psi\left(\max\left\{\begin{array}{l} d(x_2, x_3), d(x_2, Tx_2), d(x_3, Sx_3), \frac{1}{2s} d(x_2, Sx_3), \\ \dfrac{d(x_3, Sx_3)[1 + d(x_2, Tx_2)]}{1 + d(x_2, x_3)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_2, x_3), d(x_2, x_3), d(x_3, Sx_3), \frac{1}{2s} d(x_2, Sx_3), \\ \dfrac{d(x_3, Sx_3)[1 + d(x_2, x_3)]}{1 + d(x_2, x_3)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{d(x_2, x_3), d(x_3, Sx_3)\right\}\right). \qquad (12)
\end{aligned}$$

since $\frac{1}{2s} d(x_2, Sx_3) \leq \frac{1}{2}\left[d(x_2, x_3) + d(x_3, Sx_3)\right] \leq \max\left\{d(x_2, x_3), d(x_3, Sx_3)\right\}$. Assume that

$$\max\left\{d(x_2, x_3), d(x_3, Sx_3)\right\} = d(x_3, Sx_3).$$

Then from (12) we have

$$0 < d(x_3, Sx_3) \leq \psi\left(d(x_3, Sx_3)\right),$$

a contradiction to our assumption. Thus, $\max\{d(x_2,x_3), d(x_3,Sx_3)\} = d(x_2,x_3)$. Then from (12) we have

$$0 < d(x_3, Sx_3) \leq \psi\left(d(x_2,x_3)\right). \tag{13}$$

For $q_2 > 1$; by Lemma 1.9, there exists $x_4 \in Sx_3$ such that

$$d(x_3,x_4) < q_2 d(x_3,Sx_3) \leq q_2\psi\left(d(x_2,x_3)\right) \leq \psi^2\left(q\psi\left(d(x_0,x_1)\right)\right). \tag{14}$$

As ψ is increasing, from (14) we obtain that

$$0 < \psi\left(d(x_3,x_4)\right) < \psi^3\left(q\psi\left(d(x_0,x_1)\right)\right). \tag{15}$$

Following the arguments similar to those given above we construct a sequence $\{x_k\}$ such that $x_{2k} \neq x_{2k+1} \in Tx_{2k}$, and $x_{2k+2} \in Sx_{2k+1}$ with $\alpha(x_{2k}, x_{2k+1}) \geq 1$ and

$$d(x_{2k+1}, x_{2k+2}) < \psi^{2k}(q\psi\left(d(x_0,x_1)\right)) \tag{16}$$

for each $k \in \mathbb{N}_0$. As ψ is increasing, from (16) we obtain that

$$0 < \psi\left(d(x_{2k+1},x_{2k+2})\right) < \psi^{2k+1}(q\psi\left(d(x_0,x_1)\right)). \tag{17}$$

Put $q_{2k+1} = \dfrac{\psi^{2k+1}\left(q\psi\left(d(x_0,x_1)\right)\right)}{\psi\left(d(x_{2k+1},x_{2k+2})\right)}$. Then $q_{2k+1} > 1$. As $x_{2k+1} \in Tx_{2k}$, $x_{2k+2} \in Sx_{2k+1}$, $\alpha(x_{2k}, x_{2k+1}) \geq 1$, and (T,S) is α-closed, we have $\alpha(x_{2k+1}, x_{2k+2}) \geq 1$. Assume that $d(x_{2k+1}, x_{2k+2}) > 0$. As (S,T) is strictly $(\alpha - \psi)$-generalized rational contraction, we have

$$\begin{aligned}
0 &< d(x_{2k+2}, Tx_{2k+2}) \leq \alpha(x_{2k+1},x_{2k+2}) H(Sx_{2k+1}, Tx_{2k+2}) \leq \psi(M_{S,T}(x_{2k+1},x_{2k+2})) \\
&= \psi\left(\max\left\{\begin{array}{l} d(x_{2k+1},x_{2k+2}), d(x_{2k+1},Sx_{2k+1}), d(x_{2k+2},Tx_{2k+2}), \\ \frac{1}{2s}d(x_{2k+1},Tx_{2k+2}), \\ \dfrac{d(x_{2k+2},Tx_{2k+2})[1+d(x_{2k+1},Sx_{2k+1})]}{1+d(x_{2k+1},x_{2k+2})} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_{2k+1},x_{2k+2}), d(x_{2k+1},x_{2k+2}), d(x_{2k+2},Tx_{2k+2}), \\ \frac{1}{2s}d(x_{2k+1},Tx_{2k+2}), \\ \dfrac{d(x_{2k+2},Tx_{2k+2})[1+d(x_{2k+1},x_{2k+2})]}{1+d(x_{2k+1},x_{2k+2})} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{d(x_{2k+1},x_{2k+2}), d(x_{2k+2},Tx_{2k+2})\right\}\right).
\end{aligned} \tag{18}$$

Note that $\dfrac{1}{2s}d(x_{2k+1},Tx_{2k+2}) \leq \dfrac{1}{2}\left[d(x_{2k+1},x_{2k+2}) + d(x_{2k+2},Tx_{2k+2})\right] \leq \max\left\{d(x_{2k+1},x_{2k+2}), d(x_{2k+2},Tx_{2k+2})\right\}$. Assume that

$$\max\left\{d(x_{2k+1},x_{2k+2}), d(x_{2k+2},Tx_{2k+2})\right\} = d(x_{2k+2},Tx_{2k+2}).$$

Then from (18) we have

$$0 < d(x_{2k+2},Tx_{2k+2}) \leq \psi\left(d(x_{2k+2},Tx_{2k+2})\right),$$

which is a contradiction to our assumption. Thus, $\max\left\{d(x_{2k+1},x_{2k+2}), d(x_{2k+2},Tx_{2k+2})\right\} = d(x_{2k+1},x_{2k+2})$. Then from (18) we have

$$0 < d(x_{2k+2},Tx_{2k+2}) \leq \psi\left(d(x_{2k+1},x_{2k+2})\right). \tag{19}$$

For $q_{2k+1} > 1$ by Lemma 1.9, there exists $x_{2k+3} \in Tx_{2k+2}$ such that

$$d(x_{2k+2},x_{2k+3}) < q_{2k+1}d(x_{2k+2},Tx_{2k+2}) \leq q_{2k+1}\psi\left(d(x_{2k+1},x_{2k+2})\right) \leq \psi^{2k+1}\left(q\psi\left(d(x_0,x_1)\right)\right). \tag{20}$$

As ψ is increasing, from (20) we obtain that

$$0 < \psi\left(d(x_{2k+2}, x_{2k+3})\right) < \psi^{2k+2}\left(q\psi\left(d(x_0, x_1)\right)\right).$$

Hence by an induction, we have a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ such that

$$d(x_{n+1}, x_{n+2}) < \psi^n\left(q\psi\left(d(x_0, x_1)\right)\right) \tag{21}$$

for each $n \in \mathbb{N}_0$. Using the property ψ, it is clear that

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0. \tag{22}$$

Now using triangular inequality and (21) for $m \geq 1$, we obtain that

$$\begin{aligned} d(x_n, x_{n+m}) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^{m-1} d(x_{n+m-1}, x_{n+m}) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^{m} d(x_{n+m-1}, x_{n+m}) \\ &= \sum_{i=n}^{n+m-1} s^{i-n+1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{n+m-1} s^{i-n+1} \psi^{i-1}\left(q\psi\left(d(x_0, x_1)\right)\right) \\ &= \frac{1}{s^{n-2}} \sum_{i=n}^{n+m-1} s^{i-1} \psi^{i-1}\left(q\psi\left(d(x_0, x_1)\right)\right) = \frac{1}{s^{n-2}} \sum_{i=n-1}^{n+m-2} s^{i} \psi^{i}\left(q\psi\left(d(x_0, x_1)\right)\right). \end{aligned}$$

Setting $S_n = \sum_{i=0}^{n} s^i \psi^i\left(q\psi\left(d(x_0, x_1)\right)\right)$, $n \geq 1$ we obtain $d(x_n, x_{n+m}) \leq \frac{1}{s^{n-2}}\left[S_{n+m-2} - S_{n-2}\right]$, $n \geq 3$, $m \geq 1$. By the fact $s \geq 1$ and Lemma 1.18 (iii), we conclude that $\sum_{i=0}^{n} s^i \psi^i\left(q\psi\left(d(x_0, x_1)\right)\right)$ is convergent. Thus there exists $S = \lim_{n\to\infty} S_n$ which implies that

$$d(x_n, x_{n+m}) \to 0 \text{ as } n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists a point $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_n) = 0. \tag{23}$$

Since the pairs (T, S) and (S, T) are α-continuous, we have, $\lim_{n\to\infty} H(Tx_{2n}, Sx^*) = 0$ and $\lim_{n\to\infty} H(Sx_{2n+1}, Tx^*) = 0$. By Lemma 1.8, we obtain that $x^* \in Tx^* \cap Sx^*$. $\square$

Example 2.2. *Let $X = \mathbb{R}^+$ and $d(x, y) = \left|x - y\right|^2$ for all $x, y \in X$. Define the mappings $T, S : X \to CB(X)$ by,*

$$Tx = \begin{cases} [0, \frac{x}{2}] \text{ if } x \in [0, 1], \\ [2x - \frac{3}{2}, \infty) \text{ if } x \in (1, \infty) \end{cases} \quad \text{and} \quad Sx = \begin{cases} [0, \frac{x}{3}] \text{ if } x \in [0, 1], \\ [x, 2x] \text{ if } x \in (1, \infty). \end{cases}$$

Note that (X, d) is a complete dq b-metric space with $s = 2$. Define $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in [0, 1], \\ 0 \text{ if } x \notin [0, 1] \text{ or } y \notin [0, 1]. \end{cases}$$

If $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{4} \in Tx_0$, then $\alpha(x_0, x_1) \geq 1$. Also, the pairs (T, S) and (S, T) are α-closed and strictly $(\alpha - \psi)$-generalized rational contraction, where $\psi(t) = \dfrac{t}{4}$ for all $t \geq 0$. For any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, then $x \in [0, 1]$. Indeed $\{x_n\}$ is a sequence in $[0, 1]$. Note that, (T, S) and (S, T) are α-continuous . Thus all the conditions of Theorem 2.1 are satisfied. Moreover, $x^* \in \{0\} \cup (1, \frac{3}{2}]$ is the common fixed point of T and S in X.

Now, in the next Theorem, we omit the α-continuity condition on the mappings T, S.

Theorem 2.3. *Let (X, d) be a complete dq b-metric space and $T, S : X \to CB(X)$. Suppose the pairs (T, S) and (S, T) are strictly $(\alpha - \psi)$-generalized rational contraction mappings such that*

(C$_1$) (T, S) and (S, T) are α-closed;

(C$_2$) the maps $p, h : X \to \mathbb{R}$ defined by $p(x) = d(x, Tx)$ and $h(x) = d(x, Sx)$ are lower semi-continuous;

(C$_3$) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$;

(C$_4$) if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.

Proof. Following similar arguments as given in proof of Theorem 2.1, we obtain that $\{x_n\}$ is a Cauchy sequence in the complete dq b-metric space X with $\lim_{n\to\infty} d(x_n, x^*) = 0$ for some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}_0$. By hypothesis (C$_4$), we get $\alpha(x_n, x^*) \geq 1$ for each $n \in \mathbb{N}_0$. Since the pair (T, S) is strictly $(\alpha - \psi)$-generalized rational contraction, we have

$$\begin{aligned}
H(Tx_{2n}, Sx^*) &\leq \alpha(x_{2n}, x^*)H(Tx_{2n}, Sx^*)\\
&\leq \psi(M_{T,S}(x_{2n}, x^*))\\
&= \psi\left(\max\left\{\begin{array}{l} d(x_{2n}, x^*), d(x_{2n}, Tx_{2n}), d(x^*, Sx^*), \frac{1}{2s}d(x_{2n}, Sx^*),\\ \dfrac{d(x^*, Sx^*)[1 + d(x_{2n}, Tx_{2n})]}{1 + d(x_{2n}, x^*)}\end{array}\right\}\right)\\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Sx^*), \frac{1}{2s}d(x_{2n}, Sx^*),\\ \dfrac{d(x^*, Sx^*)[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x^*)}\end{array}\right\}\right).
\end{aligned}$$

On taking limit as $n \to \infty$ on both sides of above inequality, we have

$$\lim_{n\to\infty} H(Tx_{2n}, Sx^*) \leq \psi(d(x^*, Sx^*)).$$

If $d(x^*, Sx^*) > 0$. Then by definition of ψ and the condition (C$_2$), we obtain

$$\begin{aligned}
\lim_{n\to\infty} H(Tx_{2n}, Sx^*) &\leq \psi(d(x^*, Sx^*))\\
&< d(x^*, Sx^*)\\
&\leq \liminf_{n\to\infty} d(x_{2n+1}, Sx_{2n+1})\\
&\leq \lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0
\end{aligned}$$

a contradiction. Thus $d(x^*, Sx^*) = 0$ and implies $x^* \in Sx^*$. Also, since (S, T) is strictly $(\alpha - \psi)$-generalized rational contraction, we have

$$\begin{aligned}
H(Sx_{2n+1}, Tx^*) &\leq \alpha(x_{2n+1}, x^*)H(Sx_{2n+1}, Tx^*)\\
&\leq \psi(M_{S,T}(x_{2n+1}, x^*))\\
&= \psi\left(\max\left\{\begin{array}{l} d(x_{2n+1}, x^*), d(x_{2n+1}, Sx_{2n+1}), d(x^*, Tx^*),\\ \frac{1}{2s}d(x_{2n+1}, Tx^*), \dfrac{d(x^*, Tx^*)[1 + d(x_{2n+1}, Sx_{2n+1})]}{1 + d(x_{2n+1}, x^*)}\end{array}\right\}\right)\\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_{2n+1}, x^*), d(x_{2n+1}, x_{2n+2}), d(x^*, Tx^*),\\ \frac{1}{2s}d(x_{2n+1}, Tx^*), \dfrac{d(x^*, Tx^*)[1 + d(x_{2n+1}, x_{2n+2})]}{1 + d(x_{2n+1}, x^*)}\end{array}\right\}\right).
\end{aligned}$$

On taking limit as $n \to \infty$ on both sides of above inequality, we have

$$\lim_{n\to\infty} H(Sx_{2n+1}, Tx^*) \leq \psi(d(x^*, Tx^*)).$$

If $d(x^*, Tx^*) > 0$. Then by definition of ψ and the condition (C_2), we obtain

$$\begin{aligned}
\lim_{n\to\infty} H(Sx_{2n+1}, Tx^*) &\leq \psi(d(x^*, Tx^*)) \\
&< d(x^*, Tx^*) \\
&\leq \liminf_{n\to\infty} d(x_{2n+2}, Tx_{2n+2}) \\
&\leq \lim_{n\to\infty} d(x_{2n+2}, x_{2n+3}) = 0
\end{aligned}$$

a contradiction. Thus $d(x^*, Tx^*) = 0$ and implies $x^* \in Tx^*$. Hence $x^* \in Tx^* \cap Sx^*$. □

Corollary 2.4. *Let (X, d) be a complete dq b-metric space. If $T, S : X \to CB(X)$ are continuous and the pairs (T, S) and (S, T) are strictly ψ-generalized rational contraction mappings, then there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.*

Proof. Define $\alpha : X \times X \to \mathbb{R}^+$ as $\alpha(x, y) = 1$ for all $x, y \in X$. Then the result follows from Theorem 2.1 and Theorem 2.3. □

The following two Theorems generalize the main results of Samet et. al. [30] and Karapinar et. al. [17, Theorem 2.3 and 2.4].

Theorem 2.5. *Let (X, d) be a complete dq b-metric space. Suppose $T : X \to CB(X)$ is strictly $(\alpha - \psi)$-generalized rational contraction mapping such that*

(C_1) T is α-closed;

(C_2) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$;

(C_3) T is α-continuous.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. The result follows from Theorem 2.1 by choosing $T = S$. □

Example 2.6. *Let $X = \mathbb{R}^+$ and $d(x, y) = \left|x - y\right|^3$. Define $T : X \to N(X)$ by*

$$Tx = \begin{cases} [0, \frac{x}{2}] \text{ if } x \in [0, 1], \\ \{2x\} \text{ otherwise} \end{cases}$$

and $\alpha : X \times X \to \mathbb{R}^+$ as:

$$\alpha(x, y) = \begin{cases} \dfrac{1}{\left|x^2 - y^2\right|} \text{ if } x, y \in [0, 1] \text{ and } x \neq y, \\ 1 \text{ if } x, y \in [0, 1] \text{ and } x = y, \\ \tanh(x + y) \text{ otherwise.} \end{cases}$$

Note that (X, d) is a complete dq b-metric space with $s = 4$ and for any $x, y \in [0, 1]$, we obtain that $\alpha(x, y) \geq 1$ and $Tx, Ty \subseteq [0, 1]$. As $\alpha(u, v) \geq 1$ for each $u \in Tx$ and $v \in Ty$, T is α-closed. If $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{4} \in Tx_0$, then $\alpha(x_0, x_1) \geq 1$. Also, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, we obtain that $x \in [0, 1]$. Note that T is α-continuous since $Tx_n, Tx \in [0, 1]$ and $Tx_n \xrightarrow{H} Tx$ as $n \to \infty$. It is straight forward to check that T strictly $(\alpha - \psi)$-generalized rational contraction with $\psi(t) = \frac{t}{4}$ for all $t \geq 0$. Thus all the conditions of Theorem 2.5, and T has a fixed point in X. Put $x = 2$ and $y = 1$. Then, $H(Tx, Ty) = H(\{4\}, [0, \frac{1}{2}]) > \lambda d(x, y)$ for any $\lambda \in [0, 1)$. So the Nadler fixed Theorem [22, Theorem 5] is not applicable in this case.

Now, in the next Theorem, we omit the continuity condition on the mapping T.

Theorem 2.7. *Let (X, d) be a complete dq b-metric space. Suppose $T : X \to CB(X)$ is strictly $(\alpha - \psi)$-generalized rational contraction mapping such that*

(C$_1$) T is α-closed;

(C$_2$) the map $p : X \to \mathbb{R}$ defined by $p(x) = d(x, Tx)$ is lower semi-continuous;

(C$_3$) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(C$_4$) if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. Follows from Theorem 2.3 with $T = S$. □

Remark 2.8. *As every α_*-admissible mapping is α-closed mapping. Thus Theorem 2.7 generalizes the main result of Asl et. al. [3, Theorem 2.1].*

Theorem 2.9. *Let (X, d) be a complete dq b-metric space and $T : X \to CB(X)$ strictly $(\alpha - \psi)$-generalized rational contraction mapping. Suppose the following conditions hold:*

(C$_1$) T is α-closed;

(C$_2$) the map $p : X \to \mathbb{R}$ defined by $p(x) = d(x, Tx)$ is lower semi-continuous;

(C$_3$) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(C$'_4$) if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $\lim\limits_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}_0$.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. Following arguments similar to those in the proof of Theorem 2.7, we obtain that $\{x_n\}$ is Cauchy sequence in the complete dq b-metric space X with $\lim\limits_{n\to\infty} d(x_n, x^*) = 0$ for some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}_0$. By assumption (C$'_4$) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}_0$. Since $x_{n_k+1} \in Tx_{n_k}$ for all $k \in \mathbb{N}_0$ and T is a strictly $(\alpha - \psi)$-generalized rational contraction mapping, we have

$$\begin{aligned}
H(Tx_{n_k}, Tx^*) &\leq \alpha(x_{n_k}, x^*)H(Tx_{n_k}, Tx^*) \\
&\leq \psi(M_T(x_{n_k}, x^*)) \\
&= \psi\left(\max\left\{\begin{array}{l} d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{1}{2s}d(x_{n_k}, Tx^*), \\ \dfrac{d(x^*, Tx^*)[1 + d(x_{n_k}, Tx_{n_k})]}{1 + d(x_{n_k}, x^*)} \end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{\begin{array}{l} d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \frac{1}{2s}d(x_{n_k}, Tx^*), \\ \dfrac{d(x^*, Tx^*)[1 + d(x_{n_k}, x_{n_k+1})]}{1 + d(x_{n_k}, x^*)} \end{array}\right\}\right).
\end{aligned}$$

On taking limit as $k \to \infty$ on both sides of the above inequality, we have

$$\lim_{k\to\infty} H(Tx_{n_k}, Tx^*) \leq \psi(d(x^*, Tx^*)).$$

If $d(x^*, Tx^*) > 0$. Then by definition of ψ and the hypothesis (C$_2$), we obtain that

$$\lim_{k\to\infty} H(Tx_{n_k}, Tx^*) \leq \psi(d(x^*, Tx^*)) < d(x^*, Tx^*) \leq \liminf_{k\to\infty} d(x_{n_k}, Tx_{n_k}) \leq \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0$$

a contradiction. Thus $d(x^*, Tx^*) = 0$ and hence the result follows. □

Corollary 2.10. *Let (X, d) be a complete dq b-metric space. Suppose $T, S : X \to CB(X)$ such that $\alpha(x, y)H(Tx, Sy) \leq \psi(d(x, y))$ for any $\psi \in \Psi$, $x, y \in X$ with $\alpha(x, y) \geq 1$ and the following conditions hold:*

(C$_1$) (T, S) and (S, T) are α-closed;

(C$_2$) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$;

(C$_3$) (T, S) and (S, T) are α-continuous.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.

Corollary 2.11. *Let (X, d) be a complete dq b-metric space. Suppose $T : X \to CB(X)$ such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for any $\psi \in \Psi$, $x, y \in X$ with $\alpha(x, y) \geq 1$ and the following conditions hold:*

(C$_1$) T is α-closed;

(C$_2$) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$;

(C$_3$) T is α-continuous.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Corollary 2.12. *Let (X, d) be a complete dq b-metric space. Suppose $T : X \to CB(X)$ such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for any $\psi \in \Psi$, $x, y \in X$ with $\alpha(x, y) \geq 1$ and the following conditions hold:*

(C$_1$) T is α-closed;

(C$_2$) the map $p : X \to \mathbb{R}$ defined by $p(x) = d(x, Tx)$ is lower semi-continuous;

(C$_3$) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(C$_4$) if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Corollary 2.13. *Let (X, d) be a complete dq b-metric space. If $T : X \to CB(X)$ is continuous and strictly ψ-generalized rational contraction mapping, then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. Define $\alpha : X \times X \to \mathbb{R}^+$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Since $\alpha(x, y) = 1$ implies $\alpha(u, v) = 1$ for each $u \in Tx$, $v \in Ty$. Now, according to Theorem 2.5 and Theorem 2.7, T has a fixed point. □

The above corollary generalizes and extends Karapinar [17, corollaries 3.1 and 3.2] and Aydi [5, Theorem 3.2, corollaries 3.5 , 3.6, 3.7, 3.8].

We now give some important consequences of the main results presented above. The following corollary generalizes the main result of Rahman et. al. [25].

Corollary 2.14. *Let (X, d) be a complete dq b-metric space and $T : X \to CB(X)$. If there exists $\lambda \in [0, 1)$ and $0 \leq s\lambda < 1$ such that*

$$H(Tx, Ty) \leq \lambda d(x, y) \tag{24}$$

for all $x, y \in X$. Then T has a fixed point.

Proof. Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = \lambda t$ for all $t \in \mathbb{R}^+$. Clearly,

$$H(Tx, Ty) \leq \lambda d(x, y) = \psi(d(x, y)) \leq \psi(M_T(x, y)).$$

Also, T is continuous. Thus, all the condition of Corollary 2.13 are satisfied and hence the result follows. □

Corollary 2.15. *Let (X, d) be a complete dq b-metric space and $T : X \to CB(X)$ a continuous mapping. If there exists $\beta \in [0, \frac{1}{2})$ and $0 \leq s\beta < \frac{1}{2}$ such that*

$$H(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty) \right] \tag{25}$$

for all $x, y \in X$. Then T has a fixed point.

Proof. Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = 2\beta t$ for all $t \in \mathbb{R}^+$ such that

$$\begin{aligned} H(Tx, Ty) &\leq \beta \left[d(x, Tx) + d(y, Ty) \right] = 2\beta \left(\frac{d(x, Tx) + d(y, Ty)}{2} \right) \\ &= \psi \left(\frac{d(x, Tx) + d(y, Ty)}{2} \right) \leq \psi \left(\max\{d(x, Tx), d(y, Ty)\} \right) \leq \psi \left(M_T(x, y) \right). \end{aligned}$$

Thus, all the conditions of Corollary 2.13 are satisfied and hence the mapping T has a fixed point in X. □

Corollary 2.16. *Let (X, d) be a complete dq b-metric space and $T : X \to CB(X)$ a continuous mapping. If for any $x, y \in X$, the following condition holds:*

$$H(Tx, Ty) \leq \lambda d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \tag{26}$$

where $\lambda + s(\beta + \gamma) \in [0, 1)$. Then T has a fixed point.

Proof. Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = (\lambda + \beta + \gamma)t$ for all $t \in \mathbb{R}^+$ such that

$$\begin{aligned} H(Tx, Ty) &\leq \lambda d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ &\leq \lambda M_T(x, y) + \beta M_T(x, y) + \gamma M_T(x, y) \\ &\leq (\lambda + \beta + \gamma) M_T(x, y) \leq \psi \left(M_T(x, y) \right). \end{aligned}$$

Thus, all the conditions of Corollary 2.13 are satisfied and hence the mapping T has a fixed point in X. □

Remark 2.17. *If $\beta = \gamma$ in Corollary 2.16, we have*

$$H(Tx, Ty) \leq \lambda d(x, y) + \beta \left[d(x, Tx) + d(y, Ty) \right] \tag{27}$$

where $\lambda + 2\beta \in [0, 1)$.

Corollary 2.18. *Let (X, d) be a complete dq b-metric space and $T : X \to CB(X)$ a continuous mapping. If there exists $\lambda \in [0, 1)$ such that for any $x, y \in X$, we have*

$$H(Tx, Ty) \leq \lambda \max\{d(x, Tx), d(y, Ty)\}. \tag{28}$$

Then T has a fixed point.

Proof. Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = \lambda t$ for all $t \in \mathbb{R}^+$ such that

$$H(Tx, Ty) \leq \lambda \max\{d(x, Tx), d(y, Ty)\} \leq \lambda M_T(x, y) \leq \psi(M_T(x, y)).$$

Thus, all the conditions of Corollary 2.13 are satisfied and hence the mapping T has a fixed point in X. □

Corollary 2.19. *Let (X, d) be a complete dq b-metric space and $T : X \to CB(X)$ be a given continuous mapping. Suppose there exists a function $\psi \in \Psi$, for all $x, y \in X$ such that*

$$H(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\} \right).$$

Then T has a fixed point.

Corollary 2.20. *Let (X,d) be a complete dq b-metric space and $T : X \to CB(X)$ be a given continuous mapping. Suppose there exists a function $\psi \in \Psi$, for all $x, y \in X$ such that*

$$H(Tx,Ty) \leq \psi\left(d(x,y)\right).$$

Then T has a fixed point.

Corollary 2.21. *Let (X,d) be a complete dq b-metric space and $T : X \to CB(X)$ be a given continuous mapping. Suppose*

$$H(Tx,Ty) \leq \lambda \max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\}$$

holds for all $x, y \in X$ and $\lambda \in [0,1)$. Then T has a fixed point.

Remark 2.22. *Corollaries 2.14, 2.15, 2.16, and 2.18 generalize and extend Banach contraction principle [8], Kannan fixed point theorem [16], Reich fixed point theorem [27] and fixed point theorem due to Bianchini [12], respectively in the setting of dq b-metric space.*

Remark 2.23. *Note that, dislocated b-metric, quasi b-metric, b-metric, dislocated quasi metric, dislocated metric, quasi metric, and ordinary metric versions of our main results are also new in the literature.*

3. Ulam-Hyers Stability Results in dq b-Metric Spaces

In this section we prove the generalized Ulam-Hyers stability in dq b-metric spaces.
Consider the following class of functions

$$\Omega = \left\{\sigma : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that } \sigma \text{ is increasing, continuous at 0 and } \sigma(0) = 0\right\}.$$

Definition 3.1. *Let (X,d) be a dq b-metric space with $s \geq 1$ and $T : X \to CB(X)$ an operator. The fixed point inclusion is to find an $x \in X$ such that*

$$x \in Tx. \tag{29}$$

The fixed point problem (29) is said to be generalized Ulam-Hyers stable if there exists a function $\sigma \in \Omega$, such that for each $\epsilon > 0$ and for each solution v_* of the inequality

$$d(Tv,v) \leq \epsilon \tag{30}$$

there exists a solution u_* of fixed point problem (29) such that

$$d(u^*,v^*) \leq \sigma(\epsilon). \tag{31}$$

Further if there exists $c > 0$ such that $\sigma(t) := ct$, for each $t \in \mathbb{R}^+$, then the fixed point inclusion (29) is said to be Ulam-Hyers stable.

Let $F(T)$ and U be the sets of solutions of (29) and (30) respectively. For more details on Ulam-Hyers stability of fixed point problems, we refer to [13–15, 20, 24, 29, 32] and references therein.

Let (X,d) be a dq b-metric space and $T : X \to CB(X)$ be a multivalued mapping define

$$E(T) = \{x \in X : \{x\} = Tx\}.$$

Theorem 3.2. *Let (X,d) be a complete dq b-metric space and $T : X \to CB(X)$ a multivalued mapping. Assume that all the hypotheses of Corollary 2.11 hold.*

(u_1) *The fixed point inclusion (29) is σ_1^{-1}–generalized Ulam-Hyers stable provided that for $x \in F(T)$, there exists $z \in U$ such that $\alpha(x,z) \geq 1$, where $\sigma_1 : \mathbb{R}^+ \to \mathbb{R}^+$, defined as $\sigma_1(t) = t - s\psi(t)$ is strictly increasing and onto.*

(u$_2$) $F(T) = E(T) = \{x^*\}$.

(u$_3$) *If $E(T) \neq \emptyset$, then the fixed point inclusion (29) is $\sigma_2^{-1}-$generalized Ulam-Hyers stable provided that for $x \in F(T)$ there exists $z \in U$ such that $\alpha(x,z) \geq 1$, where $\sigma_2 : \mathbb{R}^+ \to \mathbb{R}^+$, defined as $\sigma_2(t) = t - s\psi(t)$ is strictly increasing and onto.*

(u$_4$) *(Estimate between the fixed point sets of two multivalued mappings) If $S : X \to CB(X)$ be a multivalued mapping such that for $x \in F(S)$ there exists $z \in F(T)$ such that $\alpha(x,z) \geq 1$ and for $x \in F(T)$ there exists $z \in F(S)$ such that $\alpha(x,z) \geq 1$, $\eta > 0$ and $H(S(x), T(x)) \leq \eta$ for all $x \in X$, then $H(F(S), F(T)) \leq \sigma_1^{-1}(s\eta)$, where σ_1 is same as in (u_1).*

(u$_5$) *(Estimate between the fixed point sets of two multivalued mappings) If $S : X \to CB(X)$ be a multivalued mapping such that for $x \in F(S)$ there exists $z \in E(T)$ such that $\alpha(x,z) \geq 1$ and for $x \in E(T)$ there exists $z \in F(S)$ such that $\alpha(x,z) \geq 1$, and $H(S(x), T(x)) \leq \eta$ for all $x \in X$, then $H(F(S), F(T)) \leq \sigma_2^{-1}(s\eta)$, where σ_2 is same as in (u_3).*

(u$_6$) *(Well-posedness of fixed point problem with respect to dq b-metric d) If $\{x_n\}$ is a sequence in X, and there exists a unique $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geq 1$, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$.*

(u$_7$) *(Well-posedness of fixed point problem with respect to Hausdorff dq b-metric H) If $\{x_n\}$ is a sequence in X, and there exists a unique $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geq 1$, and $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$.*

(u$_8$) *(Limit shadowing property of the multivalued operators) If $\{x_n\}$ is a sequence in X, and there exists a unique $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geq 1$, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then there exists a sequence of successive approximation y_n such that $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Proof. By Corollary 2.11, we have $x^* \in F(T)$, that is, $x^* \in X$ is a solution of the fixed point inclusion (29). Then by given condition there exists a $y^* \in U$ such that $\alpha(x^*, y^*) \geq 1$. Since $y^* \in U$, for any given $\epsilon > 0$, we have $d(Ty^*, y^*) \leq \epsilon$. By given assumption on T we get that

$$\begin{aligned} d(x^*, y^*) &\leq s\left[d(x^*, Ty^*) + d(Ty^*, y^*)\right] \\ &\leq s\left[H(Tx^*, Ty^*) + d(Ty^*, y^*)\right] \\ &\leq s\left[\alpha(x^*, y^*)H(Tx^*, Ty^*) + \epsilon\right] \\ &\leq s\left[\psi(d(x^*, y^*)) + \epsilon\right]. \end{aligned}$$

Note that $\sigma_1(d(x^*, y^*)) = d(x^*, y^*) - s\psi(d(x^*, y^*))$. Thus from above inequality we get $\sigma_1(d(x^*, y^*)) \leq s\epsilon$ and hence $d(x^*, y^*) \leq \sigma_1^{-1}(s\epsilon)$. Consequently the fixed point inclusion (29) is $\sigma-$generalized Ulam-Hyers stable where $\sigma = \sigma_1^{-1}$.

(u$_2$) From corollary 2.11, we have $F(T) \neq \emptyset$. Let $x^* \in E(T)$. Then $E(T) = \{x^*\}$. We need to show that $F(T) = \{x^*\}$. Let $y \in F(T)$, that is, $y \in T(y)$ with $y \neq x^*$. Then by given condition we have $\alpha(x^*, y) \geq 1$ and

$$\begin{aligned} d(x^*, y) &= d(Tx^*, y) \leq H(Tx^*, Ty) \\ &\leq \alpha(x^*, y)H(Tx^*, Ty) \\ &\leq \psi\left(d(x^*, y)\right) < d(x^*, y), \end{aligned}$$

which implies that $d(x^*, y) = 0$, and so $x^* = y$. Thus $F(T) \subseteq E(T)$. As $E(T) \subseteq F(T)$, we obtain that $E(T) = F(T)$.

(u$_3$) Let $E(T) \neq \emptyset$, then $d(x^*, y^*) = d(Tx^*, y^*) \leq s\left[H(Tx^*, Ty^*) + d(Ty^*, y^*)\right]$. Now following the same lines as in (u$_1$) result follows.

(u$_4$) Let $x^* \in F(S)$, then there exists a $y^* \in F(T)$ such that $\alpha(x^*, y^*) \geq 1$. Then by given assumption on T we get

$$\begin{aligned} d(x^*, y^*) &\leq H(Sx^*, Ty^*) \\ &\leq s\left[H(Sx^*, Tx^*) + H(Tx^*, Ty^*)\right] \\ &\leq s\left[H(Sx^*, Tx^*) + \alpha(x^*, y^*)H(Tx^*, Ty^*)\right] \\ &\leq s\left[\eta + \psi(d(x^*, y^*))\right]. \end{aligned}$$

It follows from $\sigma_1\left(d(x^*,y^*)\right) = d(x^*,y^*) - s\psi(d(x^*,y^*))$ and the above inequality that $\sigma_1\left(d(x^*,y^*)\right) \leq s\eta$. Consequently for every $x^* \in F(S)$, there exists a $y^* \in F(T)$ such that $d(x^*,y^*) \leq \sigma_1^{-1}(s\eta)$. Similarly it can be proved that for every $y^* \in F(T)$, there exists a $x^* \in F(S)$ such that $d(x^*,y^*) \leq \sigma_1^{-1}(s\eta)$. Hence by Lemma 1.10 we obtain

$$H(F(S),F(T)) \leq \sigma_1^{-1}(s\eta).$$

(u_5) This can be proved on the similar lines as in (u_3) using the definition of $E(T)$.

(u_6) Let $\{x_n\}$ be a sequence in X, there exists a unique $x^* \in E(T)$ such that $\alpha(x_n,x^*) \geq 1$, and $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Then there exists $u_n \in Tx_n$ such that $\lim_{n\to\infty} d(x_n,Tx_n) = \lim_{n\to\infty} d(x_n,u_n) = 0$. Then by given assumption we have

$$\begin{aligned} d(x_n,x^*) &\leq s\left(d(x_n,Tx_n) + d(Tx_n,x^*)\right) \\ &\leq s\left(d(x_n,Tx_n) + H(Tx_n,Tx^*)\right) \\ &\leq s\left(d(x_n,Tx_n) + \alpha(x_n,x^*)H(Tx_n,Tx^*)\right) \\ &\leq s\left(d(x_n,Tx_n) + \psi\left(d(x_n,x^*)\right)\right). \end{aligned}$$

This implies that

$$d(x_n,x^*) - s\psi\left(d(x_n,x^*)\right) \leq sd(x_n,Tx_n).$$

That is $\sigma_2\left(d(x_n,x^*)\right) \leq sd(x_n,Tx_n)$. Taking limit as n tends to ∞ and taking into account the continuity of σ_2 at 0, we get the desired result. (u_7) Follows from (u_4) as $d(x_n,Tx_n) \leq H(\{x_n\},Tx_n)$. ($u_8$) From ($u_6$) it is clear that $\lim_{n\to\infty} d(x_n,x^*) = 0$. Since $x^* \in E(T)$, so there exists a sequence of successive approximations defined as $y_n = x^*$ for all n such that $\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x_n,x^*) = 0$. □

4. Applications on a dq b-Metric Space Endowed with a Partial Order

The aim of this section is to establish the necessary conditions for existence of a common fixed point of two mappings in the setting of a partially ordered complete b-metric space.

Definition 4.1. *Let X be a nonempty set. Then $(X,d,\preceq)$ is called a partially ordered dq b-metric space if (X,d) is a dq b-metric space and $(X,\preceq)$ a partially ordered set.*

Definition 4.2. *Let $(X,d,\preceq)$ be a partially ordered dq b-metric space. A sequence $\{x_n\} \subseteq X$ is called $\preceq$ −preserving if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}_0$.*

Definition 4.3. *Let $(X,d,\preceq)$ be a partially ordered dq b-metric space. A mapping $T : X \to N(X)$ is called $\preceq$-closed if for any $x,y \in X$,*

$$x \preceq y \text{ implies that } u \preceq v \text{ for any } u \in Tx \text{ and } v \in Sy.$$

Definition 4.4. *Let $(X,d,\preceq)$ be a partially ordered dq b-metric space. A mapping $T : X \to CL(X)$ is an α-continuous on $(CL(X),H)$ if, for any sequence $\{x_n\}$ in X,*

$$\lim_{n\to\infty} d(x_n,x) = 0 \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N}_0 \text{ imply that } \lim_{n\to\infty} H(Tx_n,Tx) = 0.$$

Corollary 4.5. *Let $(X,d,\preceq)$ be a partially ordered complete dq b-metric space. Suppose $T : X \to CB(X)$ is strictly $(\preceq -\psi)$-generalized rational contraction, $\preceq$-closed and $\preceq$-continuous. If there exists $x_0 \in X$ such that $x_0 \preceq x_1$ for some $x_1 \in Tx_0$. Then there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. Define $\alpha : X \times X \to \mathbb{R}^+$ by $\alpha(x,y) = 1$ whenever $x \preceq y$ and $\alpha(x,y) = 0$ whenever $x \npreceq y$. Since $x \preceq y$ implies $u \preceq v$, $\alpha(x,y) = 1$ implies $\alpha(u,v) = 1$ for each $u \in Tx$, $v \in Ty$. Now, by using Theorem 2.5, T has a fixed point. □

Example 4.6. *Let $X = \{1, 2, 3\}$ and*

$$\begin{aligned} d(1,1) &= d(2,2) = 0;\ d(2,1) = d(1,3) = d(3,2) = 1; \\ d(1,2) &= d(3,1) = 0;\ d(2,3) = d(3,3) = 2. \end{aligned}$$

Define $x \preceq y$ by $\preceq := \{(1,1), (2,2), (3,3), (1,2), (1,3)\} \subseteq X^2$. Since $d(3,3) \neq 0$ so d is not a metric. Note that $(X, d, \preceq)$ is partially ordered complete dq b-metric space with $s = 2$. Define the mapping $T : X \to CB(X)$ by, $T1 = \{1\}$, $T2 = \{3\}$, and $T3 = \{2\}$, one can show that the mapping T is $\preceq$-closed, $\preceq$-continuous and $(\preceq -\psi)$-generalized rational contraction where $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. If $x_0 = 1$ and $x_1 = 1 \in Tx_0$, then we have $x_0 \preceq x_1$. Note that, all the conditions of Corollary 4.5 are satisfied. Moreover, $x^* = 1$ is a fixed point of T.

Remark 4.7. *In Example 4.6, the mapping T is not a Banach contraction as*

$$H(T1, T2) = H(\{1\}, \{3\}) = 1 > \lambda d(1, 2)$$

for any $\lambda \in [0, 1)$. Hence the Nadler fixed Theorem [22, Theorem 5] is not applicable which shows that our result are potential generalizations of comparable results in the literature.

Corollary 4.8. *Let $(X, d, \preceq)$ be a partially ordered complete dq b-metric space and $T : X \to CB(X)$ a strictly $(\preceq -\psi)$-generalized rational contraction. Suppose that the following conditions hold: (i) T is $\preceq$-closed; (ii) the function $p : X \to \mathbb{R}$ defined by $p(x) = d(x, Tx)$, for $x \in X$, is lower semi-continuous; (iii) there exists $x_0 \in X$ such that $x_0 \preceq x_1$ for some $x_1 \in Tx_0$; and (iv) if $\{x_n\}$ is $\preceq$-preserving sequence in X such that $\lim_{n\to\infty} x_n = x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Remark 4.9. *Corollary 4.8 extends and generalizes corollary 2.2 of Asl.et.al [3].*

Next, we apply our results for the existence of common fixed point of single valued mappings on a complete dq b-metric space.

Definition 4.10. *Let X be any nonempty set, $\alpha : X \times X \to \mathbb{R}^+$ and $f, g : X \to X$. A pair (f, g) is called α-admissible if for any $x, y \in X$, with $\alpha(x, y) \geq 1$, we have $\alpha(fx, gy) \geq 1$.*

The results presented in this section, generalize and extend the comparable results in the literature.

Theorem 4.11. *Let (X, d) be a complete dq b-metric space and $f, g : X \to X$. A pair (f, g) and (g, f) are strictly $(\alpha - \psi)$-generalized rational contraction mappings. Suppose that the following conditions hold:*

(i) (f, g) and (g, f) are α-admissible;

(ii) there exist $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) (f, g) and (g, f) are α-continuous.

Then f and g have a common fixed point.

Proof. Define the mappings $T, S : X \to CB(X)$ by $Tx = \{fx\}$ and $Sx = \{gx\}$. Then Theorem 2.1 implies the result. □

Theorem 4.12. *Let (X, d) be a complete dq b-metric space and $f, g : X \to X$. A pair (f, g) and (g, f) are strictly $(\alpha - \psi)$-generalized rational contraction mappings. Suppose that the following conditions hold:*

(i) (f, g) and (g, f) are α-admissible;

(ii) the maps $p, h : X \to \mathbb{R}$ defined by $p(x) = d(x, fx)$ and $h(x) = d(x, gx)$ for $x \in X$ are lower semi-continuous;

(iii) there exist $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iv) if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ such that $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then f and g have a common fixed point.

Proof. Define the mappings $T, S : X \to CB(X)$ by $Tx = \{fx\}$ and $Sx = \{gx\}$. Then Theorem 2.3 implies the result. □

Corollary 4.13. *Let (X, d) be a complete dq b-metric space and $f : X \to X$ a strictly $(\alpha - \psi)$-generalized rational contraction mapping. Suppose f is α-admissible, α-continuous and there exist $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Then f has a fixed point.*

Corollary 4.14. *Let (X, d) be a complete dq b-metric space and $f : X \to X$ a strictly $(\alpha - \psi)$-generalized rational contraction mapping. Suppose (i) f is α-admissible, (ii) the map $p : X \to \mathbb{R}$ defined by $p(x) = d(x, fx)$ for $x \in X$ is lower semi-continuous, (iii) there exist $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and (iv) if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ such that $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Example 4.15. *Let $X = [0, 2]$ and $d(x, y) = \left|x - y\right|^3$. Then (X, d) is a complete dq b-metric space with $s = 4$. Define $f : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as:*

$$fx = \begin{cases} \dfrac{x}{2} \text{ if } x \in [0, 1], \\ \frac{1}{2} \text{ if } x \in (1, 2], \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in [0, 1], \\ \dfrac{1}{3} \text{ otherwise,} \end{cases}$$

and $\psi(t) = \frac{t}{8}$. The mapping f is $(\alpha - \psi)$-generalized rational contraction. Indeed, we have

$$\alpha(x, y)d(fx, fy) \leq \left|\frac{x}{2} - \frac{y}{2}\right|^3 = \frac{1}{8}d(x, y) \leq \psi(M_f(x, y))$$

for all $x, y \in [0, 1]$. For $x_0 = \frac{1}{2}$, we have $\frac{1}{4} = fx_0$ such that $\alpha(x_0, fx_0) \geq 1$. The mapping f is α-admissible. Indeed, for any $x, y \in [0, 1]$, we have fx, $fy \in [0, 1]$ and thus $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. By the definition of α, we have $x_n, x \in [0, 1]$ for all $n \in \mathbb{N}$. So, $\alpha(x_n, x) \geq 1$. Also, $\lim_{n\to\infty} d(fx_n, fx) = \lim_{n\to\infty} \left|\frac{x_n}{2} - \frac{x}{2}\right|^3 = \lim_{n\to\infty} \frac{1}{8}d(x_n, x) = 0$, implies that f is α-continuous. Also, f is continuous. Thus, all the conditions of corollaries 4.13 and 4.14 are satisfied. Moreover, $x = 0$ is a fixed point of f.

The next corollay generalizes and extends Karapinar [17, Corollary 3.11, 3.12].

Corollary 4.16. *Let $(X, d, \preceq)$ be a partially ordered complete dq b-metric space. Suppose $f : X \to X$ is a strictly $(\preceq -\psi)$-generalized rational contraction, $\preceq$-closed and $\preceq$-continuous. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then there exists $x^* \in X$ such that $x^* = fx^*$.*

Corollary 4.17. *Let $(X, d, \preceq)$ be a partially ordered complete dq b-metric space and $f : X \to X$ is a strictly $(\preceq -\psi)$-generalized rational contraction and $\preceq$-closed. Suppose (i) the map $p : X \to \mathbb{R}$ defined by $p(x) = d(x, fx)$ for $x \in X$ is lower semi-continuous; (ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$; and (iii) if $\{x_n\}$ is a $\preceq$-preserving sequence in X such that $\lim_{n\to\infty} x_n = x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then there exists $x^* \in X$ such that $x^* = fx^*$.*

The above result generalizes and extends the result of Ran and Reurnings [26], Nieto and Rodrígues-López [23] and Beg [9].

Corollary 4.18. *Let $(X, d, \preceq)$ be a partially ordered complete dq b-metric space. Let $f : X \to X$ such that $d(fx, fy) \leq \psi(d(x, y))$ for all $\psi \in \Psi$, $x, y \in X$ with $x \preceq y$. Suppose f is $\preceq$-closed and $\preceq$-continuous, and if there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then there exists $x^* \in X$ such that $x^* = fx^*$.*

5. Application to Integral Equation

Motivated by [6] and the references therein, we apply corollary 4.16 to the existence of a solution of a nonlinear integral equation.

Let $X = C(J, \mathbb{R})$ be the set of all real continuous functions defined on $J = [0, L]$ where $L > 0$ and $\rho : X \times X \to \mathbb{R}^+$ defined by

$$\rho(x, y) = \sup_{t \in J} \left| x(t) - y(t) \right| \text{ for } x, y \in X.$$

Consider the dq b-metric $d : X \times X \to \mathbb{R}^+$ given as follows:

$$d(x, y) = (\rho(x, y))^p = \left(\sup_{t \in J} \left| x(t) - y(t) \right| \right)^p = \sup_{t \in J} \left| x(t) - y(t) \right|^p \text{ for all } x, y \in X \text{ and } p \geq 1.$$

It is well known that (X, d) is a complete dq b-metric space with $s = 2^{p-1}$. Let $\psi \in \Psi$ and $(\psi(t))^p \leq \psi(t^p)$ for all $p \geq 1$ and $t \in J$. Also, note that $(X, d, \preceq)$ is a partially ordered complete dq b-metric space, where $\preceq$ denotes the usual order, that is, $x \preceq y$ if $x(t) \leq y(t)$ for all $t \in J$. Consider the nonlinear integral equation as follows:

$$x(t) = q(t) + \int_0^L k(t, s) f(s, x(s)) ds \text{ for all } t \in J. \tag{32}$$

Suppose that the following conditions hold:

(C1) $q : J \to \mathbb{R}$ and $f : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, such that $f(t, x) \geq 0$ for all $t \in J$ and for all $a, b \in \mathbb{R}$,

$$\left| f(t, a) - f(t, b) \right| \leq \psi(|a - b|);$$

(C2) $k : J \times J \to \mathbb{R}$ is continuous at $t \in J$ for every $s \in J$ and measurable at $s \in J$ for all $t \in J$ such that $k(t, s) \geq 0$ and $\int_0^L k(t, s) ds \leq 1$;

(C3) there exists $x_0 \in X$ such that for all $t \in J$, we have

$$x_0(t) \leq q(t) + \int_0^L k(t, s) f(s, x_0(s)) ds.$$

Let $F : X \to X$ be a mapping defined by

$$Fx(t) = q(t) + \int_0^L k(t, s) f(s, x(s)) ds \text{ for } t \in J. \tag{33}$$

It is clear that x is a solution of integral equation (32) if and only if x is a fixed point of F.

Theorem 5.1. *Under assumptions (C1)-(C3), the integral equation (32) has a solution in X.*

Proof. Let $x, y \in X$ such that $x \preceq y$ and $t \in J$. Then

$$\begin{aligned} \left| Fx(t) - Fy(t) \right| &= \left| q(t) + \int_0^L k(t, s) f(s, x(s)) ds - q(t) - \int_0^L k(t, s) f(s, y(s)) ds \right| \\ &\leq \int_0^L k(t, s) \left| f(s, x(s)) - f(s, y(s)) \right| ds \\ &\leq \int_0^L k(t, s) \psi(\left| x(s) - y(s) \right|) ds. \end{aligned}$$

Since ψ is nondecreasing, we obtain that

$$\psi(\left|x(s)-y(s)\right|) \leq \psi(\sup_{t\in J}\left|x(t)-y(t)\right|) = \psi(\rho(x,y)).$$

This implies that

$$\left|Fx(t)-Fy(t)\right| \leq \psi(\rho(x,y)).$$

Therefore

$$\begin{aligned} d(Fx,Fy) &= \sup_{t\in J}\left|Fx(t)-Fy(t)\right|^p \\ &\leq [\psi(\rho(x,y))]^p \leq \psi((\rho(x,y))^p) \\ &\leq \psi(d(x,y)) \\ &\leq \psi(M_F(x,y)). \end{aligned}$$

Note that all the conditions of Corollary 4.16 are satisfied and hence the mapping F has a fixed point which is a solution of the integral equation (32) in X. □

Remark 5.2. *We can obtain the dq b-metric, quasi b-metric, b-metric, dq metric, quasi metric, and metric version of our main results which can be viewed as new results in the literature.*

Remark 5.3. *Similar result as the above theorem can be established if the binary relation $\preceq$ is $\preceq$-reversing.*

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