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Ćirić Type Nonunique Fixed Point Theorems on b-Metric Spaces

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Abstract. In this paper, inspired the very interesting results of Ćirić [20], we investigate the existing non-unique fixed points of certain operators in the context of b-metric spaces. Our main results unify and cover several existing results on the topic in the literature.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

In 1974, Ćirić [20] underline the importance of the notion of nonunique fixed point and proposed a criteria for certain operators which posses nonunique fixed points. Inspired this pioneer work, a number of authors have reported nonunique fixed point for the operators that fulfil different conditions, see e.g. [1, 20, 21, 26–28, 30].

Giving a more general form of the notion of metric has been considered by many authors. One of the interesting suggestion is belong to Czerwik [19] who relaxed the axiom of triangle inequality of the notion of metric and introduce the notion of b-metric. In fact, this approach was already considered previously by some authors, like Bakhtin [6], Bourbaki [18] and so on. Czerwik [19] proved the analog of Banach contraction mapping principle in the context of b-metric spaces. Following this pioneer result on b-metric, a number of authors have reported several interesting results in this direction (see e.g. [2–5],[10]-[17] and related references therein).

Definition 1.1 (Czerwik [19]). *Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

($b1$) $d(x, y) = 0$ *if and only if* $x = y$.
($b2$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
($b3$) $d(x, y) \leq s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$, *where* $s \geq 1$.

The function d is called a b-metric and the space (X, d) is called a b-metric space, in short, bMS.

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It is clear that for $s = 1$, the b-metric becomes a usual metric.

The immediate example of b-metric is the following:

Example 1.2. *Let $Y = \{x, y, z\}$ and $X = Y \cup \mathbb{N}$. Define a mapping $d : X \times X \to [0, \infty)$ such that*

$$\begin{aligned} &d(x,y) = d(y,x) = d(x,z) = d(z,x) = 1, \\ &d(y,z) = d(z,y) = A, \\ &d(x,x) = d(y,y) = d(z,z) = 0, d(n,m) = \left|\tfrac{1}{n} - \tfrac{1}{m}\right| \end{aligned}$$

where $A \in [2, \infty)$. Then, we find that

$$d(x,y) \le \frac{A}{2}\left[d(x,z) + d(z,y)\right], \text{ for } x, y, z \in Y.$$

It is evident that (X, d) is a b-metric space. Notice also that if $A > 2$ the standard triangle inequality does not hold and (X, d) is not a metric space.

The following basic and interesting examples of b-metric spaces can be found in many sources e.g. [10]-[14].

Example 1.3. *Let $X = \mathbb{R}$. Define*

$$d(x,y) = |x - y|^p \tag{1}$$

for $p > 1$. Then d is a b-metric on $\mathbb{R}$. Clearly, the first two conditions hold. Since

$$|x - y|^p \le 2^{p-1}[|x - z|^p + |z - y|^p],$$

the third condition holds with $s = 2^{p-1}$. Thus, $(\mathbb{R}, d)$ is a b-metric space with a constant $s = 2^{p-1}$.

Example 1.4. *Let $p \in (0, 1)$ and let*

$$X = l_p(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

Define

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

Then (X, d) is a b-metric space with $s = 2^{1/p}$.

Example 1.5. *Let E be a Banach space and 0_E be the zero vector of E. Let P be a cone in E with $int(P) \neq \emptyset \preceq$ be a partial ordering with respect to P. Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:*

$(M1)$ $0 \preceq d(x, y)$ *for all $x, y \in X$,*
$(M2)$ $d(x, y) = 0$ *if and only if $x = y$,*
$(M3)$ $d(x, y) \preceq d(x, z) + d(z, y)$, *for all $x, y \in X$.*
$(M4)$ $d(x, y) = d(y, x)$ *for all $x, y \in X$*

then d is called cone metric on X, and the pair (X, d) is called a cone metric space (CMS).

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D : X \times X \to [0, \infty)$ be defined by $D(x, y) = \|d(x, y)\|$, where $d : X \times X \to E$ is a cone metric space. Then (X, D) is a b-metric space with constant $s := K \ge 1$.

The notion of comparison function is defined by Rus [29] and it has been extensively studied by a number of authors to get more general form of contractive mappings.

Definition 1.6. *[8, 29] A function $\phi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is increasing and $\phi^n(t) \to 0$ as $n \to \infty$ for every $t \in [0,\infty)$, where ϕ^n is the n-th iterate of ϕ.*

Properties and examples of comparison functions can be found in [8, 29]. An important property of comparison functions is given by the following Lemma.

Lemma 1.7. *([8, 29]) If $\phi : [0,\infty) \to [0,\infty)$ is a comparison function, then*

1. *each iterate ϕ^k of ϕ, $k \geq 1$ is also a comparison function;*
2. *ϕ is continuous at 0;*
3. *$\phi(t) < t$ for all $t > 0$.*

Definition 1.8. *Let $s \geq 1$ be a real number. A function $\phi : [0,\infty) \to [0,\infty)$ is called a (b)-comparison function if*

1. *ϕ is increasing;*
2. *there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.*

The collection of all (b)-comparison functions will be denoted by Ψ. In the literature, (b)-comparison function is called (c)-comparison functions when $s = 1$. It can be shown that a (c)-comparison function is a comparison function, but the converse is not true in general. Berinde [7] also proved the following important property of (b)-comparison functions.

Lemma 1.9. *([7]) Let $\phi : [0,\infty) \to [0,\infty)$ be a (b)-comparison function. Then,*

1. *the series $\sum_{k=0}^{\infty} s^k\phi^k(t)$ converges for any $t \in [0,\infty)$;*
2. *the function $b_s : [0,\infty) \to [0,\infty)$ defined as $b_s = \sum_{k=0}^{\infty} s^k\phi^k(t)$ is increasing and continuous at $t = 0$.*

Remark 1.10. *By the Lemma 1.9, we conclude that every (b)-comparison function is a comparison function and hence, by the Lemma 1.7, any (b)-comparison function ϕ satisfies $\phi(t) < t$.*

In this manuscript, inspired by the well-known non-unique fixed point of Ćirić, we state and prove some new non-unique fixed point theorems in the setting of b-metric spaces. Our results improve the existence results in the literature, see e.g.[1, 20, 22, 23, 28].

2. Non Unique Fixed Points on *b*-Metric Spaces

We shall start to this section by recalling the notion of orbitally continuous.

Definition 2.1. *A mapping T on bMS (X,d) is said to be orbitally continuous if $\lim_{i\to\infty} T^{n_i}(x) = z$ implies $\lim_{i\to\infty} T(T^{n_i}(x)) = Tz$. A bMS (X,d) is called T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}(x)\}_{i=1}^{\infty}$, $x \in X$ converges in (X,d).*

Remark 2.2. *It is evident that orbital continuity of T yields orbital continuity of T^m for any $m \in \mathbb{N}$.*

Theorem 2.3. *Let T be an orbitally continuous self-map on the T-orbitally complete bMS (X,d). If there is $\psi \in \Psi$ such that*

$$\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} - \min\{d(x,Ty), d(Tx,y)\} \leq \psi(d(x,y)), \tag{2}$$

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. For an arbitrary $x \in X$, we shall construct an iterative sequence $\{x_n\}$ as follows:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{3}$$

We suppose that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4}$$

Indeed, if for some $n \in \mathbb{N}$ we have the inequality $x_n = Tx_{n-1} = x_{n-1}$, then, the proof is completed. By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (2), we derive that

$$\begin{aligned}\min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}\\ - \min\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \leq \psi(d(x_{n-1}, x_n)).\end{aligned} \tag{5}$$

It implies that

$$\min\{d(x_n, x_{n+1}), d(x_n, x_{n-1})\} \leq \psi(d(x_{n-1}, x_n)). \tag{6}$$

Since $\psi(t) < t$ for all $t > 0$, the case $d(x_n, x_{n-1}) \leq \psi(d(x_{n-1}, x_n))$ is impossible. Thus, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)). \tag{7}$$

Applying Remark 1.10 recurrently, we find that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)). \tag{8}$$

By Lemma 1.9, we deduce that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{9}$$

In what follow we shall prove that the sequence $\{x_n\}$ is Cauchy.

Consider $d(x_n, x_{n+k})$ for $k \geq 1$. By using the triangle inequality $(b3)$ again and again, we get the following approximation

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k})] \\ & \leq & sd(x_n, x_{n+1}) + s\{s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})]\} \\ & = & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+k}) \\ & \vdots & \\ & \leq & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots \\ & + & s^{k-1} d(x_{n+k-2}, x_{n+k-1}) + s^{k-1} d(x_{n+k-1}, x_{n+k}) \\ & \leq & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots \\ & + & s^{k-1} d(x_{n+k-2}, x_{n+k-1}) + s^k d(x_{n+k-1}, x_{n+k}), \end{array} \tag{10}$$

since $s \geq 1$. Combining (8) and (10) we derive that

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s\psi^n(d(x_0, x_1)) + s^2\psi^{n+1} d(x_0, x_1) + \ldots \\ & + & s^{k-1}\psi^{n+k-2}(d(x_0, x_1)) + s^k\psi^{n+k-1}(d(x_0, x_1)) \\ & = & \dfrac{1}{s^{n-1}}[s^n\psi^n(d(x_0, x_1)) + s^{n+1}\psi^{n+1} d(x_0, x_1) + \ldots \\ & + & s^{n+k-2}\psi^{n+k-2}(d(x_0, x_1)) + s^{n+k-1}\psi^{n+k-1}(d(x_0, x_1))]. \end{array} \tag{11}$$

Consequently, we have

$$d(x_n, x_{n+k}) \leq \frac{1}{s^{n-1}}\left[P_{n+k-1} - P_{n-1}\right], \quad n \geq 1, k \geq 1, \tag{12}$$

where $P_n = \sum_{j=0}^{n} s^j \psi^j(d(x_0, x_1))$, $n \geq 1$. From Lemma 1.9, the series $\sum_{j=0}^{\infty} s^j \psi^j(d(x_0, x_1))$ is convergent and since $s \geq 1$, upon taking limit $n \to \infty$ in (12) we get

$$\lim_{n\to\infty} d(x_n, x_{n+k}) \leq \lim_{n\to\infty} \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}] = 0. \tag{13}$$

We conclude that the sequence $\{x_n\}$ is Cauchy in (X, d).

Owing to the construction $x_n = T^n x_0$ and the fact that (X, p) is T-orbitally complete, there is $z \in X$ such that $x_n \to z$. Due to the orbital continuity of T, we conclude that $x_n \to Tz$. Hence $z = Tz$ which terminates the proof. □

Corollary 2.4. *Let T be an orbitally continuous self-map on the T-orbitally complete bMS (X, d). If there exists $k \in [0, 1)$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y), \tag{14}$$

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

If we take $s = 1$ in the previous corollary, we get the famous non-unique fixed point theorem of Ćirić.

Corollary 2.5. [Non-unique fixed point theorem of Ćirić [20]] Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). If there is $k \in [0, 1)$ such that

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y),$$

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Remark 2.6. *Regarding the Example 1.5, we deduce that the analog of Ćirić non-unique fixed point theorem, Corollary 2.5, in the setting of cone metric space with normal cone, is still valid (see e.g.[23]).*

Theorem 2.7. *Let T be an orbitally continuous self-map on the T-orbitally complete bMS (X, d).*

Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 and a self mapping $T : X \to X$ satisfies the conditions

$$0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \ \ a_1 + a_2 \neq 0, \ a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5 \tag{15}$$

$$a_1 d(Tx, Ty) + a_2\Big[d(x, Tx) + d(y, Ty)\Big] + a_3[d(y, Tx) + d(x, Ty)] \leq a_4 d(x, y) + a_5 d(x, T^2x) \tag{16}$$

hold for all $x, y \in X$. Then, T has at least one fixed point.

Proof. Take $x_0 \in X$ be arbitrary. Construct a sequence $\{x_n\}$ as follows:

$$x_{n+1} := Tx_n \quad n = 0, 1, 2, ... \tag{17}$$

When we substitute $x = x_n$ and $y = x_{n+1}$ on the inequality (16), it implies that

$$\begin{aligned} a_1 d(Tx_n, Tx_{n+1}) + a_2\Big[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\Big] + a_3[d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})] \\ \leq a_4 d(x_n, x_{n+1}) + a_5 d(x_n, T^2 x_n) \end{aligned} \tag{18}$$

for all a_1, a_2, a_3, a_4, a_5 that satisfy (15). Due to (17), the statement (18) turns into

$$\begin{aligned} a_1 d(x_{n+1}, x_{n+2}) + a_2\Big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\Big] + a_3[d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})] \\ \leq a_4 d(x_n, x_{n+1}) + a_5 d(x_n, x_{n+2}). \end{aligned} \tag{19}$$

By a simple calculation, one can get

$$(a_1 + a_2)d(x_{n+1}, x_{n+2}) + (a_3 - a_5)d(x_n, x_{n+2}) \leq (a_4 - a_2)d(x_n, x_{n+1}) \tag{20}$$

which implies

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}) \tag{21}$$

where $k = \frac{a_4 - a_2}{a_1 + a_2}$. Due to (15), we have $0 \leq k < 1$. Taking account of (21), we get inductively

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n d(x_0, x_1). \tag{22}$$

We shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \ldots + s^{p-2} \cdot d(x_{n+p-3}, x_{n+p-2}) + \\
&+ s^{p-1} \cdot d(x_{n+p-2}, x_{n+p-1}) + s^p \cdot d(x_{n+p-1}, x_{n+p}) \\
&\leq s \cdot k^n \cdot d(x_0, x_1) + s^2 \cdot k^{n+1} \cdot d(x_0, x_1) + \ldots + \\
&+ s^{p-2} \cdot k^{n+p-3} \cdot d(x_0, x_1) + s^{p-1} \cdot k^{n+p-2} \cdot d(x_0, x_1) + \\
&+ s^p \cdot k^{n+p-1} \cdot d(x_0, x_1) \\
&= \frac{1}{s^n \cdot k} \cdot \Big[s^{n+1} \cdot k^{n+1} \cdot d(x_0, x_1) + \ldots + s^{n+p-1} \cdot k^{n+p-1} \cdot d(x_0, x_1) + \\
&+ s^{n+p} \cdot k^{n+p} \cdot d(x_0, x_1) \Big] \\
&\leq \frac{1}{s^n \cdot k} \cdot \Big[s^{n+1} \cdot k^{n+1} \cdot d(x_0, x_1) + \ldots + s^{n+p} \cdot k^{n+p} \cdot d(x_0, x_1) \Big] \\
&= \frac{1}{s^n \cdot k} \cdot \sum_{i=n+1}^{n+p} s^i \cdot k^i \cdot d(x_0, x_1) \\
&< \frac{1}{s^n k} \cdot \sum_{i=n+1}^{\infty} s^i \cdot k^i \cdot d(x_0, x_1).
\end{aligned}$$

The precedent inequality is

$$d(x_n, x_{n+p}) < \frac{1}{s^n k} \cdot \sum_{i=n+1}^{\infty} s^i \cdot k^i \cdot d(x_0, x_1). \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

As in the proof of previous theorem, regarding the construction $x_n = T^n x_0$ together with the fact that (X, p) is T-orbitally complete, there is $z \in X$ such that $x_n \to z$. Again by the orbital continuity of T, we deduce that $x_n \to Tz$. Hence $z = Tz$. □

Theorem 2.7 is still valid in the context of standard metric space.

Corollary 2.8. *(See [22]) Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d).*

Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 and a self mapping $T : X \to X$ satisfies the conditions

$$0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \;\; a_1 + a_2 \neq 0, \; a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5 \tag{23}$$

$$a_1 d(Tx, Ty) + a_2 \Big[d(x, Tx) + d(y, Ty) \Big] + a_3 [d(y, Tx) + d(x, Ty)] \leq a_4 d(x, y) + a_5 d(x, T^2 x) \tag{24}$$

hold for all $x, y \in X$. Then, T has at least one fixed point.

Remark 2.9. *As we discuss in Remark 2.6, we obtain the analog of Theorem 2.7 in the context of cone metric spaces. More precisely, again taking Example 1.5 into account, one can derive that Corollary 2.8 is also still fulfilled in the setting of cone metric space with normal cone (see e.g.[22]).*

Theorem 2.10. *Let T be an orbitally continuous self-map on the T-orbitally complete bMS (X, d). Suppose that there exists $\psi \in \Psi$ such that*

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \leq \psi(d(x, y)), \tag{25}$$

for all $x, y \in X$, where

$$\begin{aligned} P(x, y) &= \min\{d(Tx, Ty)d(x, y), d(x, Tx)d(y, Ty)\}, \\ Q(x, y) &= \min\{d(x, Tx)d(x, Ty), d(y, Ty)d(Tx, y)\}, \\ R(x, y) &= \min\{d(x, Tx), d(y, Ty)\}. \end{aligned}$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. As in the proof of Theorem 2.3, we shall construct an iterative sequence $\{x_n\}$, for an arbitrary initial value $x \in X$:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{26}$$

As it is discussed in the proof of Theorem 2.3, we suppose

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{27}$$

By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (25), we derive that

$$\frac{P(x_{n-1},x_n)-Q(x_{n-1},x_n)}{R(x_{n-1},x_n)} \leq \psi(d(x_{n-1}, x_n)), \tag{28}$$

where

$$\begin{aligned} P(x_{n-1}, x_n) &= \min\{d(Tx_{n-1}, Tx_n)d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)\}, \\ Q(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n), d(x_n, Tx_n)d(Tx_{n-1}, x_n)\}, \\ R(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}. \end{aligned}$$

Due to axioms of bMS, we find that

$$\frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}} \leq \psi(d(x_{n-1}, x_n)), \tag{29}$$

If $R(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then, the inequality (29) turns into

$$d(x_{n-1}, x_n) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \tag{30}$$

which is a contraction, since $\psi(t) < t$ for all $t > 0$. Accordingly, we deduce that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)). \tag{31}$$

Applying Remark 1.10 recurrently, we find that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)). \tag{32}$$

By Lemma 1.9, we deduce that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{33}$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 2.3. □

Corollary 2.11. *Let T be an orbitally continuous self-map on the T-orbitally complete bMS (X, d). Suppose that there exists $k \in [0, 1)$ such that*

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \leq kd(x, y), \tag{34}$$

for all $x, y \in X$, where

$$\begin{aligned} P(x, y) &= \min\{d(Tx, Ty)d(x, y), d(x, Tx)d(y, Ty)\}, \\ Q(x, y) &= \min\{d(x, Tx)d(x, Ty), d(y, Ty)d(Tx, y)\}, \\ R(x, y) &= \min\{d(x, Tx), d(y, Ty)\}. \end{aligned}$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Corollary 2.12. **[Nonunique fixed point of Achari [1]]** Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). Suppose that there exists $k \in [0, 1)$ such that

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \leq kd(x, y), \tag{35}$$

for all $x, y \in X$, where

$$\begin{aligned} P(x, y) &= \min\{d(Tx, Ty)d(x, y), d(x, Tx)d(y, Ty)\}, \\ Q(x, y) &= \min\{d(x, Tx)d(x, Ty), d(y, Ty)d(Tx, y)\}, \\ R(x, y) &= \min\{d(x, Tx), d(y, Ty)\}. \end{aligned}$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 2.13. *Let T be an orbitally continuous self-map on the T-orbitally complete bMS (X, d). Suppose that there exists $k \in [0, 1)$ such that*

$$m(x, y) - n(x, y) \leq kd(x, Tx)d(y, Ty), \tag{36}$$

for all $x, y \in X$, where

$$\begin{aligned} m(x, y) &= \min\{[d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\}, \\ n(x, y) &= \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\} \end{aligned}$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. By following the lines in the proof of Theorem 2.3, we shall formulate an recursive sequence $\{x_n\}$, for an arbitrary initial value $x \in X$:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{37}$$

Regarding the analysis in the proof of Theorem 2.3, we assume that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{38}$$

By replacing $x = x_{n-1}$ and $y = x_n$ in the inequality (36), we observe that

$$m(x_{n-1}, x_n) - n(x_{n-1}, x_n) \leq kd(x_{n-1}, Tx_{n-1})d(x_n, Tx_n), \tag{39}$$

where

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{[d(Tx_{n-1}, Tx_n)]^2, d(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n), [d(x_n, Tx_n)]^2\}, \\ n(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n), d(x_{n-1}, Tx_n)d(x_n, Tx_{n-1})\}. \end{aligned}$$

By utilizing the above inequality, we get that

$$m(x_{n-1}, x_n) \leq kd(x_{n-1}, x_n)d(x_n, x_{n+1}), \tag{40}$$

where $m(x_{n-1}, x_n) = \min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)d(x_n, x_{n+1})\}$. Notice that the case $m(x_{n-1}, x_n) = d(x_{n-1}, x_n)d(x_n, x_{n+1})$ is impossible. Indeed, in this case, since $\psi(t) < t$ for all $t > 0$, the inequality (40) turns into

$$d(x_{n-1}, x_n)d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)d(x_n, x_{n+1}) \tag{41}$$

It is a contradiction since $k < 1$. Appropriately, we infer that

$$[d(x_n, x_{n+1})]^2 \leq kd(x_{n-1}, x_n)d(x_n, x_{n+1}) \tag{42}$$

which is equivalent to

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n). \tag{43}$$

Recurrently, we find that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1). \tag{44}$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 2.7. □

Theorem 2.7 is still valid in the context of standard metric space.

Corollary 2.14. [Nonunique fixed point of Pachpatte [28]] Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). Suppose that there exists $k \in [0, 1)$ such that

$$m(x, y) - n(x, y) \leq kd(x, Tx)d(y, Ty), \tag{45}$$

for all $x, y \in X$, where

$$\begin{aligned} m(x, y) &= \min\{[d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\}, \\ n(x, y) &= \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\} \end{aligned}$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Remark 2.15. *One can deduce the analog of Theorem 2.13 in the context of cone metric spaces as it mentioned in Remark 2.6.*

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