Filomat 31:12 (2017), 3611–3626
https://doi.org/10.2298/FIL1712611K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Random Iterative Algorithms and Almost Sure Stability in Banach Spaces

Abdul Rahim Khan[a], Vivek Kumar[b], Satish Narwal[c], Renu Chugh[d]

[a]*Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia*
[b]*Department of Mathematics, K. L. P. College, Rewari 123401, India*
[c]*Department of Mathematics, S. J. K. College Kalanaur, Rohtak 124113, India*
[d]*Department of Mathematics, M. D. University, Rohtak 124001, India*

Abstract. Many popular iterative algorithms have been used to approximate fixed point of contractive type operators. We define the concept of generalized ϕ-weakly contractive random operator T on a separable Banach space and establish Bochner integrability of random fixed point and almost sure stability of T with respect to several random Kirk type algorithms. Examples are included to support new results and show their validity. Our work generalizes, improves and provides stochastic version of several earlier results by a number of researchers.

1. Introduction

Random fixed points are stochastic generalization of classical or deterministic fixed points and are required for various classes of random operators arising in physical systems (see [3, 4, 14, 15, 17]). Random fixed point theory was initiated in 1950 by Prague school of probabilists. The machinery of random fixed point theory provides a convenient way of modelling many problems arising in nonlinear analysis, probability theory and for a solution of random equations in applied sciences. The study of nonlinear operators has attracted the attention of many mathematicians in various spaces (see [2, 13–15, 18, 30, 32, 33] and references therein). Several interesting random fixed point results have been established in [4, 6, 8, 13, 15, 18, 19, 27, 34]. If the exact value of a fixed point of a mapping cannot be found, we approximate it through a convenient iterative algorithm. With the developments in random fixed point theory, there has been a renewed interest in random iterative algorithms [4, 6, 8, 13, 27, 34]. In linear spaces, Mann and Ishikawa iterative algorithms have been extensively applied to fixed point problems [5, 16, 25, 29].

Initially Mann [25] iterative algorithm was employed to approximate a fixed point of a non-expansive mapping where the Picard iterative algorithm failed to converge. In 1974, Ishikawa [16] iterative algorithm has been used to obtain convergence of a Lipschitzian pseudo-contractive operator where the Mann iterative algorithm was not applicable. Later, Noor iterative algorithm [26] was introduced to solve variational inequality problems. Recently, Phuengrattana and Suantai [28] introduced SP iterative algorithm and

---

2010 *Mathematics Subject Classification*. 47H09; 47H10; 49M05; 54H25
Keywords. Random iterative schemes; Almost sure T-stability; Separable Banach space; Bochner integrability; ϕ-Weakly contractive random operator
Received: 20 May 2016; Accepted: 03 July 2016
Communicated by Ljubomir Ćirić
Email addresses: arahim@kfupm.edu.sa (Abdul Rahim Khan), ratheevivek15@yahoo.com (Vivek Kumar), narwalmaths@gmail.com (Satish Narwal), chughrenu@yahoo.com (Renu Chugh)

proved that it has better convergence rate as compared to Mann, Ishikawa and Noor iterative algorithms. Kirk [24], Rhoades [29] and Hussain et al. [12] studied Kirk type iterative algorithms with faster convergence rate than other existing iterative algorithms. Results on S-iterative algorithm for pseudo-contractive and contractive maps, respectively, were established by Sahu and Peturasel [31] and Kumar et al. [23].

Stability and convergence results for various iterative algorithms have been established in [1, 5–7, 9, 13, 20–22, 27, 28, 34]. Bochner integrability of fixed point is an interesting concept related to iterative algorithms and is used to solve different problems in functional analysis and probability theory. It is also used to study geometry of Banach spaces and differential equations in vector spaces (see [10] and references therein). Recently, Zhang et al. [34] studied almost sure T-stability of Ishikawa-type and Mann-type random algorithms for certain ϕ-weakly contractive type random operators in the setup of a separable Banach space. They also established Bochner integrability of a random fixed point for such random operators. Very recently, Okeke and Abbas [27] introduced the notion of generalized ϕ-weakly contractive random operator and obtained almost sure T-stability of random Ishikawa iterative algorithm for these operators.

We prove Bochner integrability of a random fixed point by using a verity of very general iterative algorithms like random Noor, random SP, random Kirk-Noor, random Kirk-SP for generalized ϕ-weakly contractive operators satisfying the condition (2.5). Our results are improvement and generalization of the results of Zhang et al. [34], Aweke and Abbas [27] and give random version of many important known results.

2. Preliminaries

Let Σ be a sigma algebra of subsets of a set Ω and X be a separable Banach space. Throughout this paper, we assume that (Ω, Σ, μ) is a complete probabilistic measure space, $(\Sigma, B(X))$ is the Borel measurable space.

A mapping $\xi : \Omega \to X$ is called (a) X-valued random variable if ξ is $(\Sigma, B(X))$-measurable, (b) strongly μ-measurable if, there exists a sequence $\{x_n\}$ of μ-simple functions converging to ξ, μ-almost everywhere. In view of separability of the Banach space X, the sum of two X-valued random variables is an X-valued random variable.

The following definitions and results will be needed in the sequel.

Definition 2.1. *A mapping $g : \Omega \to C$ is said to be measurable if $g^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of X and nonempty subset C of X.*

Definition 2.2. *A function $T : \Omega \times C \to C$ is said to be a random operator if $T(\cdot, x) : \Omega \to C$ is measurable for every $x \in C$.*

Definition 2.3. *A measurable mapping $p : \Omega \to C$ is said to be random fixed point of the random operator $T : \Omega \times C \to C$, if $T(w, p(w)) = p(w)$ for all $w \in \Omega$.*

We denote by $RF(T)$, the set of random fixed points of T.

Definition 2.4 ([17]). *A random variable $\xi : \Omega \to C$ is Bochner integrable if for each*

$$w \in \Omega, \quad \int_{\Omega} \|\xi(w)\| d\mu(w) < \infty, \tag{2.1}$$

where $\|\xi(w)\|$ is a non-negative real valued random variable.

The Bochner integral is a natural generalization of the familiar Lebesgue integral for vector-valued set functions.

Definition 2.5 ([17]). *A random variable ξ is Bochner integrable if and only if there exists a sequence of random variables $\{\xi_n\}_{n=1}^{\infty}$ converging strongly to ξ almost surely such that*

$$\lim_{n\to\infty} \int_{\Omega} \|\xi_n(w) - \xi(w)\| d\mu(w) = 0. \tag{2.2}$$

Definition 2.6 ([34]). *Let C be a nonempty subset of a separable Banach space X. A random operator $T : \Omega\times C \to C$ is ϕ-weakly contractive-type operator if, there exists a continuous, non-decreasing function $\phi : R^+ \to R^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$, $\phi(0) = 0$ and for each $x, y \in C$, $w \in \Omega$, we have*

$$\int_\Omega \|T(w,x) - T(w,y)\| d\mu(w) \leq \int_\Omega \|x-y\| d\mu(w) - \phi(\int_\Omega \|x-y\| d\mu(w)). \tag{2.3}$$

Definition 2.7 ([27]). *A random operator $T : \Omega \times C \to C$ is generalized ϕ-weakly contractive-type if, there exists $L(w) \geq 0$ and a continuous, non-decreasing function $\phi : R^+ \to R^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$, $\phi(0) = 0$ and for each $x, y \in C$, $w \in \Omega$, we have*

$$\int_\Omega \|T(w,x) - T(w,y)\| d\mu(w) \leq e^{L(w)\|x-y\|} \left[\int_\Omega \|x-y\| d\mu(w) - \phi\left(\int_\Omega \|x-y\| d\mu(w)\right)\right]. \tag{2.4}$$

Keeping in mind the above definitions, we introduce the following contractive condition.

Definition 2.8. *A random operator $T : \Omega\times C \to C$ is generalized ϕ-weakly contractive type if there exists $L(w) \geq 0$ and a continuous and non-decreasing function $\phi : R^+ \to R^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$, $\phi(0) = 0$ and for each $x, y \in C$, $w \in \Omega$, we have*

$$\int_\Omega \|T(w,x) - T(w,y)\| d\mu(w) \leq e^{L(w)\|x-T(w,x)\|} \left[\int_\Omega \|x-y\| d\mu(w) - \phi\left(\int_\Omega \|x-y\| d\mu(w)\right)\right]. \tag{2.5}$$

Both the conditions (2.4) *and* (2.5) *are independent of each other. If $L(w) = 0$ for each $w \in \Omega$ in* (2.4) *and* (2.5)*, then both reduce to condition* (2.3).

Motivated by the fact that three-step iterative algorithm gives better approximation [11] than one-step and two-step iterative algorithms, we consider random three-step Noor and random three-step SP iterative algorithms associated with T.

Let $T : \Omega \times C \to C$, be a random operator, where C is a nonempty convex subset of X. Let $x_0 : \Omega \to C$, be an arbitrary measurable mapping, $\{u_n(w)\}$, $\{v_n(w)\}$, $\{w_n(w)\}$ be sequences of measurable mappings from $\Omega \to C$ and $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$. The random versions of various iterative algorithms of T are defined below:

Random Noor iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned} x_{n+1}(w) &= (1-\alpha_n)x_n(w) + \alpha_n T(w, y_n(w)) + u_n(w)\\ y_n(w) &= (1-\beta_n)x_n(w) + \beta_n T(w, z_n(w)) + v_n(w)\\ z_n(w) &= (1-\gamma_n)x_n(w) + \gamma_n T(w, x_n(w)) + w_n(w), \end{aligned} \tag{RN}$$

Random SP iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned} x_{n+1}(w) &= (1-\alpha_n)y_n(w) + \alpha_n T(w, y_n(w)) + u_n(w)\\ y_n(w) &= (1-\beta_n)z_n(w) + \beta_n T(w, z_n(w)) + v_n(w)\\ z_n(w) &= (1-\gamma_n)x_n(w) + \gamma_n T(w, x_n(w)) + w_n(w), \end{aligned} \tag{RSP}$$

Random Ishikawa iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned} x_{n+1}(w) &= (1-\alpha_n)\, x_n(w) + \alpha_n T\,(w, y_n(w)) + u_n(w)\\ y_n(w) &= (1-\beta_n)\, x_n(w) + \beta_n T\,(w, x_n(w)) + v_n(w) \end{aligned} \tag{RI}$$

Random S-iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned} x_{n+1}(w) &= T\,(w, y_n(w)) + u_n(w)\\ y_n(w) &= (1-\beta_n)\, x_n(w) + \beta_n T\,(w, x_n(w)) + v_n(w) \end{aligned} \tag{RS}$$

Random Mann iterative algorithm with errors $\{x_n(w)\}$:

$$x_{n+1}(w) = (1-\alpha_n)x_n(w) + \alpha_n T(w, x_n(w)) + u_n(w) \tag{RM}$$

Remark 2.9. *Putting $\beta_n = \gamma_n = v_n(w) = w_n(w) = 0$ and $\gamma_n = w_n(w) = 0$, for all $n \in N$, in (RN), respectively, we obtain (RM) and (RI). Also, putting $\beta_n = \gamma_n = v_n(w) = w_n(w) = 0$ for all $n \in N$, in (RSP), we obtain (RM). Similarly, putting $\alpha_n = 1$, $\gamma_n = w_n(w) = 0$ and $\alpha_n = 1$, $\gamma_n = w_n(w) = 0$ in (RN) and (RSP), we get (RS).*

Hence, (RN) and (RSP) iterative algorithms are more general than (RM) and (RS) iterative algorithms. However, (RSP) iterative algorithm is most useful among all these in view of its fastness and simplicity.

For $\alpha_{n,0} \neq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,0} \neq 0$, $\alpha_{n,i}$, $\beta_{n,j}$, $\gamma_{n,k} \in [0,1]$ and fixed integers r, s, t, most general random Kirk type iterative algorithms are defined below:

Random Kirk-Noor iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned}
x_{n+1}(w) &= \alpha_{n,0}x_n(w) + \sum_{i=1}^{r} \alpha_{n,i}T^i(w, y_n) + u_n(w), \quad \sum_{i=0}^{r} \alpha_{n,i} = 1, \\
y_n(w) &= \beta_{n,0}x_n(w) + \sum_{j=1}^{s} \beta_{n,j}T^j(w, z_n) + v_n(w), \quad \sum_{j=0}^{s} \beta_{n,j} = 1 \\
z_n(w) &= \sum_{k=0}^{t} \gamma_{n,k}T^k(w, x_n) + w_n(w), \quad \sum_{k=0}^{t} \gamma_{n,k} = 1
\end{aligned} \tag{RKN}$$

Random Kirk-SP iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned}
x_{n+1}(w) &= \sum_{i=0}^{r} \alpha_{n,i}T^i(w, y_n) + u_n(w), \quad \sum_{i=0}^{r} \alpha_{n,i} = 1 \\
y_n(w) &= \sum_{j=0}^{s} \beta_{n,j}T^j(w, z_n) + v_n(w), \quad \sum_{j=0}^{s} \beta_{n,j} = 1 \\
z_n(w) &= \sum_{k=0}^{t} \gamma_{n,k}T^k(w, x_n) + w_n(w), \quad \sum_{k=0}^{t} \gamma_{n,k} = 1,
\end{aligned} \tag{RKSP}$$

Random Kirk-Ishikawa iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned}
x_{n+1}(w) &= \alpha_{n,0}x_n(w) + \sum_{i=1}^{r} \alpha_{n,i}T^i(w, y_n) + u_n(w), \quad \sum_{i=0}^{r} \alpha_{n,i} = 1 \\
y_n(w) &= \beta_{n,0}x_n(w) + \sum_{j=1}^{s} \beta_{n,j}T^j(w, x_n) + v_n(w), \quad \sum_{j=0}^{s} \beta_{n,j} = 1
\end{aligned} \tag{RKI}$$

Random Kirk-S iterative algorithm with errors $\{x_n(w)\}$:

$$\begin{aligned}
x_{n+1}(w) &= \sum_{i=1}^{r} \alpha_{n,i}T^i(w, y_n) + u_n(w), \quad \sum_{i=1}^{r} \alpha_{n,i} = 1 \\
y_n(w) &= \beta_{n,0}x_n(w) + \sum_{j=1}^{s} \beta_{n,j}T^j(w, x_n) + v_n(w), \quad \sum_{j=0}^{s} \beta_{n,j} = 1
\end{aligned} \tag{RKS}$$

Remark 2.10. *Put $r = s = t = 1$ in (RKN) and (RKSP) iterative algorithms and get (RN) and (RSP) iterative algorithms, respectively, with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1} = \beta_n$, $\gamma_{n,1} = \gamma_n$.*

Define a random iterative algorithm with the help of the functions $x_n(w)$ as follows:

$$x_{n+1}(w) = f(T; x_n(w)), \quad n = 0, 1, 2, 3, \ldots, \tag{2.6}$$

where f is some function measurable in the second variable.

Definition 2.11 ([34]). *Let $\xi^*(w)$ be a random fixed point of a random operator T and Bochner integrable with respect to the sequence $\{x_n(w)\}$. Let $\{y_n(w)\}$ be an arbitrary sequence of random variables. Set*

$$\varepsilon_n(w) = \|y_{n+1}(w) - f(T; y_n(w))\| \tag{2.7}$$

and assume that $\|\varepsilon_n(w)\| \in L^1(\Omega(\xi, \mu))$, $n = 0, 1, 2, 3, \ldots$. The iterative algorithm (2.7) *is almost surely T-stable if and only if $\lim_{n\to\infty} \int_\Omega \|\varepsilon_n(w)\| d\mu(w) = 0$ implies that $\xi^*(w)$ is Bochner integrable with respect to $\{y_n(w)\}$.*

Lemma 2.12 ([5, 27]). *Let $\{\delta_n\}$ and $\{\lambda_n\}$ be two sequences of non-negative real numbers, $\{\sigma_n\}$ be a sequence of positive numbers satisfying the conditions:*

$$\sum_{n=1}^{\infty} \sigma_n = \infty \quad and \quad \lim_{n\to\infty} \frac{\delta_n}{\sigma_n} = 0.$$

If $\lambda_{n+1} \le \lambda_n - \sigma_n \phi(\lambda_n) + \delta_n$ holds for each $n \ge 1$, where $\phi : R^+ \to R^+$ is a continuous and strictly increasing function with $\phi(0) = 0$, then $\{\lambda_n\}$ converges to 0 as $n \to \infty$.

Lemma 2.13 ([5]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying $a_{n+1} \le a_n + b_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

3. Random Noor Type Iterative Algorithms with Errors

We begin with a technical result.

Lemma 3.1. *Let C be a nonempty subset of a separable Banach space X and $T : \Omega \times C \to C$ be a random operator satisfying the condition* (2.5)*. Then, $\forall\ i \in N$ and $\forall\ x, y \in C$, we have*

$$\int_\Omega \|T^i(w, x) - T^i(w, y)\| d\mu(w) \le e^{L(w) \sum_{j=1}^{i} \|T^{j-1}(w,x) - T^j(w,x)\|} \left[\int_\Omega \|x - y\| d\mu(w) - \phi\left(\int_\Omega \|x - y\| d\mu(w) \right) \right]. \tag{3.1}$$

Proof. It is based on mathematical induction on i.

If $i = 1$, then (3.1) becomes

$$\int_\Omega \|T(w, x) - T(w, y)\| d\mu(w) \le e^{L(w)\|x - T(w,x)\|} \left[\int_\Omega \|x - y\| d\mu(w) - \phi\left(\int_\Omega \|x - y\| d\mu(w) \right) \right].$$

i.e., (3.1) reduces to (2.5) and the result holds.

Assume that (3.1) holds for $i = q$, $q \in N$, that is,

$$\int_\Omega \|T^q(w, x) - T^q(w, y)\| d\mu(w) \le e^{L(w) \sum_{j=1}^{q} \|T^{j-1}(w,x) - T^j(w,x)\|} \left[\int_\Omega \|x - y\| d\mu(w) - \phi\left(\int_\Omega \|x - y\| d\mu(w) \right) \right].$$

The statement is true for $i = q + 1$ as follows:

$$\begin{aligned} \|T^{q+1}x - T^{q+1}y\| &= \|T(T^q x) - T(T^q y)\| \\ &\le e^{L(w)\|T^q(w,x) - T^{q+1}(w,x)\|} \left[\int_\Omega \|T^q(w, x) - T^q(w, y)\| d\mu(w) - \phi\left(\int_\Omega \|T^q(w, x) - T^q(w, y)\| d\mu(w) \right) \right] \\ &\le e^{L(w)\|T^q(w,x) - T^{q+1}(w,x)\|} \int_\Omega \|T^q(w, x) - T^q(w, y)\| d\mu(w). \end{aligned}$$

□

Remark 3.2. *If $y = p(w)$ (random fixed point of T), then* (3.1) *becomes*

$$\int_\Omega \|T^i(w,x) - T^i(w,p)\| d\mu(w) \leq \left[\int_\Omega \|x-p\| d\mu(w) - \phi\left(\int_\Omega \|x-p\| d\mu(w)\right)\right].$$

Theorem 3.3. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition* (2.5) *with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RKN) admitting the following restrictions:*

(i) $\Sigma(1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0}) = \infty$

(ii) $\alpha_{n,0} < \alpha$, $\beta_{n,0} < \beta$, $\gamma_{n,0} < \gamma$

(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$.

Then the random fixed point $p(w)$ of T is Bochner integrable.

Proof. To show that $p(w)$ is Bochner integrable, we shall prove that

$$\lim_{n\to\infty} \int \|x_n(w) - p(w)\| d\mu(w) = 0$$

Using iterative algorithm (RKN) and Remark 3.2, we have

$$\begin{aligned}
&\int_\Omega \|x_{n+1}(w) - p(w)\| d\mu(w) \\
&\leq \alpha_{n,0} \int_\Omega \|(x_n(w) - p(w))\| d\mu(w) + \sum_{i=1}^{r} \alpha_{n,i} \int_\Omega \|T^i(w,y_n) - T^i(w,p)\| + \int_\Omega \|u_n(w)\| d\mu(w) \\
&\leq \alpha_{n,0} \int_\Omega \|(x_n(w) - p(w))\| d\mu(w) + \sum_{i=1}^{r} \alpha_{n,i} \left[\int_\Omega \|y_n - p\| d\mu(w) - \phi\left(\int_\Omega \|y_n - p\| d\mu(w)\right)\right] + \int_\Omega \|u_n(w)\| d\mu(w) \\
&\leq \alpha_{n,0} \int_\Omega \|(x_n(w) - p(w))\| \, d\mu(w) + \sum_{i=1}^{r} \alpha_{n,i} \left[\int_\Omega \|y_n - p\| d\mu(w)\right] + \int_\Omega \|u_n(w)\| d\mu(w) \qquad (3.2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega \|y_n(w) - p(w)\| \, d\mu(w) \\
&\quad \leq \beta_{n,0} \int_\Omega \|(x_n(w) - p(w))\| \, d\mu(w) + \sum_{j=1}^{s} \beta_{n,j} \left[\int_\Omega \|y_n - p\| d\mu(w)\right] + \int_\Omega \|v_n(w)\| d\mu(w) \qquad (3.3)
\end{aligned}$$

Also, using $\sum_{k=1}^{t} \gamma_{n,k} = 1 - \gamma_{n,0}$, we have

$$\begin{aligned}
\int_\Omega \|z_n(w) - p(w)\| \, d\mu(w) &\leq \gamma_{n,0} \int_\Omega \|x_n(w) - p(w)\| \, d\mu(w) \\
&\quad + \sum_{k=1}^{t} \gamma_{n,k} \left[\int_\Omega \|x_n - p\| d\mu(w) - \phi\left(\int_\Omega \|x_n - p\| d\mu(w)\right)\right] + \int_\Omega \|w_n(w)\| d\mu(w) \\
&\leq \int_\Omega \|x_n(w) - p(w)\| \, d\mu(w) - (1-\gamma_{n,0})\phi\left(\int_\Omega \|x_n - p\| d\mu(w)\right) + \int_\Omega \|w_n(w)\| d\mu(w) \\
&\qquad (3.4)
\end{aligned}$$

Putting (3.3) and (3.4) in (3.2), we get

$$\begin{aligned}
&\int_\Omega \|x_{n+1}(w) - p(w)\| \, d\mu(w) \\
&\leq \int_\Omega \| (x_n(w) - p(w)) \| \, d\mu(w) - (1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0})\phi\left(\int_\Omega \|x_n(w) - p(w)\| \, d\mu(w)\right) \\
&\quad + \int_\Omega \|w_n(w)\| d\mu(w) + \int_\Omega \|v_n(w)\| d\mu(w) + \int_\Omega \|u_n(w)\| d\mu(w) \qquad (3.5)
\end{aligned}$$

Using conditions (ii)-(iii), we obtain:

$$\lim_{n\to\infty} \frac{\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] \, d\mu(w)}{(1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0})} \leq \lim_{n\to\infty} \frac{\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] \, d\mu(w)}{(1-\alpha)(1-\beta)(1-\gamma)} = 0.$$

Now putting $\int_\Omega \|x_n(w) - p(w)\| \, d\mu(w) = \lambda_n$, $(1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0}) = \sigma_n$ and $\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] \, d\mu(w) = \delta_n$ in (3.5) and using Lemma 2.12, we get $\lim_{n\to\infty} \int_\Omega \|x_n(w) - p(w)\| \, d\mu(w) = 0$. □

Theorem 3.4. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition* (2.5) *with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RKN) admitting the following restrictions:*

(i) $\Sigma(1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0}) = \infty$

(ii) $\alpha_{n,0} < \alpha$, $\beta_{n,0} < \beta$, $\gamma_{n,0} < \gamma$

(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$.

Then, $\{x_n\}$ is almost surely T-stable.

Proof. Let $\{p_n(w)\}$ be any sequence of random variable

$$\|\varepsilon_n(w)\| = \left\| p_{n+1}(w) - \alpha_{n,0} p_n(w) + \sum_{i=1}^{r} \alpha_{n,i} T^i(w, q_n) + u_n(w) \right\|, \quad n = 0, 1, 2, \ldots, \qquad (3.6)$$

where

$$\begin{aligned}
&q_n(w) = \beta_{n,0} p_n(w) + \sum_{j=1}^{s} \beta_{n,j} T^j(w, r_n) + v_n(w), \\
&r_n(w) = \sum_{k=0}^{t} \gamma_{n,k} T^k(w, p_n) + w_n(w) \quad \text{and} \\
&\lim_{n\to\infty} \int_\Omega \|\varepsilon_n(w)\| \, d\mu(w) = 0.
\end{aligned}$$

Now we prove that $p(w)$ is Bochner integrable with respect to the sequence $\{p_n(w)\}$.

Using (3.6) and Remark 3.2, we have

$$\begin{aligned}
\int_\Omega \|p_{n+1}(w) - p(w)\| d\mu(w) &\leq \int_\Omega \|p_{n+1}(w) - \alpha_{n,0} p_n(w) + \sum_{i=1}^{r} \alpha_{n,i} T^i(w, q_n) + u_n(w)\| d\mu(w) \\
&\quad + \alpha_{n,0} \int_\Omega \|(p_n(w) - p(w))\| \, d\mu(w) \\
&\quad + \sum_{i=1}^{r} \alpha_{n,i} \int_\Omega \|T^i(w, q_n) - T^i(w, p)\| + \int_\Omega \|u_n(w)\| d\mu(w)
\end{aligned}$$

$$\leq \int_\Omega \|\varepsilon_n(w)\| d\mu(w) + \alpha_{n,0} \int_\Omega \|(p_n(w) - p(w))\| \, d\mu(w)$$
$$+ \sum_{i=1}^{r} \alpha_{n,i} \left[\int_\Omega \|q_n - p\| d\mu(w) - \phi\left(\int_\Omega \|y_n - p\| d\mu(w) \right) \right] + \int_\Omega \|u_n(w)\| d\mu(w)$$
$$\leq \int_\Omega \|\varepsilon_n(w)\| d\mu(w) + \alpha_{n,0} \int_\Omega \|(p_n(w) - p(w))\| \, d\mu(w)$$
$$+ \sum_{i=1}^{r} \alpha_{n,i} \left[\int_\Omega \|q_n - p\| d\mu(w) \right] + \int_\Omega \|u_n(w)\| d\mu(w) \tag{3.7}$$

Also

$$\int_\Omega \|q_n(w) - p(w)\| \, d\mu(w) \leq \beta_{n,0} \int_\Omega \|(p_n(w) - p(w))\| \, d\mu(w) + \sum_{j=1}^{s} \beta_{n,j} \left[\int_\Omega \|r_n - p\| d\mu(w) \right] + \int_\Omega \|v_n(w)\| d\mu(w)$$
$$\int_\Omega \|q_n(w) - p(w)\| \, d\mu(w) \leq \beta_{n,0} \int_\Omega \|(p_n(w) - p(w))\| \, d\mu(w) + (1 - \beta_{n,0}) \left[\int_\Omega \|r_n - p\| d\mu(w) \right] + \int_\Omega \|v_n(w)\| d\mu(w) \tag{3.8}$$

and

$$\int_\Omega \|r_n(w) - p(w)\| \, d\mu(w) \leq \int_\Omega \|p_n(w) - p(w)\| \, d\mu(w) - (1 - \gamma_{n,0})\phi\left(\int_\Omega \|p_n - p\| d\mu(w) \right) + \int_\Omega \|w_n(w)\| d\mu(w) \tag{3.9}$$

Now estimates (3.7)-(3.9) yield:

$$\int_\Omega \|p_{n+1}(w) - p(w)\| d\mu(w)$$
$$\leq \int_\Omega \|p_n(w) - p(w)\| d\mu(w) - (1 - \alpha_{n,0})(1 - \beta_{n,0})(1 - \gamma_{n,0})\phi\left(\int_\Omega \|p_n(w) - p(w)\| \, d\mu(w) \right)$$
$$+ \int_\Omega \|w_n(w)\| d\mu(w) + \int_\Omega \|v_n(w)\| d\mu(w) + \int_\Omega \|u_n(w)\| d\mu(w) + \int_\Omega \|\varepsilon_n(w)\| d\mu(w) \tag{3.10}$$

Using $\lim_{n\to\infty} \int_\Omega \|\varepsilon_n(w)\| \, d\mu(w) = 0$ and conditions (ii)-(iii), we have

$$\lim_{n\to\infty} \frac{\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] d\mu(w) + \int_\Omega \|\varepsilon_n(w)\| \, d\mu(w)}{(1 - \alpha_{n,0})(1 - \beta_{n,0})(1 - \gamma_{n,0})}$$
$$\leq \lim_{n\to\infty} \frac{\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] d\mu(w) + \int_\Omega \|\varepsilon_n(w)\| \, d\mu(w)}{(1 - \alpha)(1 - \beta)(1 - \gamma)} = 0$$

Now taking $\lambda_n = \int_\Omega \|p_n(w) - p(w)\| \, d\mu(w)$, $\sigma_n = (1 - \alpha_{n,0})(1 - \beta_{n,0})(1 - \gamma_{n,0})$ and $\delta_n = \int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] d\mu(w) + \int_\Omega \|\varepsilon_n(w)\| \, d\mu(w)$ in (3.10) and using Lemma 2.12, we get $\lim_{n\to\infty} \int_\Omega \|p_n(w) - p(w)\| \, d\mu(w) = 0$.

Conversely, let $p(w)$ be Bochner integrable with respect to the sequence $\{p_n(w)\}$. Then we have

$$\int_\Omega \|\varepsilon_n(w)\| d\mu(w) = \int_\Omega \|p_{n+1}(w) - \alpha_{n,0} p_n(w) + \sum_{i=1}^{r} \alpha_{n,i} T^i(w, q_n) + u_n(w)\| d\mu(w)$$
$$\leq \int_\Omega \|(p_{n+1}(w) - p(w))\| \, d\mu(w) + \alpha_{n,0} \int_\Omega \|(p_n(w) - p(w))\| \, d\mu(w)$$

$$+\sum_{i=1}^{r}\alpha_{n,i}\int_{\Omega}\|T^i(w,q_n)-T^i(w,p)\|+\int_{\Omega}\|u_n(w)\|d\mu(w) \tag{3.11}$$

The estimates (3.8), (3.9) and (3.11) yield:

$$\begin{aligned}&\int_{\Omega}\|\varepsilon_n(w)\|d\mu(w)\\ &\leq \int_{\Omega}\|p_{n+1}(w)-p(w)\|\,d\mu(w)-(1-\alpha_{n,0})(1-\beta_{n,0}))(1-\gamma_{n,0})(1-\gamma_{n,0})\phi\left(\int_{\Omega}\|p_n(w)-p(w)\|\,d\mu(w)\right)\\ &\quad+\int_{\Omega}\|w_n(w)\|d\mu(w)+\int_{\Omega}\|v_n(w)\|d\mu(w)+\int_{\Omega}\|u_n(w)\|d\mu(w)\end{aligned} \tag{3.12}$$

Using condition (iii) and Bochner integrability of $p(w)$, (3.12) yields $\lim_{n\to\infty}\int_{\Omega}\|\varepsilon_n(w)\|\,d\mu(w)=0$. This shows that $\{x_n(w)\}$ is almost surely T-stable. □

Example 3.5. *Let $\Omega=[0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X=R$, $C=[0,2]$ and define random operator $T:\Omega\times C\to C$ as $T(w,x)=\frac{w-x}{2}$. Then the measurable mapping $p:\Omega\to X$ defined by $p(w)=\frac{w}{3}$, for every $w\in\Omega$, serves as a random fixed point of T. Also, for $\phi(t)=\frac{t}{2}$ and $L(w)=3$, we have*

$$\begin{aligned}\int_{\Omega}\|T(w,x)-T(w,y)\|d\mu(w)&=\left[\int_{\Omega}\|x-y\|d\mu(w)-\phi\left(\int_{\Omega}\|x-y\|d\mu(w)\right)\right]\\ &\leq e^{3\|x-T(w,x)\|}\left[\int_{\Omega}\|x-y\|d\mu(w)-\phi\left(\int_{\Omega}\|x-y\|d\mu(w)\right)\right],\end{aligned}$$

Hence T satisfies the condition (2.5). Taking parameters $\alpha_{n,0}=1-\frac{n^2}{1+n^2}$, $\beta_n=1-\frac{n^3}{1+n^3}$, $\gamma_n=1-\frac{n^3}{1+n^4}$ and choosing error terms $u_n(w)=\frac{w}{(n+1)^2}$, $v_n(w)=\frac{w}{(n+1)^2}$, $w_n(w)=\frac{w}{(n+1)^3}$ we have

$$0<\alpha_{n,0},\beta_{n,0},\gamma_{n,0}\leq 0.5,\quad \Sigma(1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0})=\Sigma\frac{n^8}{(1+n^4)(1+n^3)(1+n^2)}=\infty$$

and $\lim_{n\to\infty}u_n(w)=0$, $\lim_{n\to\infty}v_n(w)=0$, $\lim_{n\to\infty}w_n(w)=0$. So, all the conditions of Theorem 3.3 and Theorem 3.4 are satisfied and hence the random fixed point $p(w)$ of $T(w,x)$ is Bochner integrable and (RKN) is almost surely T-stable.

Special cases of Theorems 3.3 and 3.4 provide the following series of new important results for random operators.

Theorem 3.6. *Let C be a nonempty closed and convex subset of X and $T:\Omega\times C\to C$ a random operator satisfying the condition (2.5) with $RF(T)\neq\phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RKI) admitting the following restrictions:*

(i) $\Sigma(1-\alpha_{n,0})(1-\beta_{n,0})=\infty$
(ii) $\alpha_{n,0}<\alpha$, $\beta_{n,0}<\beta$,
(iii) $\lim_{n\to\infty}u_n(w)=0$, $\lim_{n\to\infty}v_n(w)=0$.

Then $p(w)$ is Bochner integrable and (RKI) is almost surely T-stable.

Proof. Put $t=0$, $w_n(w)=0$ in the proofs of Theorems 3.3 and 3.4. □

Theorem 3.7. *Let C be a nonempty closed and convex subset of X and $T:\Omega\times C\to C$ a random operator satisfying the condition (2.5) with $RF(T)\neq\phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RKS) admitting the following restrictions:*

(i) $\Sigma(1-\beta_{n,0})=\infty$

(ii) $\beta_{n,0} < \beta$,
(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$.

Then $p(w)$ is Bochner integrable and (RKS) is almost surely T-stable.

Proof. Set $t = 0$, $\alpha_{n,0} = 0$, $w_n(w) = 0$ in the proofs of Theorems 3.3 and 3.4. □

Theorem 3.8. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition (2.5) with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RN) admitting the following restrictions:*

(i) $\Sigma\alpha_n\beta_n\gamma_n = \infty$
(ii) $0 < \alpha \leq \alpha_n$, $0 < \beta \leq \beta_n$ *and* $0 < \gamma \leq \gamma_n (n \geq 1)$
(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$.

Then $p(w)$ is Bochner integrable and (RN) is almost surely T-stable.

Proof. Put $r = s = t = 1$, $\alpha_{n,1} = \alpha_n$, $\alpha_{n,0} = 1 - \alpha_n$, $\beta_{n,1} = \beta_n$, $\beta_{n,0} = 1 - \beta_n$, $\gamma_{n,1} = \gamma_{n,0} = 1 - \gamma_n$ in the proofs of Theorems 3.3 and 3.4. □

Theorem 3.9. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition (2.5) with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RS) admitting the following restrictions:*

(i) $\Sigma\beta_n = \infty$
(ii) $0 < \beta \leq \beta_n (n \geq 1)$
(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$.

Then $p(w)$ is Bochner integrable and (RS) is almost surely T-stable.

Proof. Put $r = s = 1$, $t = 0$, $\alpha_{n,0} = 0$, $\beta_{n,1} = \beta_n$, $\beta_{n,0} = 1 - \beta_n$ in the proofs of Theorems 3.3 and 3.4. □

Theorem 3.10 ([34]). *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition (2.5) with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RI) admitting the following restrictions:*

(i) $\Sigma\alpha_n\beta_n = \infty$
(ii) $0 < \alpha \leq \alpha_n$, $0 < \beta \leq \beta_n (n \geq 1)$

Then $p(w)$ is Bochner integrable and (RI) is almost surely T-stable.

Proof. Put $r = s = 1$, $t = 0$, $\alpha_{n,1} = \alpha_n$, $\alpha_{n,0} = 1 - \alpha_n$, $\beta_{n,1} = \beta_n$, $\beta_{n,0} = 1 - \beta_n$, $u_n(w) = 0$, $v_n(w) = 0$, $w_n(w) = 0$, $L = 0$ in the proofs of Theorems 3.3 and 3.4. □

We now extend Theorem 3.3 for three generalized ϕ-weakly contractive random operators as follows:

Theorem 3.11. *Let C be a nonempty closed and convex subset of X and let $T_i : \Omega \times C \to C$, $i = 1, 2, 3$ be three random operators satisfying the condition (2.5) with $CRF = \bigcap_{i=1}^{3} RF(T_i) \neq \phi$. Let $p(w)$ be a common random fixed point of the random operators $\{T_i, i = 1, 2, 3\}$ and $\{x_n(w)\}$ be the random Kirk-Noor algorithm of three operators with errors defined as follows:*

$$\begin{aligned} x_{n+1}(w) &= \alpha_{n,0}x_n(w) + \sum_{i=1}^{r} \alpha_{n,i}T_1^i(w, y_n) + u_n(w), \quad \sum_{i=0}^{r} \alpha_{n,i} = 1, \\ y_n(w) &= \beta_{n,0}x_n(w) + \sum_{j=1}^{s} \beta_{n,j}T_2^j(w, z_n) + v_n(w), \quad \sum_{j=0}^{s} \beta_{n,j} = 1 \\ z_n(w) &= \sum_{k=0}^{t} \gamma_{n,k}T_3^k(w, x_n) + w_n(w), \quad \sum_{k=0}^{t} \gamma_{n,k} = 1 \end{aligned} \tag{RKNTO}$$

where $\{u_n(w)\}$, $\{v_n(w)\}$, $\{w_n(w)\}$ are sequences of measurable mappings from Ω to C with $\Sigma \int_\Omega u_n(w)\,d\mu(w) < \infty$, $\Sigma \int_\Omega v_n(w)\,d\mu(w) < \infty$, $\Sigma \int_\Omega w_n(w)\,d\mu(w) < \infty$ and $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$. Then common random fixed point of the random operators $\{T_i,\ i = 1, 2, 3\}$ is Bochner integrable if and only if for all $w \in \Omega$, $\lim_{n\to\infty} \inf \int_\Omega d(x_n(w), CRF)d\mu(w) = 0$, where $d(x_n(w), CRF) = \inf\{\|x_n(w) - \xi(w)\| : \xi \in CRF\}$, provided $\int_\Omega \|T_i(w, \xi(w)) - \xi(w)\|d\mu(w) = 0$ implies $\|T_i(w, \xi(w)) - \xi(w)\| = 0$.

Proof. The necessity is obvious and hence omitted. Now to prove the sufficiency part, we show that $\lim_{n\to\infty} \int \|x_n(w) - p(w)\|\,d\mu(w) = 0$, where $p(w) \in CRF$.

Following the same steps as in the proof of Theorem 3.3, we have the following estimate:

$$\begin{aligned}
&\int_\Omega \|x_{n+1}(w) - p(w)\|\,d\mu(w)\\
&\leq \int_\Omega \|(x_n(w) - p(w))\|\,d\mu(w) - (1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0})\phi\left(\int_\Omega \|x_n(w) - p(w)\|\,d\mu(w)\right)\\
&\quad + \int_\Omega \|w_n(w)\|d\mu(w) + \int_\Omega \|v_n(w)\|d\mu(w) + \int_\Omega \|u_n(w)\|d\mu(w)\\
&\leq \int_\Omega \|(x_n(w) - p(w))\|\,d\mu(w) + \int_\Omega \|w_n(w)\|d\mu(w) + \int_\Omega \|v_n(w)\|d\mu(w) + \int_\Omega \|u_n(w)\|d\mu(w)
\end{aligned} \tag{3.13}$$

It follows from (3.13), in view of $d(x_n(w), CRF) = \inf\{\|x_n(w) - \xi(w)\| : \xi \in CRF\}$:

$$\int_\Omega d(x_{n+1}(w), CRF)d\mu(w) \leq \int_\Omega d(x_n(w), CRF)d\mu(w) + b_n(w), \tag{3.14}$$

where $b_n(w) = (\int_\Omega \|w_n(w)\| + \int_\Omega \|v_n(w)\| + \int_\Omega \|u_n(w)\|)d\mu(w)$.

Clearly, $\sum_{n=0}^{\infty} b_n < \infty$. So by Lemma 2.13, $\lim_{n\to\infty} \int_\Omega d(x_n(w), RF)d\mu(w)$ exists.

Therefore, using the given condition in the theorem, we have for all $w \in \Omega$,

$$\lim_{n\to\infty} \int_\Omega d(x_n(w), RF)d\mu(w) = 0.$$

Now, if $a_n = \int_\Omega \|x_n(w) - p(w)\|\,d\mu(w)$ in (3.13), then it follows that for any natural number m and for all $n \geq m$,

$$\begin{aligned}
\|a_{n+m}(w)\| &\leq \|a_{n+m-1}(w)\| + b_{n+m-1}(w)\\
&\leq \|a_{n+m-2}(w)\| + b_{n+m-2}(w) + b_{n+m-1}(w)\\
&\leq \cdots \leq \|a_n(w)\| + \sum_{k=n}^{n+m-1} b_k(w).
\end{aligned} \tag{3.15}$$

Therefore, we have

$$\begin{aligned}
\|a_{n+m}(w) - a_n(w)\| &\leq \|a_n(w)\| + \sum_{k=n}^{n+m-1} b_k(w) + \|a_n(w)\|\\
&= 2\|a_n(w)\| + \sum_{k=n}^{n+m-1} b_k(w)
\end{aligned} \tag{3.16}$$

As $\sum_{n=0}^{\infty} b_n < \infty$ and $\lim_{n\to\infty} \int_\Omega d(x_n(w), CRF)d\mu(w) = 0$, so there exists $m_1 \in N$ such that for all $n \geq m_1$, we have $\int_\Omega d(x_n(w), CRF)d\mu(w) < \frac{\varepsilon}{4}$ and $\sum_{k=n}^{\infty} b_k(w) < \frac{\varepsilon}{2}$.

Hence there exists $q \in CRF$ such that

$$\int_{\Omega} \|x_n(w) - q(w)\| d\mu(w) < \frac{\varepsilon}{4} \quad \text{for all } n \geq m_1.$$

So from (3.16), we have that for all $w \in \Omega$, for all $n \geq m_1$ and for any positive integer m,

$$\|a_{n+m}(w) - a_n(w)\| \leq 2\|a_n(w)\| + \sum_{k=n}^{n+m-1} b_k(w) < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,$$

or

$$\int_{\Omega} \|x_{n+m}(w) - x_n(w)\| d\mu(w) < \varepsilon,$$

from which it follows that $\left\{\int_{\Omega} \|x_n(w)\| d\mu(w)\right\}$ is a Cauchy sequence for each $w \in \Omega$. So, $\int_{\Omega} \|x_n(w)\| d\mu(w) \to \int_{\Omega} \|\xi(w)\| d\mu(w)$ as $n \to \infty$ for each $w \in \Omega$, where $\int_{\Omega} \|\xi(w)\| d\mu(w) : \Omega \to X$, being the limit of the sequence of measurable functions is also measurable.

Now we prove that $\xi(w) \in CRF$. As for each $w \in \Omega$, $\int_{\Omega} \|x_n(w)\| d\mu(w) \to \int_{\Omega} \|\xi(w)\| d\mu(w)$, when $n \to \infty$, so there exists $m_2 \in N$ such that

$$\int_{\Omega} \|x_n(w) - \xi(w)\| d\mu(w) < \frac{\varepsilon}{4} \quad \text{for all } n \geq m_2.$$

Let $m_3 = \max\{m_1, m_2\}$. Then for all $w \in \Omega$ and $n \geq m_3$, we have

$$\begin{aligned}
&\int_{\Omega} \|T_1(w, \xi(w)) - \xi(w)\| d\mu(w) \\
&\quad \leq \int_{\Omega} \|T_1(w, \xi(w)) - \xi^*(w)\| d\mu(w) + \int_{\Omega} \|\xi^*(w) - \xi(w)\| d\mu(w) \\
&\quad \leq \int_{\Omega} \|\xi^*(w) - \xi(w)\| d\mu(w) - \varphi(\int_{\Omega} \|\xi^*(w) - \xi(w)\| d\mu(w)) + \int_{\Omega} \|\xi^*(w) - \xi(w)\| d\mu(w) \\
&\quad \leq 2\int_{\Omega} \|\xi^*(w) - \xi(w)\| d\mu(w) \\
&\quad \leq 2\int_{\Omega} \|\xi^*(w) - x_n(w)\| d\mu(w) + 2\int_{\Omega} \|\xi^*(w) - x_n(w)\| d\mu(w) \\
&\quad < 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon
\end{aligned}$$

which yields $T_1(w, \xi(w)) = \xi(w)$ for each $w \in \Omega$. As ξ is measurable, so $\xi \in RF(T_1)$. In the same way, we can show that $\xi \in RF(T_2)$ and $\xi \in RF(T_3)$. Hence we have $\xi \in CRF$. Thus common random fixed point of T_1, T_2, T_3 is Bochner integrable.

4. Random SP Type Iterative Algorithm With Errors

Theorem 4.1. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition* (2.5) *with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RKSP) admitting the following restrictions:*

(i) $\Sigma(1 - \gamma_{n,0}) = \infty$ *or* $\Sigma(1 - \beta_{n,0}) = \infty$ *or* $\Sigma(1 - \alpha_{n,0}) = \infty$

(ii) $\gamma_{n,0} < \gamma$ *or* $\beta_{n,0} < \beta$ *or* $\alpha_{n,0} < \alpha$

(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0.$

Then $p(w)$ of T is Bochner integrable and (RKSP) is almost surely T-stable.

Proof. Using iterative algorithm (RKSP) and following the corresponding steps in the proof of Theorem 3.3, we have

$$\begin{aligned}
&\int_\Omega \|x_{n+1}(w) - p(w)\| d\mu(w) \\
&\leq \alpha_{n,0} \int_\Omega \|(y_n(w) - p(w))\| d\mu(w) + \sum_{i=1}^{r} \alpha_{n,i} \left[\int_\Omega \|y_n - p\| d\mu(w)\right] + \int_\Omega \|u_n(w)\| d\mu(w) \\
&\leq \int_\Omega \|(y_n(w) - p(w))\| \, d\mu(w) + \int_\Omega \|u_n(w)\| d\mu(w) \qquad (4.1)
\end{aligned}$$

Similarly,

$$\int_\Omega \|y_n(w) - p(w)\| \, d\mu(w) \leq \int_\Omega \|(z_n(w) - p(w))\| d\mu(w) + \int_\Omega \|v_n(w)\| d\mu(w) \qquad (4.2)$$

Also,

$$\begin{aligned}
&\int_\Omega \|z_n(w) - p(w)\| \, d\mu(w) \\
&\leq \int_\Omega \|x_n(w) - p(w)\| \, d\mu(w) - (1 - \gamma_{n,0})\phi\left(\int_\Omega \|x_n - p\| d\mu(w)\right) + \int_\Omega \|w_n(w)\| d\mu(w) \qquad (4.3)
\end{aligned}$$

Using estimates (4.1)-(4.3), we arrive at

$$\begin{aligned}
&\int_\Omega \|x_{n+1}(w) - p(w)\| \, d\mu(w) \\
&\leq \int_\Omega \|(x_n(w) - p(w))\| \, d\mu(w) - (1 - \gamma_{n,0})\phi\left(\int_\Omega \|x_n(w) - p(w)\| \, d\mu(w)\right) \\
&\quad + \int_\Omega \|w_n(w)\| d\mu(w) + \int_\Omega \|v_n(w)\| d\mu(w) + \int_\Omega \|u_n(w)\| d\mu(w) \qquad (4.4)
\end{aligned}$$

Using conditions (ii)-(iii), we have

$$\lim_{n\to\infty} \frac{\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] d\mu(w)}{(1 - \gamma_{n,0})} \leq \lim_{n\to\infty} \frac{\int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] d\mu(w)}{(1 - \gamma)} = 0.$$

Now, if $\lambda_n = \int_\Omega \|x_n(w) - p(w)\| \, d\mu(w)$, $\sigma_n = 1 - \gamma_{n,0}$ and

$$\delta_n = \int_\Omega [\|w_n(w)\| + \|u_n(w)\| + \|v_n(w)\|] d\mu(w)$$

in (4.4), then using Lemma 2.12, we get $\lim_{n\to\infty} \int_\Omega \|x_n(w) - p(w)\| \, d\mu(w) = 0$.
The almost sure T-stability of (RKSP) can be proved as in the proof of Theorem 3.4. □

Remark 4.2. *As $(1 - \alpha_{n,0})(1 - \beta_{n,0})(1 - \gamma_{n,0}) \leq 1 - \gamma_{n,0}$ implies $\Sigma(1 - \alpha_{n,0})(1 - \beta_{n,0})(1 - \gamma_{n,0}) \leq \Sigma(1 - \gamma_{n,0})$, so $\Sigma(1-\alpha_{n,0})(1-\beta_{n,0})(1-\gamma_{n,0}) = \infty$ implies $\Sigma(1-\gamma_{n,0}) = \infty$; hence we conclude that random SP iterative algorithm with errors requires weaker restriction ($\Sigma(1 - \gamma_{n,0}) = \infty$) on parameters as compared to random Noor iterative algorithm with errors which requires $\Sigma(1 - \alpha_{n,0})(1 - \beta_{n,0})(1 - \gamma_{n,0}) = \infty$, as far as Bochner integrability of fixed point $p(w)$ is concerned.*

Special cases of Theorem 4.1 provide the following new important random fixed points results.

Theorem 4.3. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition* (2.5) *with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RSP) admitting the following restrictions:*

(i) $\Sigma\gamma_n = \infty$
(ii) $0 < \gamma \leq \gamma_n (n \geq 1)$
(iii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$.

Then $p(w)$ is Bochner integrable and (RSP) is almost surely T-stable.

Proof. Put $r = s = t = 1$, $\alpha_{n,1} = \alpha_n$, $\alpha_{n,0} = 1 - \alpha_n$, $\beta_{n,1} = \beta_n$, $\beta_{n,0} = 1 - \beta_n$, $\gamma_{n,1} = \gamma_{n,0} = 1 - \gamma_n$ in the proof of Theorem 4.1. □

Theorem 4.4. *Let C be a nonempty closed and convex subset of X and $T : \Omega \times C \to C$ a random operator satisfying the condition* (2.5) *with $RF(T) \neq \phi$. Let $p(w)$ be a random fixed point of T and $\{x_n(w)\}$ be (RM) admitting the following restrictions:*

(i) $\Sigma\alpha_n = \infty$
(ii) $0 < \alpha \leq \alpha_n (n \geq 1)$
(iii) $\lim_{n\to\infty} u_n(w) = 0$.

Then $p(w)$ is Bochner integrable and (RM) is almost surely T-stable.

Proof. Put $r = 1$, $s = t = 0$, $\alpha_{n,1} = \alpha_n$, $\alpha_{n,0} = 1 - \alpha_n$ in the proof of Theorem 4.1. □

Theorem 4.5. *Let C be a nonempty closed and convex subset of X and let $T_i : \Omega \times C \to C$, $i = 1, 2, 3$ be three random operators satisfying the condition* (2.5) *with $CRF = \bigcap_{i=1}^{3} RF(T_i) \neq \phi$. Let $p(w)$ be a common random fixed point of the random operators $\{T_i, i = 1, 2, 3\}$ and $\{x_n(w)\}$ be the random SP algorithm of three operators with errors defined as follows:*

$$\begin{aligned} x_{n+1}(w) &= (1 - \alpha_n)y_n(w) + \alpha_n T_1(w, y_n(w)) + u_n(w) \\ y_n(w) &= (1 - \beta_n)y_n(w) + \beta_n T_2(w, z_n(w)) + v_n(w) \\ z_n(w) &= (1 - \gamma_n)y_n(w) + \gamma_n T_3(w, x_n(w)) + w_n(w), \end{aligned} \tag{RSPTO}$$

where $\{u_n(w)\}$, $\{v_n(w)\}$, $\{w_n(w)\}$ are sequences of measurable mappings from Ω to C with $\Sigma \int_\Omega u_n(w)\, d\mu(w) < \infty$, $\Sigma \int_\Omega v_n(w)\, d\mu(w) < \infty$, $\Sigma \int_\Omega w_n(w)\, d\mu(w) < \infty$ and $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$. Then the common random fixed point of the random operators $\{T_i, i = 1, 2, 3\}$ is Bochner integrable if and only if for all $w \in \Omega$, $\lim_{n\to\infty} \inf \int_\Omega d(x_n(w), CRF) d\mu(w) = 0$, provided

$$\int_\Omega \|T_i(w, \xi(w)) - \xi(w)\| d\mu(w) = 0 \quad \textit{implies} \quad \|T_i(w, \xi(w)) - \xi(w)\| = 0.$$

Proof. Verbatim repetition of the proof of Theorem 3.11 and is omitted. □

5. Conclusions

We have studied Bochner integrability of random fixed point and almost sure stability with respect to random Kirk type algorithms of generalized ϕ-weakly contractive operators on a separable Banach space. Our results include generalization, refinement and random version of some well-known results:

(1) Our Theorems 3.3 and 3.4 extend and generalize, respectively, Theorems 1 and 3 by Zhang et al. [34] and provide random version of Theorems 3, 4, 5, 10-11 by Rhoades [29] and many results given in the book of Berinde [5].

(2) Our fixed point result, Theorem 3.11, corrects and sets analogue of Theorems 3.1 and 3.4 by Okeke and Abbas [27].
(3) A random analogue of Theorems 2.6 and 2.4 by Hussain et al. [12] is given in Theorems 3.4 and 4.1, respectively.
(4) Theorem 3.7 extends and provides random version of Theorems 9-10 by Gursoy and Karakaya [9].
(5) Stochastic generalization of Theorem 8 by Kumar et al. [23] is presented in Theorem 3.9.

Acknowledgements

The author A. R. Khan would like to acknowledge the support provided by Deanship of Scientific Research (DSR) at King Fahd University of Petroleum and Minerals (KFUPM) for funding this work through project No. IN141047.

References

[1] A. Alotaibi, V. Kumar, N. Hussain, Convergence comparison and stability of Jungck-Kirk type algorithms for common fixed point problems, Fixed Point Theory Appl. 2013 (2013) 173.
[2] M. Abbas, N. Saleem, M. D. L. Sen, Optimal coincidence point results in partially ordered non-Archimedean fuzzy metric spaces, Fixed Point Theory Appl. 1 (2016) 1-18.
[3] A. T. Bharucha-Reid, Fixed point theorems in probabilistic analysis, Bull. Am. Math. Soc. 82 (1976) 641-657.
[4] I. Beg and M. Abbas, Equivalence and stability of random fixed point iterative procedures, J. Appl. Math. Stoch. Anal. Article ID 23297 (2006) 1-19.
[5] V. Berinde, Iterative approximations of fixed points, Editura Efemeride, Baia Mare (2007).
[6] R. Chugh, V. Kumar, S. Narwal, Some strong convergence results of random iterative algorithms with errors in Banach spaces, Commu. Korean Math. Soc. 31 (2016) 147-161.
[7] R. Chugh, V. Kumar, Convergence of SP iterative scheme with mixed errors for accretive Lipschitzian and strongly accretive Lipschitzian operators in Banach space, Int. J. Comput. Math. 90 (2013) 1865-1880.
[8] S. S. Chang, Y. J. Cho, J. K. Kim, H. Y. Zhou, Random Ishikawa iterative sequence with applications, Stoch. Anal. Appl. 23 (2005) 69-77.
[9] F. Gursoy, V. Karakaya, Some convergence and stability results for two new Kirk type hybrid fixed point iterative algorithms, J. Function Spaces Article ID 684191 (2014) 1-8.
[10] I. Ganiev, G. S. Gharib, The Bochner integral for measurable sections and its properties, Ann. Funct. Anal. 4 (2013) 1-10.
[11] R. Glowinski, P. L. Tallec, Augmented Lagrangian and operator-splitting methods in nonlinear mechanics, SIAM, Philadelphia, 1989.
[12] N. Hussain, R. Chugh, V. Kumar, A. Rafiq, On the rate of convergence of Kirk-type iterative schemes, J. Appl. Math. 2012 (2012), Article ID 526503, 22 pages.
[13] N. Hussain, S. Narwal, R. Chugh, V. Kumar, On convergence of random iterative schemes with errors for strongly pseudo-contractive Lipschitzian maps in real Banach spaces, J. Nonlinear Sci. Appl. 9 (2016), 3157-3168.
[14] O. Hans, Random operator equations, in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, Part I, pp. 185-202, University of California Press, California (1961).
[15] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces, J. Math. Anal. Appl. 67 (2) (1979) 261-273.
[16] S. Ishikawa, Fixed points by a new iteration method, Proc. Am. Math. Soc., 44, 147-150 (1974).
[17] M. C. Joshi, R. K. Bose, Some Topics in Nonlinear Functional Analysis, Wiley Eastern, New Delhi (1985).
[18] A. R. Khan, N. Hussain, Random coincidence point theorem in Frechet spaces with applications, Stoch. Anal. Appl. 22 (2004) 155-167.
[19] A. R. Khan, N. Hussain, N. Yasmin, N. Shafqat, Random coincidence point results for weakly increasing functions in partially ordered metric spaces, Bull. Iranian Math. Soc. 41 (2015) 407-422.
[20] A. R. Khan, V. Kumar, N. Hussain, Analytical and numerical treatment of Jungck-type iterative schemes, Appl. Math. Comput. 231 (2014) 521-535.
[21] A. R. Khan, F. Gursoy, V. Kumar, Stability and data dependence results for Jungck-Khan iterative scheme, Turk. J. Math. 40 (2016) 631-640.
[22] A. R. Khan, A. A. Domlo, H. Fukhar-ud-din, Common fixed points Noor iteration for a finite family of asymptotically quasi non-expansive mappings in Banach spaces, J. Math. Anal. Appl. 341 (2008) 1-11.
[23] V. Kumar, A. Latif, A. Rafiq, N. Hussain, S-iteration process for quasi contractive mappings, J. Inequal. Appl. 2013 (2013), 206.
[24] W. A. Kirk, On successive approximations for non-expansive mappings in Banach spaces, Glasgow Math. J. 12 (1971) 6-9.
[25] W. R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4 (1953) 506-510.
[26] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.
[27] G. A. Okeke, M. Abbas, Convergence and almost sure T-stability for a random iterative sequence generated by a generalized random operator, J. Inequal. Appl. 2015 (2015), 146, DOI 10.1186/s 13660-015-0666-8.

[28] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval, J. Comp. Appl. Math. 235 (2011) 3006-3014.
[29] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001) 2683-2693.
[30] Z. Raza, N. Saleem, M. Abbas, Optimal coincidence points of proximal quasi-contraction mappings in non-Archimedean fuzzy metric spaces, J. Nonlinear Sci. Appl. 9 (2016) 3787-3801.
[31] D. R. Sahu, A. Petrusel, Strong convergence of iterative methods by strictly pseudo-contractive mappings in Banach spaces, Nonlinear Anal. 74 (2011) 6012-6023.
[32] N. Saleem, M. Abbas, Z. Raza, Optimal coincidence best approximation solution in non-Archimedean Fuzzy Metric Spaces, Iran. J. Fuzzy Syst. 13 (2016) 113-124.
[33] N. Saleem, B. Ali, M. Abbas, Z. Raza, Fixed points of Suzuki type generalized multi-valued mappings in fuzzy metric spaces with applications, Fixed Point Theory Appl. 1 (2015) 1-18.
[34] S. S. Zhang, X. R. Wang, M. Liu, J. H. Zhu, Almost sure T-stability and convergence for random iterative algorithms, Appl. Math. Mech. 32 (6) (2011) 805-810.