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# Moore-Penrose Inverse in Indefinite Inner Product Spaces

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**Abstract.** We present the definition and some properties for the Moore-Penrose inverse in possibly degenerate indefinite inner product spaces. The extensions of appropriate results, given for matrices in Euclidean and nondegenerate indefinite inner product spaces are established. All this is done by using the concept of linear relations.

## 1. Introduction

Let $\mathbb{C}^n$ be the space equipped with an indefinite inner product induced by a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ via

$$[x, y] = \langle Hx, y\rangle,$$

where $\langle ., .\rangle$ denotes the standard Euclidean scalar product on $\mathbb{C}^n$. If the Hermitian matrix $H$ is invertible, then the indefinite inner product is nondegenerate. In that case, for every matrix $A \in \mathbb{C}^{n\times n}$ there is the unique matrix $A^{[*]_H}$ satisfying

$$[A^{[*]_H}x, y] = [x, Ay], \quad \text{for all } x, y \in \mathbb{C}^n.$$

Spaces with a degenerate inner product (when Gram matrix $H$ is singular) are not so familiar. In that kind of spaces the $H$-adjoint of the matrix $A \in \mathbb{C}^{n\times n}$ need not exist. Examples can be found in ([3, 8]). In ([2]) it was shown that the orthogonal complement of a subspace is not necessarily the direct complement.

In ([4]) the definition and the basic properties for the Moore-Penrose inverse were given. It was shown that in a nondegenerate indefinite inner product space a matrix need not have a Moore-Penrose inverse. By ([4], Theorem 1.) the Moore-Penrose inverse for matrix $A$ exists if and only if $rank(A) = rank(AA^{[*]}) = rank(A^{[*]}A)$. When it exists, it is unique. Also, if the Moore-Penrose inverse exists, then $R(A)$ and $N(A^{[*]})$ are orthogonal complementary subspaces of $\mathbb{C}^n$.

Our aim in this paper is to propose a more general definition of the Moore-Penrose inverse. As in ([3, 7, 8]), we will consider $H$-adjoint $A^{[*]}$ not as a matrix, but as a linear relation in $\mathbb{C}^n$, i.e. a subspace of $\mathbb{C}^{2n}$. Also, a matrix $A \in \mathbb{C}^{n\times n}$ can be interpreted as a linear relation via its graph $\Gamma(A)$, where: $\Gamma(A) := \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix} : x \in \mathbb{C}^n \right\} \subseteq$

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$\mathbb{C}^{2n}$. The $H$-adjoint of $A$ is the linear relation $A^{[*]} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{C}^{2n} : [y, \omega] = [z, x] \text{ for all } \begin{pmatrix} x \\ \omega \end{pmatrix} \in A \right\}$. We just mention that we can always find a basis of $\mathbb{C}^n$ such that the matrices $H$ and $A$ have the forms:

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Here $H_1$ is an invertible Hermitian matrix and the inner product induced by it is nondegenerate. From ([8], Proposition 2.6) we have

$$A^{[*]_H} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ {A_1}^{[*]_{H_1}} y_1 \\ z_2 \end{pmatrix} : {A_2}^* H_1 y_1 = 0 \right\}.$$

Here we will suppress the subscripts $H$ and $H_1$ whenever it is clear from the context what is meant. About inner product spaces see ([1, 2, 5, 6]).

We give some important notions for linear relations.

**Definition 1.1.** *For linear relations $A, B \subseteq \mathbb{C}^{2n}$ we define:*
*$domA = \left\{ x : \begin{pmatrix} x \\ y \end{pmatrix} \in A \right\}$ - the domain of $A$,*
*$mulA = \left\{ y : \begin{pmatrix} 0 \\ y \end{pmatrix} \in A \right\}$ - the multivalued part of $A$,*
*$A^{-1} = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in A \right\}$ - the inverse of $A$,*
*$A + B = \left\{ \begin{pmatrix} x \\ y+z \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in A, \begin{pmatrix} x \\ z \end{pmatrix} \in B \right\}$ - the sum of $A$ and $B$,*
*$AB = \left\{ \begin{pmatrix} x \\ z \end{pmatrix} : \text{there exists some } y \in C^n \text{ with } \begin{pmatrix} y \\ z \end{pmatrix} \in A, \begin{pmatrix} x \\ y \end{pmatrix} \in B \right\}$ - the product of $A$ and $B$.*
*If $domA = \mathbb{C}^n$, we say that $A$ has full domain. In all the cases $x, y, z$ are understood to be from $\mathbb{C}^n$.*

**Theorem 1.2.** *Let $A, B \subseteq \mathbb{C}^{2n}$ be linear relations. Then*

1. *$A \subseteq B$ implies $B^{[*]} \subseteq A^{[*]}$;*
2. *$A^{[*]} + B^{[*]} \subseteq (A + B)^{[*]}$;*
3. *$mulA^{[*]} = (domA)^{[\perp]}$; if $A$ is a matrix, then $mulA^{[*]} = kerH$;*
4. *$(A^{[*]})^{[*]} = A + (kerH \times kerH)$.*

In [4] the notion of the Moore-Penrose inverse of matrices in nondegenerate indefinite inner product spaces is introduced. In this paper we give a generalization of the notion of the Moore-Penrose inverse $A^{[\dagger]}$ to degenerate indefinite inner product spaces. It is done via linear relations.

This paper is organized as follows. After giving basic notions and results concerning indefinite inner product spaces and linear relations in Section 1, in Section 2 we give definition of the Moore-Penrose inverse for the linear relations and investigate its properties for square matrices in degenerate case. Also, we show that the Moore-Penrose inverse is not unique in general (Example 2.4). This section includes our main result - the description of the Moore-Penrose inverses of $A^{[*]}A$ and $AA^{[*]}$ under the additional assumption that $A$ and $A^{[\dagger]}$ are matrices. More precisely, the following is shown:

• $(A^{[\dagger]})^{[*]} \subseteq (A^{[*]})^{[\dagger]}$ (Theorem 2.6),

• $A^{[*]}AA^{[\dagger]} = A^{[*]}$ and $A^{[\dagger]}AA^{[*]} \subseteq A^{[*]}$ (Theorem 2.8),

• $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ is the $\{1, 2, (3)\}$-inverse of $A^{[*]}A$ (Theorem 2.11),

• $(A^{[\dagger]})^{[*]}A^{[\dagger]}$ is the $\{1, 2, (4)\}$-inverse of $AA^{[*]}$ (Theorem 2.12) ,

• necessary and sufficient conditions are given such that $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ is the Moore-Penrose inverse of $A^{[*]}A$ and that $(A^{[\dagger]})^{[*]}A^{[\dagger]}$ is the Moore-Penrose inverse of $AA^{[*]}$ (Theorem 2.14 and 2.15).

## 2. Definition and Properties of the Moore-Penrose Inverse

In this section we give a new, more general definition for the Moore-Penrose inverse. That definition includes linear relations and matrices. Most of the results in this paper are given for matrices. The case when the Moore-Penrose inverse is not a matrix but a linear relation will be the object of some later researches.

In ([4]) it was shown that the Moore-Penrose inverse for a matrix $A \in \mathbb{C}^{n\times m}$ is the unique matrix $X \in \mathbb{C}^{m\times n}$ that satisfies following equations:

$$AXA = A, \quad XAX = X, \quad AX = (AX)^{[*]} \quad and \quad XA = (XA)^{[*]}.$$

If $A$ and $X$ are $n \times n$ complex matrices, then $(AX)^{[*]}$ and $(XA)^{[*]}$ are matrices if and only if $H$ is invertible ([3]). That means that in the space with an indefinite inner product induced by Hermitian but not invertible matrix $H$, the third and the fourth condition from the definition of the Moore-Penrose inverse are never satisfied. These conditions are equal to $(AX)^*H = HAX$ and $(XA)^*H = HXA$, respectively. By ([3], Proposition 2.5) it is equivalent to $AX$ and $XA$ are $H$-symmetric. This motivates the following definition of the Moore-Penrose inverse.

**Definition 2.1.** *Let $A \subseteq \mathbb{C}^{2n}$ be a linear relation. A linear relation $X \subseteq \mathbb{C}^{2n}$ is the Moore-Penrose inverse of $A$ if it satisfies the following four equations:*

$$AXA = A \tag{1}$$

$$XAX = X \tag{2}$$

$$AX \subseteq (AX)^{[*]} \tag{3}$$

$$XA \subseteq (XA)^{[*]} \tag{4}$$

*The Moore-Penrose inverse of $A$ is usually denoted by $A^{[\dagger]}$.*

We recall the notion of the weighted generalized Moore-Penrose inverse (where the weights are Hermitian and possibly singular matrices) and then show its equality with the Moore-Penrose inverse in indefinite inner product spaces.

**Definition 2.2.** *Let $A$,$M$ and $N$ be matrices of order $m \times n$, $m \times m$ and $n \times n$, respectively, where $M$ and $N$ are Hermitian. An $n \times m$ matrix $X$ is said to be a generalized weighted Moore-Penrose inverse of $A$ if the following conditions are satisfied: $AXA = A$, $XAX = X$, $(MAX)^* = MAX$ and $(NXA)^* = NXA$, where $*$ denotes conjugate transpose.*

**Lemma 2.3.** *Let $A \in \mathbb{C}^{n\times n}$ be a matrix and let $H$ be Hermitian matrix of the same size. Then the Moore-Penrose inverse and generalized weighted Moore-Penrose inverse of $A$ (with respect to $H$) coincide.*

*Proof.* The (3) and (4) are direct corollary of ([3], Proposition 2.5.), i.e,

$$HAX = (HAX)^* \text{ and } HXA = (HXA)^*.$$

□

Unlike the nondegenerate case, in degenerate inner product spaces a Moore-Penrose inverse does not have to be unique. The next example illustrates it.

**Example 2.4.** *Let* $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

*By direct computation the matrix X that satisfies* (1), (2), (3) *and* (4) *has the form* $X = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$, *where c is an arbitrary complex number.*

Through this paper we will assume that matrices $A$, $X$ and $H$ are given in their characteristic forms: $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ and $H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $H_1$ is invertible, as in [8].

**Theorem 2.5.** *Let* $A \in \mathbb{C}^{n\times n}$ *be a matrix. Then* $X \in \mathbb{C}^{n\times n}$ *is a Moore-Penrose inverse of A if and only if the following conditions hold:*
*(i)* $A_1X_2 + A_2X_4 = 0$,
*(ii)* $A_1X_1 + A_2X_3$ *is* $H_1$*-selfadjoint,*
*(iii)*$X_1A_2 + X_2A_4 = 0$,
*(iv)*$X_1A_1 + X_2A_3$ *is* $H_1$*-selfadjoint,*
*(v)* $A_1X_1A_1 + A_2X_3A_1 = A_1$,
*(vi)* $A_1X_1A_2 + A_2X_3A_2 = A_2$,
*(vii)* $A_3X_1A_1 + A_4X_3A_1 + A_3X_2A_3 + A_4X_4A_3 = A_3$,
*(viii)* $A_4X_3A_2 + A_4X_4A_4 = A_4$,
*(ix)* $X_1A_1X_1 + X_2A_3X_1 = X_1$,
*(x)* $X_1A_1X_2 + X_2A_3X_2 = X_2$,
*(xi)* $X_3A_1X_1 + X_4A_3X_1 + X_3A_2X_3 + X_4A_4X_3 = X_3$,
*(xii)* $X_4A_3X_2 + X_4A_4X_4 = X_4$.

*Proof.* From the third condition for the Moore-Penrose inverse, by Lemma 2.3. we get:

$$HAX = (HAX)^* \iff \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} =$$

$$= \begin{bmatrix} H_1(A_1X_1 + A_2X_3) & H_1(A_1X_2 + A_2X_4) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (H_1(A_1X_1 + A_2X_3))^* & 0 \\ (H_1(A_1X_2 + A_2X_4))^* & 0 \end{bmatrix}, \text{ i.e,}$$

(i) $A_1X_2 + A_2X_4 = 0$,
(ii) $A_1X_1 + A_2X_3$ is $H_1$-selfadjoint.

Similarly, from the condition (4) we have:
(iii)$X_1A_2 + X_2A_4 = 0$,
(iv)$X_1A_1 + X_2A_3$ is $H_1$-selfadjoint.

From these two results and the conditions (1) and (2) we have:

$$\begin{bmatrix} A_1X_1A_1 + A_2X_3A_1 & A_1X_1A_2 + A_2X_3A_2 \\ A_3X_1A_1 + A_4X_3A_1 + A_3X_2A_3 + A_4X_4A_3 & A_4X_3A_2 + A_4X_4A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \textit{i.e,}$$

(v) $A_1X_1A_1 + A_2X_3A_1 = A_1$,
(vi) $A_1X_1A_2 + A_2X_3A_2 = A_2$,

(vii) $A_3X_1A_1 + A_4X_3A_1 + A_3X_2A_3 + A_4X_4A_3 = A_3$,
(viii) $A_4X_3A_2 + A_4X_4A_4 = A_4$.

Similarly,
$$\begin{bmatrix} X_1A_1X_1X_2A_3X_1 & X_1A_1X_2 + X_2A_3X_2 \\ X_3A_1X_1 + X_4A_3X_1 + X_3A_2X_3 + X_4A_4X_3 & X_4A_3X_2 + X_4A_4X_4 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$
(ix) $X_1A_1X_1 + X_2A_3X_1 = X_1$,
(x) $X_1A_1X_2 + X_2A_3X_2 = X_2$,
(xi) $X_3A_1X_1 + X_4A_3X_1 + X_3A_2X_3 + X_4A_4X_3 = X_3$,
(xii) $X_4A_3X_2 + X_4A_4X_4 = X_4$. □

In the rest of the section we give some basic properties of a Moore-Penrose inverse.

**Theorem 2.6.** *If $A^{[\dagger]} \in C^{n\times n}$ is a Moore-Penrose inverse of $A \in C^{n\times n}$, then $(A^{[\dagger]})^{[*]} \subseteq (A^{[*]})^{[\dagger]}$.*

*Proof.* Let us put $X = A^{[\dagger]} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$. We have that
$$X^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ X_1{}^{[*]_{H_1}} y_1 \\ z_2 \end{pmatrix} : X_2^* H_1 y_1 = 0 \right\}$$
and
$$A^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]_{H_1}} y_1 \\ z_2 \end{pmatrix} : A_2^* H_1 y_1 = 0 \right\}.$$

We check the conditions from the definition of the Moore-Penrose inverse of $A^{[*]}$:
(1) $A^{[*]}X^{[*]}A^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]}X_1{}^{[*]}A_1{}^{[*]}y_1 \\ z_2 \end{pmatrix} : \begin{matrix} A_2^*H_1y_1 = 0, \\ X_2^*H_1A_1{}^{[*]}y_1 = 0, \\ A_2^*H_1X_1{}^{[*]}A_1{}^{[*]}y_1 = 0 \end{matrix} \right\}.$

Let $A_2^*H_1y_1 = 0$ hold. Then
$X_2{}^*H_1A_1{}^{[*]}y_1 = X_2{}^*A_1{}^*H_1y_1 = (A_1X_2)^*H_1y_1 \overset{(i)}{=} (-A_2X_4)^*H_1y_1 = -X_4{}^*A_2{}^*H_1y_1 = 0$, from our assumption.
Also, $A_2{}^*H_1X_1{}^{[*]}A_1{}^{[*]}y_1 = A_2{}^*X_1{}^*A_1{}^*H_1y_1 = (A_1X_1A_2)^*H_1y_1 \overset{(vi)}{=}$
$(A_2 - A_2X_3A_2)^*H_1y_1 = A_2^*H_1y_1 - (A_2X_3A_2)^*H_1y_1 = -A_2{}^*X_3{}^*A_2{}^*H_1y_1 = 0$, and
$A_1{}^{[*]}X_1{}^{[*]}A_1{}^{[*]}y_1 = (A_1X_1A_1)^{[*]}y_1 \overset{(v)}{=} (A_1 - A_2X_3A_1)^{[*]}y_1 =$
$A_1{}^{[*]}y_1 - (A_2X_3A_1)^{[*]}y_1 = A_1{}^{[*]}y_1 - H_1{}^{-1}A_1{}^*X_3{}^*A_2{}^*H_1y_1 = A_1{}^{[*]}y_1$.

Thus, $A^{[*]}X^{[*]}A^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\} = A^{[*]}$.

(2) Similarly, $X^{[*]}A^{[*]}X^{[*]} = X^{[*]}$.

(3) $A^{[*]}X^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]}X_1{}^{[*]}y_1 \\ z_2 \end{pmatrix} : \begin{matrix} X_2^*H_1y_1 = 0, \\ A_2^*H_1X_1{}^{[*]}y_1 = 0 \end{matrix} \right\} == \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (X_1A_1)^{[*]}y_1 \\ z_2 \end{pmatrix} : X_2^*H_1y_1 = 0 \right\}$ because of

$A_2{}^*H_1X_1{}^{[*]}y_1 = A_2{}^*X_1{}^*H_1y_1 = (X_1A_2)^*H_1y_1 \overset{\text{(iii)}}{=} -A_4{}^*X_2{}^*H_1y_1 = 0.$

On the other hand, we have

$$\left(A^{[*]}X^{[*]}\right)^{[*]} = \left\{\begin{pmatrix} y \\ z \end{pmatrix} : [y,\omega] = [z,x] \text{ for all } \begin{pmatrix} x \\ \omega \end{pmatrix} \in A^{[*]}X^{[*]}\right\}.$$

It is easy to see that $A^{[*]}X^{[*]} \subseteq \left(A^{[*]}X^{[*]}\right)^{[*]}$ if and only if $[y,\omega] = [z,x]$ for all $\begin{pmatrix} x \\ \omega \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \in A^{[*]}X^{[*]}$.

Let $\begin{pmatrix} x \\ \omega \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \in A^{[*]}X^{[*]}$. As $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ are from the domain of $A^{[*]}X^{[*]}$, we have that $X_2^*H_1x_1 = X_2^*H_1y_1 = 0$.
Also, $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} (X_1A_1)^{[*]}x_1 \\ \omega_2 \end{pmatrix}$ and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (X_1A_1)^{[*]}y_1 \\ z_2 \end{pmatrix}$.

Now, we have,
$[y,\omega] = \omega^*Hy = \omega_1{}^*H_1y_1 = ((X_1A_1)^{[*]}x_1)^*H_1y_1 = x_1^*H_1X_1A_1y_1$ and
$[z,x] = x^*Hz = x_1^*H_1z_1 = x_1^*H_1(X_1A_1)^{[*]}y_1 \overset{(iv)}{=} x_1^*H_1(X_1A_1 + X_2A_3 - (X_2A_3)^{[*]})y_1 = x_1^*H_1X_1A_1y_1 + x_1^*H_1X_2A_3y_1 - x_1^*A_3^*X_2^*H_1y_1 = x_1^*H_1X_1A_1y_1$, as $X_2^*H_1x_1 = X_2^*H_1y_1 = 0$, proving that $[y,\omega] = [z,x]$. Thus, $A^{[*]}X^{[*]} \subseteq \left(A^{[*]}X^{[*]}\right)^{[*]}$ holds.

(4) In the same way it can be shown that $X^{[*]}A^{[*]} \subseteq \left(X^{[*]}A^{[*]}\right)^{[*]}$. □

**Remark 2.7.** *We mention that in degenerate inner product spaces, in general we just have* $(AXA)^{[*]} \supseteq A^{[*]}X^{[*]}A^{[*]}$ *([3]). From the previous theorem we see that if* $X \in \mathbb{C}^{n\times n}$ *is a Moore-Penrose inverse then equality holds.*

In nondegenerate indefinite inner product spaces the well known property of the Moore-Penrose inverse $A^{[*]} = A^{[*]}AA^{[\dagger]} = A^{[\dagger]}AA^{[*]}$ holds. We will show that it is not true in general in the degenerate case. As usual, in the proof of the theorem we will use $X = A^{[\dagger]} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$.

**Theorem 2.8.** *If* $A^{[\dagger]} \in \mathbb{C}^{n\times n}$ *is a Moore-Penrose inverse of* $A \in \mathbb{C}^{n\times n}$, *then* $A^{[*]} = A^{[*]}AA^{[\dagger]}$ *and* $A^{[\dagger]}AA^{[*]} \subseteq A^{[*]}$.

*Proof.* Using (i) we have

$$A^{[*]}AA^{[\dagger]} = \left\{\begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]}(A_1X_1 + A_2X_3)y_1 \\ z_2 \end{pmatrix} : A_2^*H_1(A_1X_1 + A_2X_3)y_1 = 0\right\}.$$

The conditions that we have can be simplified as follows ((ii),(vi),(v)):
$A_2^*H_1(A_1X_1 + A_2X_3)y_1 = 0 \Longleftrightarrow A_2^*H_1(A_1X_1 + A_2X_3)^{[*]}y_1 = 0 \Longleftrightarrow$
$A_2^*(A_1X_1 + A_2X_3)^*H_1y_1 = 0 \Longleftrightarrow (A_1X_1A_2 + A_2X_3A_2)^*H_1y_1 = 0 \Longleftrightarrow A_2^*H_1y_1 = 0.$
Also, $A_1{}^{[*]}(A_1X_1 + A_2X_3)y_1 = (A_1X_1A_1 + A_2X_3A_1)^{[*]}y_1 = A_1^{[*]}y_1$.

So, we have $A^{[*]}AA^{[\dagger]} = \left\{\begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^*H_1y_1 = 0\right\} = A^{[*]}.$

Let us check the second statement:

$$A^{[\dagger]}AA^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (X_1A_1 + X_2A_3)A_1^{[*]}y_1 \\ (X_3A_1 + X_4A_3)A_1^{[*]}y_1 + (X_3A_2 + X_4A_4)z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\}.$$

Now, we have $(X_1A_1 + X_2A_3)A_1^{[*]}y_1 \overset{\text{(iv)}}{=} (A_1X_1A_1 + A_1X_2A_3)^{[*]}y_1 \overset{\text{(v,i)}}{=}$ $(A_1 - A_2X_3A_1 - A_2X_4A_3)^{[*]}y_1 = A_1^{[*]}y_1$, for $A_2^*H_1y_1 = 0$.

Thus, $A^{[\dagger]}AA^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1^{[*]}y_1 \\ (X_3A_1 + X_4A_3)A_1^{[*]}y_1 + (X_3A_2 + X_4A_4)z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\}$

$$\subseteq \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\} = A^{[*]}.$$

The opposite inclusion holds just in the case when $X_3A_2 + X_4A_4$ is invertible. □

**Example 2.9.** *To see that a similar statement that $A^{[\dagger]}AA^{[*]} = A^{[*]}$ does not hold, consider the following example:* $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then* $A^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_1 \\ z_2 \end{pmatrix} \right\}$, *and* $A^{[\dagger]}AA^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_1 \\ cy_1 \end{pmatrix} \right\}$, *where $z_2$ and $c$ are arbitrary, but fixed complex constants.*

*So, we have just $A^{[\dagger]}AA^{[*]} \subseteq A^{[*]}$, while the opposite is not true, for example, in* $y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

**Definition 2.10.** *A matrix $X \in \mathbb{C}^{n\times n}$ is $\{1, 2, (3)\}$- inverse ($\{1, 2, (4)\}-$inverse) of $A \in \mathbb{C}^{n\times n}$ if the following conditions hold: $AXA = A$, $XAX = X$ and $AX \subseteq (AX)^{[*]}$, ($AXA = A$, $XAX = X$ and $XA \subseteq (XA)^{[*]}$).*

**Theorem 2.11.** *If $A^{[\dagger]} \in \mathbb{C}^{n\times n}$ is a Moore-Penrose inverse of $A \in \mathbb{C}^{n\times n}$, then $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ is a {1,2,(3)}- inverse of $A^{[*]}A$.*

*Proof.* To simplify the notation, we denote $A^{[\dagger]}$ by $X$. Let us check the conditions for the Moore-Penrose inverse:

(1) $A^{[*]}AXX^{[*]}A^{[*]}A = A^{[*]}X^{[*]}A^{[*]}A = A^{[*]}A$.
(2) $XX^{[*]}A^{[*]}AXX^{[*]} = XX^{[*]}A^{[*]}X^{[*]} = XX^{[*]}$.
(3) $A^{[*]}AXX^{[*]} = A^{[*]}X^{[*]} \subseteq (A^{[*]}X^{[*]})^{[*]}$.

The first parts follow from Theorem 2.8 and the second ones from Theorem 2.7. □

**Theorem 2.12.** *If $A^{[\dagger]} \in \mathbb{C}^{n\times n}$ is a Moore-Penrose inverse of $A \in \mathbb{C}^{n\times n}$, then $(A^{[\dagger]})^{[*]}A^{[\dagger]}$ is a {1,2,(4)}- inverse of $AA^{[*]}$.*

*Proof.* We check the conditions (1), (2) and (4) from the definition of the Moore-Penrose inverse. The equalities that arise here are explained in detail below.

(1) $AA^{[*]}(A^{[\dagger]})^{[*]}A^{[\dagger]}AA^{[*]} = AA^{[*]}X^{[*]}XAA^{[*]} =$

$$AA^{[*]}\left\{ \begin{pmatrix} y_1 \\ y_2 \\ X_1^{[*]}(X_1A_1 + X_2A_3)A_1^{[*]}y_1 \\ z_2 \end{pmatrix} : \begin{array}{c} A_2^*H_1y_1 = 0 \\ X_2^*H_1(X_1A_1 + X_2A_3)A_1^{[*]}y_1 = 0 \end{array} \right\}$$

$= AA^{[*]} \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (A_1X_1)^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\} = A \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1^{[*]}(A_1X_1)^{[*]}y_1 \\ z_2 \end{pmatrix} : \begin{matrix} A_2^*H_1y_1 = 0 \\ A_2^*H_1(A_1X_1)^{[*]}y_1 = 0 \end{matrix} \right\} =$

$A \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (A_1X_1A_1)^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\} = A \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^*H_1y_1 = 0 \right\} = AA^{[*]}$, where we used the equations from the Theorem 2.5.

Let $A_2^*H_1y_1 = 0$. Then:
$X_2^*H_1(X_1A_1 + X_2A_3)A_1^{[*]}y_1 = X_2^*H_1(X_1A_1 + X_2A_3)^{[*]}A_1^{[*]}y_1 =$
$X_2^*(X_1A_1 + X_2A_3)^*H_1A_1^{[*]}y_1 = (X_1A_1X_2 + X_2A_3X_2)^*H_1A_1^{[*]}y_1 =$
$X_2^*H_1A_1^{[*]}y_1 = X_2^*A_1^*H_1y_1 = (A_1X_2)^*H_1y_1 = -(A_2X_4)^*H_1y_1 = 0.$

Also, $X_1^{[*]}(X_1A_1 + X_2A_3)A_1^{[*]}y_1 = X_1^{[*]}(X_1A_1 + X_2A_3)^{[*]}A_1^{[*]}y_1 = (X_1A_1X_1 + X_2A_3X_1)^{[*]}A_1^{[*]}y_1 = X_1^{[*]}A_1^{[*]}y_1$,

and $A_2^*H_1(A_1X_1)^{[*]}y_1 = A_2^*(A_1X_1)^*H_1y_1 = (A_1X_1A_2)^*H_1y_1 = (A_2 - A_2X_3A_2)^*H_1y_1 = 0.$

Finally, $(A_1X_1A_1)^{[*]}y_1 = (A_1 - A_2X_3A_1)^{[*]}y_1 = A_1^{[*]}y_1.$

(2) For checking the second condition we have: $X^{[*]}XAA^{[*]}X^{[*]}X =$

$X^{[*]}XA \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) \\ \omega_2 \end{pmatrix} : \begin{matrix} X_2^*H_1(X_1y_1 + X_2y_2) = 0 \\ A_2^*H_1X_1^{[*]}(X_1y_1 + X_2y_2) = 0 \end{matrix} \right\} =$

$X^{[*]}XA \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) \\ \omega_2 \end{pmatrix} : X_2^*H_1(X_1y_1 + X_2y_2) = 0 \right\} =$

$X^{[*]}X \left\{ \begin{pmatrix} y_1 \\ y_2 \\ A_1A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) + A_2\omega_2 \\ A_3A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) + A_4\omega_2 \end{pmatrix} : X_2^*H_1(X_1y_1 + X_2y_2) = 0 \right\} =$

$X^{[*]} \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (X_1A_1 + X_2A_3)A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) \\ * \end{pmatrix} : X_2^*H_1(X_1y_1 + X_2y_2) = 0 \right\} =$

$X^{[*]} \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (X_1A_1)^{[*]}(X_1y_1 + X_2y_2) \\ * \end{pmatrix} : X_2^*H_1(X_1y_1 + X_2y_2) = 0 \right\} =$

$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ X_1^{[*]}(X_1A_1)^{[*]}(X_1y_1 + X_2y_2) \\ z_2 \end{pmatrix} : \begin{matrix} X_2^*H_1(X_1y_1 + X_2y_2) = 0 \\ X_2^*H_1(X_1A_1)^{[*]}(X_1y_1 + X_2y_2) = 0 \end{matrix} \right\} =$

$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ X_1^{[*]}(X_1y_1 + X_2y_2) \\ z_2 \end{pmatrix} : X_2^*H_1(X_1y_1 + X_2y_2) = 0 \right\} = X^{[*]}X.$

Similarly as in (1), let $X_2^* H_1(X_1y_1 + X_2y_2) = 0$.
Now, $A_2^* H_1 X_1^{[*]}(X_1y_1 + X_2y_2) = (X_1A_2)^* H_1(X_1y_1 + X_2y_2) = -A_4^* X_2^* H_1(X_1y_1 + X_2y_2) = 0$
and $X_2^* H_1(X_1A_1)^{[*]}(X_1y_1 + X_2y_2) = (X_1A_1X_2)^* H_1(X_1y_1 + X_2y_2) = (X_2 - X_2A_3X_2)^* H_1(X_1y_1 + X_2y_2) = 0$.

Also, we have
$X_1A_1A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) + X_1A_2\omega_2 + X_2A_3A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) + X_2A_4\omega_2 =$
$(X_1A_1 + X_2A_3)A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) + (X_1A_2 + X_2A_4)\omega_2 =$
$(X_1A_1 + X_2A_3)^{[*]}A_1^{[*]}X_1^{[*]}(X_1y_1 + X_2y_2) =$
$(X_1A_1(X_1A_1 + X_2A_3))^{[*]}(X_1y_1 + X_2y_2) = (X_1A_1X_1A_1 + (X_2 - X_2A_3X_2)A_3)^{[*]}(X_1y_1 + X_2y_2) =$
$(X_1A_1X_1A_1)^{[*]}(X_1y_1 + X_2y_2) = ((X_1 - X_2A_3X_1)A_1)^{[*]}(X_1y_1 + X_2y_2) = (X_1A_1)^{[*]}(X_1y_1 + X_2y_2)$,

and

$X_1^{[*]}(X_1A_1)^{[*]}(X_1y_1 + X_2y_2) = (X_1A_1X_1)^{[*]}(X_1y_1 + X_2y_2) = (X_1 - X_2A_3X_2)^{[*]}(X_1y_1 + X_2y_2) = X_1^{[*]}(X_1y_1 + X_2y_2)$.

(3) We show that $(A^{[\dagger]})^{[*]}A^{[\dagger]}AA^{[*]} \subseteq ((A^{[\dagger]})^{[*]}A^{[\dagger]}AA^{[*]})^{[*]}$, i.e. $X^{[*]}XAA^{[*]} \subseteq (X^{[*]}XAA^{[*]})^{[*]}$ is satisfied. Here we use the equalities from (1).

$$X^{[*]}XAA^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ (A_1X_1)^{[*]}y_1 \\ z_2 \end{pmatrix} : A_2^* H_1 y_1 = 0 \right\}$$

We know that $X^{[*]}XAA^{[*]} \subseteq (X^{[*]}XAA^{[*]})^{[*]}$ if and only if
$((A_1X_1)^{[*]}x_1)^* H_1 y_1 = x_1^* H_1(A_1X_1)^{[*]}y_1$, for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $A_2^* H_1 y_1 = A_2^* H_1 x_1 = 0$.
The left side is $((A_1X_1)^{[*]}x_1)^* H_1 y_1 = x_1^* H_1 A_1 X_1 y_1$.
The right side is: $x_1^* H_1(A_1X_1)^{[*]}y_1 = x_1^* H_1(A_1X_1 + A_2X_3 - (A_2X_3)^{[*]})y_1 =$
$x_1^* H_1A_1X_1y_1 + x_1^* H_1A_2X_3y_1 - x_1^* X_3^* A_2^* H_1 y_1 = x_1H_1A_1X_1y_1$, because of $A_2^* H_1 x_1 = A_2^* H_1 y_1 = 0$.

The left and the right side are equal, so we see that $X^{[*]}XAA^{[*]} \subseteq (X^{[*]}XAA^{[*]})^{[*]}$ is satisfied. □

We can show that in general case the fourth (third) condition is not satisfied.

**Example 2.13.** *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ *and* $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then the Moore-Penrose inverse for* $A$ *is given by* $X = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

*We check if the fourth condition holds:*

$A^{[\dagger]}(A^{[\dagger]})^{[*]}A^{[*]}A = XX^{[*]}A^{[*]}A = \left\{ \begin{pmatrix} y_1 \\ -y_1 \\ -z_2 \\ z_2 \end{pmatrix} \right\}$, *where* $y_1$ *and* $z_2$ *are arbitrary complex numbers. The definition of H-symmetry implies that if* $A^{[\dagger]}(A^{[\dagger]})^{[*]}A^{[*]}A \subseteq (A^{[\dagger]}(A^{[\dagger]})^{[*]}A^{[*]}A)^{[*]}$ *then* $(z_2)^* y_1 = (y_1)^* z_2$. *Obviously, it is not satisfied for* $y_1 = 1$ *and* $z_2 = i$.

**Theorem 2.14.** *If* $X = A^{[\dagger]} \in \mathbb{C}^{n\times n}$ *is a Moore-Penrose inverse of* $A \in \mathbb{C}^{n\times n}$ *then* $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ *is a Moore-Penrose inverse of* $A^{[*]}A$ *if and only if* ${X_2}^* H_1 y_1 = 0$ *for all* $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *which satisfies* ${A_2}^* H_1(A_1y_1 + A_2y_2) = 0$.

*Proof.* From Theorem 2.11, we already have that $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ is {1,2,(3)}- inverse of $A^{[*]}A$. Putting $X = A^{[\dagger]}$, we have

$$XX^{[*]}A^{[*]}A = XX^{[*]}\left\{\begin{pmatrix} y_1 \\ y_2 \\ A_1{}^{[*]}(A_1y_1 + A_2y_2) \\ z_2 \end{pmatrix} : A_2^*H_1(A_1y_1 + A_2y_2) = 0\right\} =$$

$$X\left\{\begin{pmatrix} y_1 \\ y_2 \\ X_1{}^{[*]}A_1{}^{[*]}(A_1y_1 + A_2y_2) \\ z_2 \end{pmatrix} : \begin{matrix} A_2^*H_1(A_1y_1 + A_2y_2) = 0 \\ X_2{}^*H_1A_1{}^{[*]}(A_1y_1 + A_2y_2) = 0 \end{matrix}\right\} =$$

$$X\left\{\begin{pmatrix} y_1 \\ y_2 \\ A_1y_1 + A_2y_2 \\ z_2 \end{pmatrix} : A_2^*H_1(A_1y_1 + A_2y_2) = 0\right\} = \left\{\begin{pmatrix} y_1 \\ y_2 \\ X_1(A_1y_1 + A_2y_2) + X_2z_2 \\ X_3(A_1y_1 + A_2y_2) + X_4z_2 \end{pmatrix} : A_2^*H_1(A_1y_1 + A_2y_2) = 0\right\}.$$

Here we use conditions from the Theorem 2.5: we have $X_2{}^*H_1A_1{}^{[*]}(A_1y_1+A_2y_2) = X_2^*A_1^*H_1(A_1y_1+A_2y_2) = -X_4^*A_2^*H_1(A_1y_1 + A_2y_2) = 0$, from the first condition.

Also, $X_1{}^{[*]}A_1{}^{[*]}(A_1y_1 + A_2y_2) = (A_1X_1)^{[*]}(A_1y_1 + A_2y_2) = (A_1X_1 + A_2X_3 - (A_2X_3)^{[*]})(A_1y_1 + A_2y_2) =$ $A_1X_1A_1y_1 + A_2X_3A_1y_1 + A_1X_1A_2y_2 + A_2X_3A_2y_2 =$ $(A_1X_1A_1 + A_2X_3A_1)y_1 + (A_1X_1A_2 + A_2X_3A_2)y_2 = A_1y_1 + A_2y_2$.

The necessary and sufficient condition for being the {(4)}-inverse (i.e. $XX^{[*]}A^{[*]}A \subseteq (XX^{[*]}A^{[*]}A)^{[*]}$ ) is: $(X_1(A_1x_1 + A_2x_2) + X_2\omega_2)^*H_1y_1 = x_1{}^*H_1(X_1(A_1y_1 + A_2y_2) + X_2z_2)$, when $A_2{}^*H_1(A_1y_1 + A_2y_2) = 0$ and $A_2{}^*H_1(A_1x_1 + A_2x_2) = 0$ for every $z_2$ and every $\omega_2$.

The last condition is equivalent to $x_1^*(X_1A_1)^*H_1y_1 + x_2^*(X_1A_2)^*H_1y_1 + \omega_2{}^*X_2^*H_1y_1 = x_1^*H_1X_1A_1y_1 + x_1^*H_1X_1A_2y_2 + x_1^*H_1X_2^*z_2$ for all $z_2$ and $\omega_2$ of appropriate sizes.

It is not difficult to see that the last one is satisfied if and only if $X_2{}^*H_1y_1 = X_2^*H_1x_1 = 0$ and $x_1^*(X_1A_1)^*H_1y_1 + x_2^*(X_1A_2)^*H_1y_1 = x_1^*H_1X_1A_1y_1 + x_1^*H_1X_1A_2y_2$.

Moreover, from Theorem 2.5. and $X_2{}^*H_1y_1 = X_2^*H_1x_1 = 0$ we have $x_1^*(X_1A_1)^*H_1y_1 = x_1^*H_1(X_1A_1)^{[*]}y_1 = x_1^*H_1X_1A_1y_1$, as in the proof of Theorem 2.12.

Now, $x_2^*(X_1A_2)^*H_1y_1 = -x_2^*(X_2A_4)^*H_1y_1 = -x_2^*A_4^*X_2^*H_1y_1 = 0$ and $x_1^*H_1X_1A_2y_2 = -x_1^*H_1X_2A_4y_2 = 0$.

So, under the condition that $X_2^*H_1y_1 = 0$ for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, such that $A_2^*H_1(A_1y_1 + A_2y_2) = 0$ we have that left and the right side of the previous equation are equal. Thus, we have that it is necessary and sufficient condition for $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ being a Moore-Penrose inverse of $A^{[*]}A$. □

**Theorem 2.15.** *If $X = A^{[\dagger]} \in \mathbb{C}^{n\times n}$ is a Moore-Penrose inverse of $A \in \mathbb{C}^{n\times n}$ then $(A^{[\dagger]})^{[*]}A^{[\dagger]}$ is a Moore-Penrose inverse of $AA^{[*]}$ if and only if $A_2{}^*H_1y_1 = 0$ for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ which satisfies $X_2{}^*H_1(X_1y_1 + X_2y_2) = 0$.*

*Proof.* Similar to the previous result. □

**Corollary 2.16.** *If $X = A^{[\dagger]} \in \mathbb{C}^{n\times n}$ is a Moore-Penrose inverse of $A \in \mathbb{C}^{n\times n}$ and $A_2 = 0$ and $X_2 = 0$ (equivalently to $A^{[*]}$ and $X^{[*]}$ have full domains), then $(A^{[*]}A)^{[\dagger]} = A^{[\dagger]}(A^{[\dagger]})^{[*]}$, and $(AA^{[*]})^{[\dagger]} = (A^{[\dagger]})^{[*]}A^{[\dagger]}$.*

*Proof.* Since $A_2 = X_2 = 0$ we have that $A_2^*H_1(A_1y_1 + A_2y_2) = 0$, $X_2^*H_1(X_1y_1 + X_2y_2) = 0$, $A_2^*H_1y_1 = 0$

and $X_2^* H_1 y_1 = 0$ for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, so by Theorem 2.14. and Theorem 2.15, $(A^{[*]}A)^{[\dagger]} = A^{[\dagger]}(A^{[\dagger]})^{[*]}$, and $(AA^{[*]})^{[\dagger]} = (A^{[\dagger]})^{[*]}A^{[\dagger]}$ hold. $\square$

The next example shows that $A^{[\dagger]} = (A^{[*]}A)^{[\dagger]}A^{[*]}$ and $A^{[\dagger]} = A^{[*]}(AA^{[*]})^{[\dagger]}$ does not hold.

**Example 2.17.** *Let* $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $A = I_2$. *Then* $A^{[\dagger]} = X = I_2 = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_1 \\ y_2 \end{pmatrix} \right\}$.

*Accordingly to Corollary 2.16, we have* $(A^{[*]}A)^{[\dagger]}A^{[*]} = A^{[\dagger]}(A^{[\dagger]})^{[*]}A^{[*]} = XX^{[*]}A^{[*]} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_1 \\ z_2 \end{pmatrix} \right\}$. *Also,* $A^{[*]}(AA^{[*]})^{[\dagger]} = A^{[*]}(A^{[\dagger]})^{[*]}A^{[\dagger]} = A^{[*]}X^{[*]}X = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_1 \\ z_2 \end{pmatrix} \right\}$. *Hence, we have* $(A^{[*]}A)^{[\dagger]}A^{[*]} \neq A^{[\dagger]}$ *and* $A^{[*]}(AA^{[*]})^{[\dagger]} \neq A^{[\dagger]}$.

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