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On Weighted Adjacency Operators Associated to Directed Graphs

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Abstract. The weighted adjacency operators associated to directed graphs are defined and some operator properties such as normality and hyponormality are investigated. As well, the weighted adjacency operators associated to a forested circuit are defined and their fundamental operator properties are studied. We produce some examples showing the properties of p-hyponormality and p-paranormality are distinct for $p \in (0, \infty)$. Basic connections between directed graphs and Hilbert space operators are discussed.

1. Introduction

Let $\mathcal{G} = (V, E)$ be a directed graph with vertex set V and edge set E. In [14] Mohar defined the spectrum of a locally finite countable graph and extended some known properties in spectral theory of finite graphs to infinite graphs. In [6] Fujii-Sasaoka-Watatani defined adjacency operators associated to infinite directed graphs and discussed some relations between graphs and bounded adjacency operators (cf. [4],[5],[7]). On the other hand, Jablonski-Jung-Stochel ([9]) introduced and investigated classes of weighted shifts on directed trees which generalize the classical weighted shifts. These operators provided some interesting examples and counterexamples concerning moment sequences and subnormal operators (cf. [10]). The structure of such operators has been well developed in recent years. In this paper we introduce a new notion of weighted adjacency operators associated to directed graphs and investigate some operator properties such as normality, p-hyponormality and p-paranormality, etc. These properties are sometimes referred to as weak hyponormalities. In particular, we consider FC operators which will be defined below. This model plays an important role for providing examples showing distinctions among the classes of p-hyponormality and p-paranormality with $p \in (0, \infty)$. Note that gaps between normal and weak hyponormal operators have been studied by several operator theorists (see [2],[8],[11],[12]). The underlying goal of this paper is to obtain a connection between directed graphs and Hilbert space operators.

The organization of this paper is as follows. In Section 2 we recall some terminology and notation concerning directed graphs and weak hyponormalities for operators which will be used throughout the paper. In Section 3 we construct a weighted adjacency operator $A_{\mathcal{G}}$ associated to a directed graph $\mathcal{G} = (V, E)$ in the unbounded case generally and investigate some fundamental properties. However, we mainly

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concentrate our consideration on the case of bounded operators for the study of such properties. In Section 4 the normality of $A_{\mathcal{G}}$ will be characterized, and we investigate some related properties in operator theory. In Section 5 some properties related to hyponormality are investigated. In Section 6 FC operators will be defined and studied. Finally, by using such properties we obtain some distinction examples for the classes of p-hyponormal and p-paranormal operators for $p \in (0, \infty)$.

2. Preliminaries and Notations

We first recall some fundamental notions in graph theory which will be used frequently in this paper. Let $\mathcal{G} = (V, E)$ be a directed graph with a vertex set V and an edge set $E \subset V \times V$. We define the following subsets of V by

$$N^+(u) = \{v \in V : (u, v) \in E\}, \quad N^-(u) = \{v \in V : (v, u) \in E\}$$

and

$$N^+(u, v) = N^+(u) \cap N^+(v), \quad N^-(u, v) = N^-(u) \cap N^-(v).$$

Moreover, for a set $W \subset V$, we set $N^+(W) = \underset{u \in W}{\cup} N^+(u)$ and $N^-(W) = \underset{u \in W}{\cup} N^-(u)$. In addition, we set

$$D^+(W) = \overset{\infty}{\underset{n=0}{\cup}} N^+_{\langle n \rangle}(W),$$

where

$$N^+_{\langle 0 \rangle}(W) = W, \quad N^+_{\langle 1 \rangle}(W) = N^+(W), \quad N^+_{\langle n+1 \rangle}(W) = N^+(N^+_{\langle n \rangle}(W)).$$

Similarly, we set

$$D^-(W) = \overset{\infty}{\underset{n=0}{\cup}} N^-_{\langle n \rangle}(W),$$

where

$$N^-_{\langle 0 \rangle}(W) = W, \quad N^-_{\langle 1 \rangle}(W) = N^-(W), \quad N^-_{\langle n+1 \rangle}(W) = N^-(N^-_{\langle n \rangle}(W)).$$

Note that if $\mathcal{G}$ is a directed tree (see below), $D^+(W)$ is the set of descendants of W and $D^-(W)$ is the set of ancestors of W. For each vertex $v \in V$, we let $d^+(v)$ denote the cardinality of $N^+(v)$. We write $d^+(u, v)$ for the cardinality of $N^+(u, v)$. The number $d^+(v)$ is referred to as the *outdegree* of v. Similarly, we use $d^-(v)$ for the cardinality of $N^-(v)$ and $d^-(u, v)$ for the cardinality of $N^-(u, v)$, and $d^-(v)$ is referred to as the *indegree* of v. And we denote the *valency* (or *degree*) of v by $d(v) = d^+(v) + d^-(v)$. A directed graph $\mathcal{G} = (V, E)$ has *bounded valency* if $\{d(v)\}_{v \in V}$ is bounded. A graph $\mathcal{G}$ is called *locally finite* if every vertex of $\mathcal{G}$ has finite valency. Set

$$\mathcal{G}_{\text{root}} := \{u \in V : N^-(u) = \emptyset\};$$

a member of $\mathcal{G}_{\text{root}}$ is called a *root* of $\mathcal{G}$. We write $V^\circ = V \setminus \mathcal{G}_{\text{root}}$ and $V' = \{u \in V : N^+(u) \neq \emptyset\}$. A vertex v of $\mathcal{G}$ is called a *leaf* if $v \notin V'$. For $n \geq 2$, a finite sequence $\{u_j\}_{j=1}^n$ of distinct vertices is said to be a *circuit* in $\mathcal{G}$ if $(u_j, u_{j+1}) \in E$, $1 \leq j \leq n-1$, and $(u_n, u_1) \in E$. A *loop* in $\mathcal{G}$ is an edge whose endpoints are equal. We set $\widetilde{E} = \{\{u, v\} \subseteq V : (u, v) \in E \text{ or } (v, u) \in E\}$. A directed graph $\mathcal{G}$ is *connected* if for any two distinct vertices u and v, there exists a finite sequence $\{v_j\}_{j=1}^n$ of vertices ($n \geq 2$) such that $u = v_1, \{v_j, v_{j+1}\} \in \widetilde{E}$, $1 \leq j \leq n-1$, and $v_n = v$. (See [3], [15] and [17] for more information on directed graphs and the definition of directed tree.)

Secondly we recall some terminologies concerning operators on a complex Hilbert space. Let A be an operator on a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$ the domain and the range of A. Let $B(\mathcal{H})$ be the usual Banach algebra of all bounded linear operators T with $\mathcal{D}(T) = \mathcal{H}$. For $X, Y \in B(\mathcal{H})$, we set $[X, Y] = XY - YX$. An operator $T \in B(\mathcal{H})$ is *normal* if $[T^*, T] = 0$, and $T \in B(\mathcal{H})$ is *hyponormal* if $[T^*, T]$ is positive, i.e., $[T^*, T] \geq 0$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. As well, T is said to be ∞-*hyponormal* if it is p-hyponormal for all $p > 0$ ([13]). According to the Löwner-Heinz inequality ([18],[8]), every q-hyponormal operator is p-hyponormal for $p \leq q$. Recall that $T \in B(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is the appropriate partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. We say T is *p-paranormal* if $\| |T|^p U |T|^p x\| \geq \| |T|^p x\|^2$ for all unit vectors $x \in \mathcal{H}$. Every q-paranormal operator is p-paranormal for $q \leq p$. It is obvious that every p-hyponormal operator is p-paranormal, $p \in (0, \infty)$. Hence any p-hyponormal operator is q-paranormal for all $q \in (0, \infty)$.

Let $\mathcal{G} = (V, E)$ be a directed graph. Let $\ell^2(V)$ be the Hilbert space of all square summable complex functions on V with the standard inner product

$$\langle f, g\rangle = \sum_{u \in V} f(u)\overline{g(u)}, \qquad f, g \in \ell^2(V).$$

For $u \in V$, we define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$. Denote by $\mathcal{E}_V$ the algebraic linear span of the set $\{e_u : u \in V\}$.

In the case of a disconnected directed graph $\mathcal{G}$, every weighted adjacency operator associated to $\mathcal{G}$ can be decomposed into the direct sum of weighted adjacency operators associated to some connected directed graphs. So we will consider connected directed graphs throughout this paper. We write $\mathbb{C}$ [$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$, $\mathbb{N}_0$, resp.] for the set of complex numbers [real numbers, positive real numbers, positive integers, nonnegative integers, resp.]. And $\mathbb{Z}$ [$\mathbb{Z}_+$, $\mathbb{Z}_-$, resp.] is the integer [positive integer, negative integer, resp.] set. For a subset $\mathcal{M}$ of a normed space $\mathcal{X}$, we let $\vee\mathcal{M}$ denote the closed linear span of $\mathcal{M}$.

3. Basic Construction

In this section we give a definition of the weighted adjacency operator $A_{\mathcal{G}}$ associated to a directed graph $\mathcal{G} = (V, E)$ and discuss the boundedness of $A_{\mathcal{G}}$. The construction of unweighted adjacency operators $A_{\mathcal{G}}$ was introduced in [6].

3.1. Definition of weighted adjacency operators. Let $\mathcal{G} = (V, E)$ be a directed graph with a vertex set V and edge set E. For a given set $\Lambda = \{\lambda_v\}_{v \in V^\circ}$ in $\mathbb{C}$, we define an operator $A := A_{\mathcal{G}}$ in $\ell^2(V)$ by

$$Af = \sum_{u \in V^\circ} \sum_{v \in N^-(u)} f(v)\lambda_u e_u \tag{3.1}$$

with

$$\mathcal{D}(A) = \{f = \sum_{v \in V} f(v)e_v \in \ell^2(V) : \sum_{u \in V^\circ} |\lambda_u|^2 \left| \sum_{v \in N^-(u)} f(v)\right|^2 < \infty\}. \tag{3.2}$$

(When $J = \emptyset$, we adopt the convention that $\sum_{\iota \in J} \theta_\iota = 0$. Also we adopt the convention that $0 \cdot \pm\infty = 0$.) This operator $A_{\mathcal{G}}$ is called the *weighted adjacency operator* associated to a directed graph $\mathcal{G} = (V, E)$. We will write the triple (V, E, Λ) for the weighted directed graph with weight set Λ, and abuse notation slightly so that $\mathcal{G}$ may stand for either (V, E) or (V, E, Λ) as convenient. And we define an operator $B := B_{\mathcal{G}}$ in $\ell^2(V)$ by

$$Bf = \sum_{u \in V'} \sum_{v \in N^+(u)} f(v)\overline{\lambda_v} e_u \tag{3.3}$$

with

$$\mathcal{D}(B) = \{f = \sum_{v\in V} f(v)e_v \in \ell^2(V) : \sum_{u\in V'} \Big|\sum_{v\in N^+(u)} f(v)\overline{\lambda_v}\Big|^2 < \infty\}. \tag{3.4}$$

It is easy to see that both $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are linear subspaces of $\ell^2(V)$. It follows from the definition of the operator A that e_u is in $\mathcal{D}(A)$ if and only if $\sum_{v\in N^+(u)} |\lambda_v|^2 < \infty$. If e_u is in $\mathcal{D}(A)$, then

$$Ae_u = \sum_{v\in N^+(u)} \lambda_v e_v \quad \text{and} \quad \|Ae_u\|^2 = \sum_{v\in N^+(u)} |\lambda_v|^2, \quad u \in V. \tag{3.5}$$

If e_u is in $\mathcal{D}(B)$, then

$$Be_u = \begin{cases} \overline{\lambda_u} \sum_{v\in N^-(u)} e_v & \text{if } u \in V^\circ, \\ 0 & \text{if } u \in \mathcal{G}_{\text{root}}, \end{cases} \tag{3.6}$$

and

$$\|Be_u\|^2 = |\lambda_u|^2 d^-(u), \quad u \in V^\circ.$$

In particular, if $u \in V$ is such that $\lambda_u \neq 0$, then e_u is in $\mathcal{D}(B)$ if and only if $d^-(u) < \infty$. Note that if $\sum_{v\in N^+(u)} |\lambda_v|^2 < \infty$ for all $u \in V$, A is densely defined. If $d^-(u) < \infty$ for all $u \in V$, B is densely defined. If $\mathcal{E}_V \subset \mathcal{D}(A)$, we let A_0 denote the operator with $\mathcal{D}(A_0) = \mathcal{E}_V$ defined by $A_0 f = Af$ for $f \in \mathcal{D}(A_0)$. If $\mathcal{E}_V \subset \mathcal{D}(B)$, we define B_0 with $\mathcal{D}(B_0) = \mathcal{E}_V$ similarly. Obviously, A_0 and B_0 are densely defined.

Proposition 3.1. *Let $A = A_{\mathcal{G}}$ and $B = B_{\mathcal{G}}$ be the operators associated to the graph $\mathcal{G} = (V, E, \Lambda)$ as above. Then the following assertions hold.*

(i) *If $\sum_{v\in N^+(u)} |\lambda_v|^2 < \infty$ for all $u \in V$, then $A^* \subset B = A_0^*$.*

(ii) *If $d^-(u) < \infty$ for all $u \in V$, then $B^* \subset A = B_0^*$.*

In each case of (i) *and* (ii), *A and B are closed. In particular, if $\mathcal{E}_V \subset \mathcal{D}(A) \cap \mathcal{D}(B)$, then $B_0 \subset \overline{B_0} \subset A^*$ and $A_0 \subset \overline{A_0} \subset B^*$.*

Proof. (i) It follows from (3.3) and (3.5) that $\langle A_0 e_u, g\rangle = \langle e_u, Bg\rangle$ for all $u \in V$ and $g \in \mathcal{D}(B)$, which is equivalent to the assertion that A_0 and B are formally adjoints of each other (cf. [16]). Hence $B \subset A_0^*$. To show that $B = A_0^*$, it is sufficient to see that $\mathcal{D}(A_0^*) \subset \mathcal{D}(B)$. Let $g \in \mathcal{D}(A_0^*)$. For $e_u \in \mathcal{D}(A_0)$,

$$\langle e_u, A_0^* g\rangle = \langle A_0 e_u, g\rangle = \sum_{v\in N^+(u)} \lambda_v \overline{g(v)}. \tag{3.7}$$

Since $e_u \in \mathcal{D}(A_0)$ for all $u \in V$, by (3.7), we get

$$\sum_{u\in V'} \Big|\sum_{v\in N^+(u)} g(v)\overline{\lambda_v}\Big|^2 = \sum_{u\in V'} \big|\langle A_0^* g, e_u\rangle\big|^2 = \big\|A_0^* g\big\|^2 < \infty.$$

So $g \in \mathcal{D}(B)$. Thus $B = A_0^*$.

(ii) By (3.1) and (3.6), B_0 and A are formally adjoints of each other. So $A \subset B_0^*$. To show that $A = B_0^*$, we claim that $\mathcal{D}(B_0^*) \subset \mathcal{D}(A)$. Let $f \in \mathcal{D}(B_0^*)$. By the definition of the domain of B_0^*, there exists a function $h_f \in \ell^2(V)$ such that

$$\langle f, B_0 g\rangle = \langle h_f, g\rangle, \quad g \in \mathcal{D}(B_0). \tag{3.8}$$

By (3.6) and (3.8), we have that

$$h_f(u) = \langle h_f, e_u \rangle = \langle f, \overline{\lambda_u} \sum_{v \in N^-(u)} e_v \rangle = \lambda_u \sum_{v \in N^-(u)} \langle f, e_v \rangle = \lambda_u \sum_{v \in N^-(u)} f(v). \tag{3.9}$$

Hence

$$\sum_{u \in V^\circ} |\lambda_u|^2 \left| \sum_{v \in N^-(u)} f(v) \right|^2 = \sum_{u \in V^\circ} \left| h_f(u) \right|^2 = \left\| h_f \right\|^2 < \infty.$$

Thus, $f \in \mathcal{D}(A)$. The remaining parts are routine. □

Note that if $\mathcal{G}$ is locally finite, then the hypotheses of (i) and (ii) of Proposition 3.1 that $\sum_{v \in N^+(u)} |\lambda_v|^2 < \infty$ and $d^-(u) < \infty$ for all $u \in V$, are satisfied. And recall that the closedness of weighted shifts associated to directed trees was obtained by a different proof (cf. [9, Prop. 3.1.2]; see [9] for more information concerning weighted shifts on directed trees).

3.2. Boundedness. Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph. If $A_{\mathcal{G}}$ is a bounded operator on $\mathcal{D}(A_{\mathcal{G}}) = \ell^2(V)$, then clearly $B_{\mathcal{G}} = A_{\mathcal{G}}^*$. In this subsection, we discuss the boundedness of $A_{\mathcal{G}}$.

Proposition 3.2. *Let $A = A_{\mathcal{G}}$ be the operator associated to a given weighted directed graph $\mathcal{G} = (V, E, \Lambda)$. Let us set constants as follows:*

$$\alpha := \alpha_{\mathcal{G}} = \sup_{u \in V} \left(\sum_{v \in N^+(u)} |\lambda_v|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \beta := \beta_{\mathcal{G}} = \sup_{v \in V} \left(\sum_{u \in N^+(v)} |\lambda_u|^2 d^-(u) \right)^{\frac{1}{2}} \tag{3.10}$$

as determined by the weights Λ. If $\beta < \infty$, then A is bounded. Moreover, we get $\alpha \leq \|A\| \leq \beta$.

Proof. According to Hölder's inequality, we have that

$$\left| \sum_{v \in N^-(u)} f(v) \right| \leq \sum_{v \in N^-(u)} \left| f(v) \right| \leq \left(\sum_{v \in N^-(u)} \left| f(v) \right|^2 \right)^{\frac{1}{2}} d^-(u)^{\frac{1}{2}}. \tag{3.11}$$

For any $f \in \mathcal{D}(A)$, by (3.11), we get

$$\begin{aligned} \left\| Af \right\|^2 &= \sum_{u \in V^\circ} |\lambda_u|^2 \left| \sum_{v \in N^-(u)} f(v) \right|^2 \leq \sum_{u \in V^\circ} \sum_{v \in N^-(u)} |\lambda_u|^2 d^-(u) \left| f(v) \right|^2 \\ &= \sum_{v \in V} \sum_{u \in N^+(v)} |\lambda_u|^2 d^-(u) \left| f(v) \right|^2 \\ &= \sum_{v \in V} \left| f(v) \right|^2 \sum_{u \in N^+(v)} |\lambda_u|^2 d^-(u) \\ &\leq \beta^2 \sum_{v \in V} \left| f(v) \right|^2 = \beta^2 \left\| f \right\|^2. \end{aligned} \tag{3.12}$$

On the other hand, if A is bounded, then

$$\|A\|^2 \geq \|Ae_u\|^2 = \sum_{v \in N^+(u)} |\lambda_v|^2, \qquad u \in V. \tag{3.13}$$

Taking the supremum over u in (3.13), we get $\|A\| \geq \alpha$. Hence the proof is complete. □

The following corollary comes immediately from the proof of Proposition 3.2.

Corollary 3.3. *Under the same notation as Proposition 3.2, suppose* $M := \left(\sup_{v\in V} d^-(v)\right)^{\frac{1}{2}} < \infty$. *Then A is bounded if and only if* $\alpha < \infty$. *In this case, we have that* $\alpha \leq \|A\| \leq M\alpha$.

Recall that if $A_{\mathcal{G}}$ is a bounded weighted shift associated to a directed tree with $\Lambda = \{\lambda_v\}_{v\in V^\circ}$ in $\mathbb{C}$ (that is, $d^-(v) \leq 1$ for all $v \in V$) then $\left\|A_{\mathcal{G}}\right\| = \alpha_{\mathcal{G}} = \beta_{\mathcal{G}}$ (cf. [9, Prop. 3.1.8]). And we give a weighted adjacency operator $A_{\mathcal{G}}$ satisfying $\alpha_{\mathcal{G}} < \|A_{\mathcal{G}}\| < \beta_{\mathcal{G}}$ as follows.

Example 3.4. We construct a weighted directed graph $\mathcal{G} = (V, E, \Lambda)$ in Figure 3.1 and a bounded weighted adjacency operator $A = A_{\mathcal{G}}$ associated to $\mathcal{G}$ such that $\alpha < \|A\| < \beta$, where $\alpha = \alpha_{\mathcal{G}}$ and $\beta = \beta_{\mathcal{G}}$ as in (3.10). Let us consider a vertex set $V = \{v_j\}_{j=0}^4$ and let $\Lambda = \{\lambda_{v_j}\}_{j=1}^4$ be weights with $\lambda_{v_1} = \frac{1}{2}$, $\lambda_{v_2} = \frac{1}{2}$, $\lambda_{v_3} = 2$ and $\lambda_{v_4} = 3$.

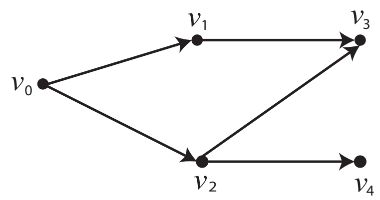

Figure 3.1

First we compute the bounds α and β of $\|A\|$. By considering their definitions in (3.10) we have $\alpha = \sqrt{13}$ and $\beta = \sqrt{17}$. Next, we claim $\alpha < \|A\| < \beta$. To compute $\|A\|$, we consider $f \in \ell^2(V)$ such that $\|f\| = 1$ and seek to maximize $\|Af\|$. And we set

$$x := f(v_0),\ y := f(v_1),\ z := f(v_2),\ a := f(v_3),\ b := f(v_4);$$

obviously,

$$|x|^2 + |y|^2 + |z|^2 + |a|^2 + |b|^2 = 1. \tag{3.14}$$

By a simple computation, we get

$$\begin{aligned}\|Af\|^2 &= |\lambda_{v_1}|^2|f(v_0)|^2 + |\lambda_{v_2}|^2|f(v_0)|^2 + |\lambda_{v_3}|^2|f(v_1) + f(v_2)|^2 + |\lambda_{v_4}|^2|f(v_2)|^2 \\ &= \frac{1}{2}|x|^2 + 4|y+z|^2 + 9|z|^2. \end{aligned} \tag{3.15}$$

To maximize $\|Af\|^2$, clearly we should take $a = f(v_3) = 0 = f(v_4) = b$. Consider now (3.15) subject to (3.14) but for fixed $|x|$ and $|z|$; clearly then $|y|$ is determined. To maximize $|y + z|$ for fixed $|y|$ and $|z|$, it suffices to take real numbers y and z (or of the same argument). To see this, let $y = r_1e^{i\theta_1}$ and $z = r_2e^{i\theta_2}$. Observe that maximizing $|y + z| = |r_1e^{i\theta_1} + r_2e^{i\theta_2}|$ is equivalent to maximizing $|r_1 + r_2e^{i(\theta_2-\theta_1)}| = |r_1 + r_2e^{i\theta'}|$, where $\theta' := \theta_2 - \theta_1$. But this is largest if $\theta' = 0$ for fixed r_1 and r_2.

Thus for fixed $|x|$ and $|z|$ the maximum occurs for $y = \gamma z$ with $\gamma > 0$. In this case (3.14) becomes $|x|^2 + (1 + \gamma^2)|z|^2 = 1$, and solving for $|x|^2$ and inserting into the right hand side of (3.15) it is easy to check that the expression is increasing in $|z|$, so in pursuit of a maximum, we should set $x = 0$. In this event we seek to maximize

$$\|Af\|^2 = 4|y+z|^2 + 9|z|^2$$

subject to $|y|^2 + |z|^2 = 1$. Since we know $y = \gamma z$ with $\gamma > 0$, we may as well assume that y and z are real. But it is well known that $\left|y+z\right|$ has maximum $\sqrt{2}$ at $y = z = \pm\frac{1}{\sqrt{2}}$ if $y^2 + z^2 = 1$. Then

$$\|Af\|^2 < 4(\sqrt{2})^2 + 9, \quad \|f\| \leq 1.$$

Hence $\|Af\|^2 < 17 = \beta^2$ for $\|f\| = 1$. A computation shows that with $z = \frac{9}{10}$ and $y = \sqrt{1-\left(\frac{9}{10}\right)^2}$, $\|Af\| > \sqrt{14} > \alpha$. So $\alpha < \|A\| < \beta$, as desired for our example.

For a weight set $\Lambda = \{\lambda_u\}_{u\in V^\circ} \subset \mathbb{C}\setminus\{0\}$, we set $\|\Lambda\|_\infty = \sup_{u\in V^\circ}|\lambda_u|$ and $\|\Lambda^*\|_\infty = \sup_{u\in V^\circ}\left|\lambda_u^{-1}\right|$. If $A_{\mathcal{G}}$ is bounded on $\ell^2(V)$, then $\|\Lambda\|_\infty < \infty$. For a directed graph $\mathcal{G} = (V, E)$ with bounded valency, we set $k^+ := \max_{v\in V} d^+(v)$ and $k^- := \max_{v\in V} d^-(v)$ for brevity. Then the norm of a weighted adjacency operator $A_{\mathcal{G}}$ can be estimated in a particular case as the following proposition; see [6, Th. 2] for the case of unweighted adjacency operators.

Proposition 3.5. *Let $A = A_{\mathcal{G}}$ be the bounded weighted adjacency operator on $\ell^2(V)$ associated to a weighted directed graph $\mathcal{G} = (V, E, \Lambda)$. Suppose Λ is bounded away from zero, i.e., $\|\Lambda^*\|_\infty < \infty$. Then $\mathcal{G}$ has bounded valency. Moreover, $\|A\| \le \|\Lambda\|_\infty \sqrt{k^+k^-}$. Further, if there exist sets $\{v_j\}_{j=1}^{k^-}$ and $\{u_j\}_{j=1}^{k^+}$ of vertices such that $(v_i, u_j) \in E$ and $|\lambda_{u_j}| = \|\Lambda\|_\infty$, $1 \le i \le k^-$, $1 \le j \le k^+$, then $\|A\| = \|\Lambda\|_\infty \sqrt{k^+k^-}$.*

Proof. Observe that

$$d^+(v)\,\|\Lambda^*\|_\infty^{-2} \le \sum_{w\in N^+(v)} |\lambda_w|^2 \le \|A\|^2,$$

i.e., $d^+(v) \le \|A\|^2\,\|\Lambda^*\|_\infty^2$ for all $v \in V$. Similarly, we get $d^-(v) \le \|A\|^2\,\|\Lambda^*\|_\infty^2$ for all $v \in V$. Hence $\mathcal{G}$ has bounded valency, which implies that $k^\pm < \infty$. Let $f \in \ell^2(V)$. Then it follows from the inequality (3.12) that

$$\|Af\|^2 \le \sum_{v\in V}|f(v)|^2 \sum_{u\in N^+(v)} |\lambda_u|^2 d^-(u) \le k^-k^+\,\|\Lambda\|_\infty^2\,\|f\|^2.$$

So $\|A\| \le \|\Lambda\|_\infty \sqrt{k^+k^-}$. If there exist sets $\{v_j\}_{j=1}^{k^-}$ and $\{u_j\}_{j=1}^{k^+}$ as above, then taking $f = \sum_{i=1}^{k^-} e_{v_i}$, we have

$$\|Af\|^2 = \sum_{i=1}^{k^-}\sum_{j=1}^{k^-}\sum_{w\in N^+(v_i,v_j)} |\lambda_w|^2 = k^-k^-k^+\,\|\Lambda\|_\infty^2 = k^-k^+\,\|\Lambda\|_\infty^2\,\|f\|^2.$$

Then $\|A\| \ge \sqrt{k^-k^+}\,\|\Lambda\|_\infty$. Hence the proof is complete. □

Note that it is easy to construct many examples of weighted directed graphs $\mathcal{G} = (V, E, \Lambda)$ satisfying $\|A\| = \|\Lambda\|_\infty \sqrt{k^+k^-}$.

4. Normality

Our goal of this section is to find a characterization of normality and some related properties of the weighted adjacency operator $A_{\mathcal{G}}$ associated to $\mathcal{G} = (V, E, \Lambda)$. We first give a characterization of normal weighted adjacency operators.

Proposition 4.1. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph. Let $A = A_{\mathcal{G}} \in B(\ell^2(V))$ be the weighted adjacency operator associated to $\mathcal{G}$. Then the following conditions are equivalent:*
(i) *A is normal,*
(ii) *$\langle A^*Ae_u, e_v\rangle = \langle AA^*e_u, e_v\rangle$ for all $u, v \in V$,*
(iii) *it holds that*

$$\sum_{w\in N^+(u,v)} |\lambda_w|^2 = \begin{cases} d^-(u,v)\overline{\lambda_u}\lambda_v & \text{if } u, v \in V^\circ, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. (i)⇔(ii) It is evident from the definitions.

(ii)⇔(iii) By a direct computation with (3.5) and (3.6), we have

$$\langle A^*Ae_u, e_v\rangle = \langle \sum_{w\in N^+(u)} \sum_{z\in N^-(w)} |\lambda_w|^2 e_z, e_v\rangle = \sum_{w\in N^+(u,v)} |\lambda_w|^2 \tag{4.2}$$

and

$$\begin{aligned}\langle AA^*e_u, e_v\rangle &= \begin{cases} \langle \sum\limits_{w\in N^-(u)} \sum\limits_{z\in N^+(w)} \overline{\lambda_u}\lambda_z e_z, e_v\rangle & \text{if } u, v \in V^\circ, \\ 0 & \text{otherwise,}\end{cases} \\ &= \begin{cases} d^-(u,v)\overline{\lambda_u}\lambda_v & \text{if } u, v \in V^\circ, \\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{4.3}$$

which prove this equivalence. □

In Proposition 4.1, if $\lambda_u = 1$ for all $u \in V^\circ$(the unweighted adjacency operator case), then $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal if and only if $\mathcal{G}$ is normally symmetric, i.e., $d^+(u,v) = d^-(u,v)$ for all $u, v \in V$ (cf. [6, p.731]). But this result is not preserved in the case of general weighted directed graphs, as we show next.

Example 4.2. Let $\mathcal{G} = (V, E)$ be the directed graph with vertex set $V = \mathbb{Z}_+ \cup \mathbb{Z}_- \cup \{0, 0^*\}$ and edge set $E = \{(i, i+1) : i \in \mathbb{Z}\} \cup \{(-1, 0^*), (0^*, 1)\}$; see Figure 4.1. Define the weight set $\Lambda = \{\lambda_v\}_{v\in V}$ by

$$\lambda_k = \begin{cases} 1, & k \in \{0, 0^*, 1\}, \\ \sqrt{2}, & \text{otherwise.}\end{cases}$$

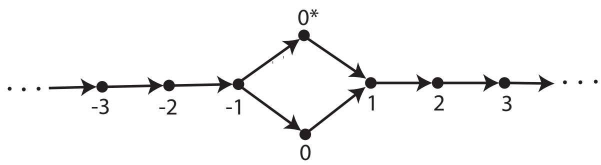

Figure 4.1

According to Proposition 4.1, the associated weighted adjacency operator $A_{\mathcal{G}}$ to $\mathcal{G}$ is normal. But $\mathcal{G}$ is not normally symmetric because $d^+(1) = 1$ and $d^-(1) = 2$.

The following lemma which will be used in Theorem 4.10 is elementary.

Lemma 4.3. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph. Suppose that the weighted adjacency operator $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal. Assume all λ_z are non-zero for all $z \in V^\circ$. Then for $u, v \in V$, $N^+(u,v) = \emptyset$ if and only if $N^-(u,v) = \emptyset$. If $N^+(u,v) \neq \emptyset$ for some $u, v \in V^\circ$, then $\overline{\lambda_u}\lambda_v \in \mathbb{R}_+$, and so $\arg(\lambda_u) = \arg(\lambda_v)$.*

Observe that if we allow zero weights, the weighted directed graph $\mathcal{G} = (V, E, \Lambda)$ associated with a normal weighted adjacency operator $A_{\mathcal{G}}$ can have only a trivial root or a root $v \in V$ such that all of vertices $w \in N^+(v)$ have zero weights, and can have only a trivial leaf or leaves with zero weights.

Notice that Lemma 4.3 can be improved slightly under some weak assumptions as in the following proposition, whose proof we also omit.

Proposition 4.4. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph. Suppose that the weighted adjacency operator $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal and that for all $u, v \in V^\circ$, there exist vertices $w_1, \cdots, w_n$ in V° such that $w_1 = u, w_n = v$ and $N^+(w_i, w_{i+1}) \neq \emptyset$, $1 \leq i \leq n-1$. Then $\arg(\lambda_u) = \arg(\lambda_v) =: \theta$ for all $u, v \in V^\circ$. Furthermore, if we set $c = e^{i\theta}$ and $r_u = \frac{1}{c}\lambda_u \in \mathbb{R}$ for all $u \in V^\circ$, then $A_{\mathcal{G}} = cA_{\mathcal{G}'}$, where $\mathcal{G}' = (V, E, \{r_u\}_{u\in V^\circ})$.*

If for some $u, v \in V^\circ$ there exist vertices $w_1, \cdots, w_n$ in V° such that $w_1 = u, w_n = v$ and $N^+(w_i, w_{i+1}) \neq \emptyset$, $1 \leq i \leq n-1$, we say that u and v are *step-related*. We say that $\mathcal{G}$ is step-related if every $u, v \in V^\circ$ are step-related.

Recall that $\mathcal{G} = (V, E)$ is *strongly connected* if for any two vertices v, u, there exists a finite sequence $w_1, \cdots, w_n$ of vertices such that $w_1 = v$, $w_n = u$ and $(w_j, w_{j+1}) \in E$ for all $j = 1, \cdots, n-1$. Such a sequence is called a *directed path* from v to u (cf. [3]). Note that the set of vertices u in V such that there exists a directed path from v to u is $D^+(\{v\})$, if we allow the trivial path from v to itself. For convenience we shorten $D^+(\{v\})$ to $D^+(v)$.

Proposition 4.5. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph with $d^-(u) = d^+(u)$ for all $u \in V$. Suppose $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal. Suppose also that there exists a vertex $u^* \in V^\circ$ such that $|\lambda_{u^*}| = \sup\{|\lambda_u| : u \in V^\circ\}$. Then $|\lambda_v| = |\lambda_{u^*}|$ for all $v \in D^+(u^*)$.*

Proof. We consider first the subset $N^+(u^*)$ to launch our argument in $D^+(u^*)$. Since A is normal, it follows from (4.1) that

$$d^+(u^*)|\lambda_{u^*}|^2 = d^-(u^*)|\lambda_{u^*}|^2 = \sum_{w \in N^+(u^*)} |\lambda_w|^2.$$

Since $|\lambda_{u^*}| = \sup\{|\lambda_u| : u \in V^\circ\}$, obviously $|\lambda_w| = |\lambda_{u^*}|$ for all $w \in N^+(u^*)$. Repeating the argument using such a w in $N^+(u^*)$ and members of $N^+(w)$, and so on, yields the claim. □

In the situation of Proposition 4.5, any vertex $w \in V^\circ$ such that $N^+(w) \subseteq D^+(u^*)$ also satisfies $|\lambda_w| = |\lambda_{u^*}|$. And the following example shows that the converse implication of Proposition 4.5 does not hold.

Example 4.6. Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph as in Figure 4.2 with weights $\lambda_{v_1} = i$, $\lambda_{v_2} = i$. By Proposition 4.1, it follows that the associated weighted adjacency operator $A_{\mathcal{G}}$ is normal.

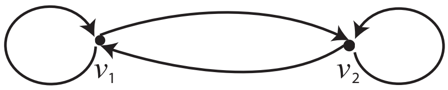

Figure 4.2

However, considering a weighted directed graph $\mathcal{G}' = (V, E, \Lambda')$ with weights $\Lambda' = \{\lambda_{v_1} = i, \lambda_{v_2} = 1\}$ using the same directed graph (V, E), Proposition 4.1 shows that the associated weighted adjacency operator $A_{\mathcal{G}'}$ is not normal.

Suppose that there exists $u^* \in V^\circ$ such that $|\lambda_{u^*}| = \sup\{|\lambda_u| : u \in V^\circ\}$ (that is, the sup is attained). Set

$$l = \sup\{|\lambda_u| : u \in V^\circ\}, \quad \mathcal{L} = \{u \in V^\circ : |\lambda_u| = l\}.$$

Then the conclusion of Proposition 4.5 is that $D^+(u^*) \subseteq \mathcal{L}$ (in other words, $|\lambda_v| = |\lambda_{u^*}| = l$ for all $v \in D^+(u^*)$).

Corollary 4.7. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph such that $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal, $l = \sup\{|\lambda_u| : u \in V^\circ\}$ is attained, and $d^-(u) = d^+(u)$ for all $u \in V$. Let $u, v \in V^\circ$ be such that $d^+(u, v) = d^-(u, v) \neq 0$ and $N^+(u, v) \subseteq \mathcal{L}$. Then $D^+(u) \cup D^+(v) \subseteq \mathcal{L}$.*

Proof. Note $|\lambda_z| = l$ for all $z \in N^+(u, v)$. This and the normality of $A_{\mathcal{G}}$ prove that

$$d^+(u, v)l^2 = \sum_{w \in N^+(u,v)} |\lambda_w|^2 = d^-(u, v)\overline{\lambda_u}\lambda_v.$$

Since $d^+(u, v) = d^-(u, v) \neq 0$, we get $l^2 = \overline{\lambda_u}\lambda_v$. Using the equality $l = \sup\{|\lambda_u| : u \in V^\circ\}$, we can obtain $|\lambda_u| = l = |\lambda_v|$ without difficulties. It follows from Proposition 4.5 that every vertex w in $D^+(u)$ and $D^+(v)$ has the same modulus l. □

Note that by Lemma 4.3, in the condition of Corollary 4.7 and assuming all λ_z are non-zero, $\arg(\lambda_u) = \arg(\lambda_v)$.

Corollary 4.8. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph such that $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal, $l = \sup\{|\lambda_u| : u \in V^\circ\}$ is attained, and $d^-(u) = d^+(u)$ for all $u \in V$. Suppose also that $\mathcal{G}$ is strongly connected. Then Λ has all weights of same magnitude*: $|\lambda_v| = l = \sup\{|\lambda_u| : u \in V^\circ\}$ for all $v \in V$.

Proof. Use the fact that the strong connectness of $\mathcal{G}$ implies $D^+(u^*) = V$ for any $u^* \in V$, in particular for a u^* such that $|\lambda_{u^*}| = l$. □

If in addition $\mathcal{G} = (V, E, \Lambda)$ is step-related as in Proposition 4.4, the only weighting on V yielding A normal has all weights of the form cK for some $K \geq 0$, where c is the constant as in Proposition 4.4 and K is the uniform magnitude in Corollary 4.8. If we assume all weights real and positive here, the only normal weightings have $\lambda_u \equiv K$ for some $K > 0$. In addition, we may remove the assumption of strongly connected in Corollary 4.8 by adding some additional hypotheses to obtain the same conclusion.

Corollary 4.9. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph such that $A_{\mathcal{G}} \in B(\ell^2(V))$ is normal, $l = \sup\{|\lambda_u| : u \in V^\circ\}$ is attained, and $d^-(u) = d^+(u) \neq 0$ for all $u \in V$. Suppose also that for all $w \in \mathcal{L}$, $\mathcal{L} \cap N^-(w) \neq \varnothing$, and that for all $u \in \mathcal{L}$ and for all $v \in V \backslash \mathcal{L}$, $d^+(u, v) = d^-(u, v)$. Then Λ has all weights of the same magnitude.*

Proof. Let $w \in V$. Choose some $u \in \mathcal{L}$. Since $\mathcal{G}$ is connected, there exist vertices $u_1 = u, u_2, \ldots, u_n = w$ such that $\{u_i, u_{i+1}\} \in \widetilde{E}$, for all $i = 1, \ldots, n-1$. We will show inductively that $u_i \in \mathcal{L}$ for all $i = 1, \ldots, n$. Consider u_2, and suppose for a contradiction that $u_2 \notin \mathcal{L}$. If $(u_1, u_2) \in E$, then $u_2 \in D^+(u_1) \subseteq \mathcal{L}$ by Proposition 4.5. If $(u_2, u_1) \in E$, then pick $v \in \mathcal{L}$ such that $v \in N^-(u_1)$ by our assumption. Since $N^+(v) \subseteq \mathcal{L}$ by Proposition 4.5, surely $N^+(u_2, v) \subseteq \mathcal{L}$. Also, with $v \in \mathcal{L}$ and $u_2 \in V \backslash \mathcal{L}$ we have $d^+(u_2, v) = d^-(u_2, v)$ by our assumption and neither is zero since $u_1 \in N^+(v, u_2)$. Arguing as in the proof of Corollary 4.7, we obtain $|\lambda_{u_2}| = l$ and $u_2 \in \mathcal{L}$. Thus $(u_1, u_2) \notin E$ and $(u_2, u_1) \notin E$, but this contradicts the construction of u_2. Therefore $u_2 \in \mathcal{L}$. Repeating the argument shows successively $u_i \in \mathcal{L}$ for all $i = 1, \ldots, n$. In particular, $w = u_n \in \mathcal{L}$. □

Note that if $\mathcal{L}$ is strongly connected and not a singleton set, it has the property that for all $w \in \mathcal{L}$, $\mathcal{L} \cap N^-(w) \neq \varnothing$. However if we let $\mathcal{L}$ be the directed graph with $V = \mathbb{Z}$ and $E = \{(i, i+1) : i \in \mathbb{Z}\}$, we see that $\mathcal{L}$ has this property but is not strongly connected.

If all weights in Λ are non-zero and have the same magnitude, we say that Λ has *positive uniform magnitude*. Under this condition, the matrix form of the normal weighted adjacency operator $A = A_{\mathcal{G}} \in B(\ell^2(V))$ can be represented as in (4.8) below. To do so, we first recall from Lemma 4.3 that $\mathcal{G}_{\text{root}} = \varnothing$ (when $V^\circ \neq \varnothing$), i.e., $V = V^\circ$. For a subset W of V, define

$$N(W) = N^+(W) \cup N^-(W) \text{ and, recursively,}$$
$$N_{\langle 0 \rangle}(W) = W, \; N_{\langle n+1 \rangle}(W) = N(N_{\langle n \rangle}(W)), \quad n \geq 0.$$

For brevity, we let $N_{\langle n \rangle}(u)$ denote $N_{\langle n \rangle}(\{u\})$. It follows from our standing assumption of the connectedness of $\mathcal{G}$ that for any fixed vertex $u \in V$

$$V = \cup_{n=0}^{\infty} N_{\langle n \rangle}(u). \tag{4.4}$$

Since A is bounded and Λ is the set of non-zero weights, by (3.5), (3.6) and [1, Cor. 19.5], $N_{\langle 1 \rangle}(u)$ is countable for any $u \in V$. So $N_{\langle n \rangle}(u)$ is countable for $u \in V$, and thus V is countable. For $u \in V$, we set

$$V_{\lambda_u} = \{v \in V : \lambda_v = \lambda_u\}. \tag{4.5}$$

Then we may write $V = \sqcup_{j \in \Gamma} V_{\lambda_j}$, a disjoint union of sets as in (4.5). For any $i \in \Gamma$ and $u \in V_{\lambda_i}$, suppose u has in arrows from V_{λ_j} and V_{λ_k} for $j \neq k$; in other words, there exist two vertices $v \in V_{\lambda_j}$ and $w \in V_{\lambda_k}$ such that $u \in N^+(v, w) \neq \varnothing$. Since A is normal, by Lemma 4.3 we have $\arg(\lambda_v) = \arg(\lambda_w)$. Since $\Lambda = \{\lambda_u\}_{u \in V^\circ}$ has positive uniform magnitude, we have $\lambda_j = \lambda_v = \lambda_w = \lambda_k$, which is a contradiction. Hence, any u in V_{λ_i} may have in arrows from at most one of the V_{λ_j} for $j \in \Gamma$. Now we let

$$V_{j,k} = \{w \in V_{\lambda_j} | w \text{ has in arrows from } V_{\lambda_k}\}, \quad j, k \in \Gamma. \tag{4.6}$$

Setting

$$\mathcal{M}_{j,k} = \vee\{e_u : u \in V_{j,k}\}, \tag{4.7}$$

$\ell^2(V)$ can be represented as $\oplus_{j\in\Gamma}(\oplus_{k\in\Gamma}\mathcal{M}_{j,k})$. For $e_w \in \mathcal{M}_{j,k}$,

$$\langle Ae_w, e_v\rangle = \langle \sum_{u\in N^+(w)} \lambda_u e_u, e_v\rangle = \begin{cases} \lambda_v, & \text{if } v \in N^+(w), \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the matrix of A has form as in (4.8) (since rows indicate "in arrows"). We write our conclusion below.

Theorem 4.10. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph with $V^\circ \neq \varnothing$ and let $A_{\mathcal{G}} \in B(\ell^2(V))$ be a normal weighted adjacency operator. Suppose that $\Lambda = \{\lambda_u\}_{u\in V^\circ}$ has positive uniform magnitude. Then the matrix form of $A_{\mathcal{G}}$ can be represented by a finite or infinite block matrix*

$$A_{\mathcal{G}} \cong \begin{array}{c} \begin{array}{cccccccccccc} \overbrace{\mathcal{M}_{1,1}} & \overbrace{\mathcal{M}_{1,2}} & \overbrace{\mathcal{M}_{1,3}} & \cdots & \overbrace{\mathcal{M}_{2,1}} & \overbrace{\mathcal{M}_{2,2}} & \overbrace{\mathcal{M}_{2,3}} & \cdots & \overbrace{\mathcal{M}_{3,1}} & \overbrace{\mathcal{M}_{3,2}} & \overbrace{\mathcal{M}_{3,3}} & \cdots \end{array} \\ \begin{pmatrix} * & * & * & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & * & * & * & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ * & * & * & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & * & * & * & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ * & * & * & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & * & * & * & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{4.8}$$

relative to the decomposition $\oplus_{j\in\Gamma}(\oplus_{k\in\Gamma}\mathcal{M}_{j,k})$, where "$*$" is some finite or infinite matrix (not all $*$ need be the same).

We investigate the matrix form (4.8) in Theorem 4.10 in a related example as follows.

Example 4.11. Let $\mathcal{G} = (V, E)$ be the directed graph with vertex set $V = \{(i, j) : i \in \mathbb{Z}, 1 \le j \le 4 - i(\text{mod } 2)\}$ and edge set

$$E = \{((i, k), (i + 1, l)) : i \in \mathbb{Z},\ 1 \le k \le 4 - i(\text{mod } 2),\ 1 \le l \le 4 - (i + 1)(\text{mod } 2)\};$$

see Figure 4.3. Define a weight set $\Lambda = \{\lambda_v\}_{v\in V}$ by

$$\lambda_v = \begin{cases} a, & i \text{ is odd,} \\ b, & i \text{ is even,} \end{cases} \qquad v = (i, j) \in V,$$

where $a, b \in \mathbb{C} \setminus \{0\}$.

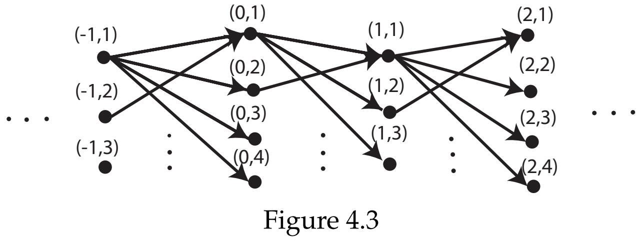

Figure 4.3

By Proposition 4.1, the associated weighted adjacency operator $A_{\mathcal{G}}$ is normal if and only if $|a| = |b|$, i.e., $\Lambda = \{\lambda_u\}_{u\in V}$ has positive uniform magnitude. Under the notation in the proof of Theorem 4.10, if we consider $\Gamma = \{(0,1),(1,1)\}$, then $V = V_{\lambda_{(0,1)}} \sqcup V_{\lambda_{(1,1)}}$, where

$$V_{\lambda_{(0,1)}} = \{(i,j) : i \text{ is even}, 1 \le j \le 4\} \text{ and } V_{\lambda_{(1,1)}} = \{(i,j) : i \text{ is odd}, 1 \le j \le 3\}$$

as in (4.5). Also, we may obtain easily that

$$V_{(0,1),(0,1)} = \varnothing, V_{(0,1),(1,1)} = V_{\lambda_{(0,1)}}, V_{(1,1),(0,1)} = V_{\lambda_{(1,1)}} \text{ and } V_{(1,1),(1,1)} = \varnothing$$

as in (4.6). Let us write $\widetilde{I}_{m\times n}$ for the $m \times n$ matrix whose entries are all 1 and write I_∞ for the countably infinite identity matrix. Consider some ordered bases of the subspaces $\mathcal{M}_{(0,1),(1,1)}$ and $\mathcal{M}_{(1,1),(0,1)}$ as in (4.7); for example, the lexicographic ordering on the subscripts of the included $e_{(i,j)}$, yielding

$$\cdots, e_{(-2,1)}, e_{(-2,2)}, e_{(-2,3)}, e_{(-2,4)}, \boxed{e_{(0,1)}, e_{(0,2)}, e_{(0,3)}, e_{(0,4)}}, e_{(2,1)}, e_{(2,2)}, e_{(2,3)}, e_{(2,4)}, \cdots,$$

and

$$\cdots, e_{(-1,1)}, e_{(-1,2)}, e_{(-1,3)}, \boxed{e_{(1,1)}, e_{(1,2)}, e_{(1,3)}}, e_{(3,1)}, e_{(3,2)}, e_{(3,3)}, \cdots,$$

respectively, where "□" indicates the center-block of the ordered basis. Then we obtain a block matrix expression for $A_{\mathcal{G}}$ of

$$A_{\mathcal{G}} \cong \begin{pmatrix} 0 & aI_\infty \otimes \widetilde{I}_{3\times 4} \\ bU \otimes \widetilde{I}_{4\times 3} & 0 \end{pmatrix}$$

relative to the decomposition $\mathcal{M}_{(1,1),(0,1)} \oplus \mathcal{M}_{(0,1),(1,1)}$, where U is the usual unweighted bilateral shift of multiplicity one.

We recapture the cardinality of the vertex set V of the weighted directed graph $\mathcal{G} = (V, E, \Lambda)$ below by a proof similar to that of part of Theorem 4.10.

Remark 4.12. Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph with non-zero weights $\Lambda = \{\lambda_u\}_{u\in V^\circ}$. Suppose $A_{\mathcal{G}}$ is a bounded weighted adjacency operator on $\ell^2(V)$. Under our standing assumption that $\mathcal{G}$ is connected, by (4.4), we can see that V is countable.

Finally, we note that while much of the content of the corollaries relates normality of $A_{\mathcal{G}}$ to positive uniform magnitude, Example 4.2 shows that normality alone is not sufficient to guarantee this.

5. Hyponormality

Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph. Recall that $A_{\mathcal{G}} \in B(\ell^2(V))$ is hyponormal if and only if $\|Af\|^2 \ge \|A^*f\|^2$ for all $f \in \ell^2(V)$, and using (3.1) and (3.3), this is

$$\sum_{u\in V'} \left| \sum_{v\in N^+(u)} f(v)\overline{\lambda_v} \right|^2 \le \sum_{u\in V^\circ} |\lambda_u|^2 \left| \sum_{v\in N^-(u)} f(v) \right|^2, \quad f \in \ell^2(V). \tag{5.1}$$

This inequality can be reformulated under some special conditions as follows.

Theorem 5.1. *Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph and let $A = A_{\mathcal{G}} \in B(\ell^2(V))$ be the weighted adjacency operator associated to the graph $\mathcal{G}$. Suppose $d^+(u, v) = 0$ for all $u, v \in V$ with $u \ne v$. Then the following assertions are equivalent:*

(i) *A is hyponormal,*

(ii) *for every $u \in V$, the following two conditions hold:*

1° *if* $v \in N^+(u)$ *and* $\|Ae_v\| = 0$, *then* $\lambda_v = 0$,

2° $\sum\limits_{v \in N^+_\Lambda(u)} \frac{|\lambda_v|^2}{\|Ae_v\|^2} \le 1$,

where $N^+_\Lambda(u) = \{w \in N^+(u) \mid \|Ae_w\| > 0\}$.

Proof. The proof of this theorem is essentially same as the proof of [9, Th. 5.1.2]. So we omit the details. □

The corresponding result in the case of p-hyponormality will be stated in Theorem 6.4. And we consider below some properties induced by hyponormality.

Proposition 5.2. *Let* $\mathcal{G} = (V, E, \Lambda)$ *be a weighted directed graph with* $V^\circ \neq \emptyset$ *and with all weights non-zero. Let* $A = A_{\mathcal{G}} \in B(\ell^2(V))$ *be a weighted adjacency operator associated to the graph* $\mathcal{G}$. *If* A *is hyponormal, then the following assertions hold.*

(i) $N^+(u) \neq \emptyset$ *for all* $u \in V$, *i.e.,* $\mathcal{G}$ *is leafless.*

(ii) *For all* $v \in V^\circ$ *such that* $d^+(v) \le d^-(v)$ *and* $|\lambda_v| = \sup\{|\lambda_w| : w \in V^\circ\}$, *we have* $|\lambda_z| = |\lambda_v|$ *for all* $z \in N^+(v)$ *and* $d^-(v) = d^+(v)$.

(iii) *If* $d^+(u) \le d^-(u)$ *for all* $u \in V^\circ$ *and* $v^* \in V^\circ$ *satisfies* $|\lambda_{v^*}| = \sup\{|\lambda_w| : w \in V^\circ\}$, *then* $|\lambda_z| = |\lambda_{v^*}|$ *and* $d^-(z) = d^+(z)$ *for all* $z \in D^+(v^*)$. *In particular, if* $\mathcal{G}$ *is strongly connected, we get* $|\lambda_w| = |\lambda_{v^*}|$ *and* $d^-(w) = d^+(w)$ *for all* $w \in V$.

Proof. (i) Suppose that there exists $u \in V$ such that $N^+(u) = \emptyset$. Then $u \in V^\circ$ (recall that we assume $\mathcal{G}$ is connected) and

$$0 \le \|A^* e_u\|^2 \le \|Ae_u\|^2 = 0.$$

But $\|A^* e_u\|^2 = |\lambda_u|^2 \, d^-(u)$, so $d^-(u) = 0$, which contradicts $u \in V^\circ$.

(ii) By the definition of hyponormality, (4.2) and (4.3), we obtain

$$\sum_{w \in N^+(v)} |\lambda_w|^2 \ge d^-(v) |\lambda_v|^2.$$

The hypothesis of (ii) implies the first inequality in the string

$$d^+(v) \, |\lambda_v|^2 \ge \sum_{w \in N^+(v)} |\lambda_w|^2 \ge d^-(v) \, |\lambda_v|^2 \, .$$

So $d^+(v) \ge d^-(v)$. Thus $d^+(v) = d^-(v)$ and $\sum\limits_{w \in N^+(v)} |\lambda_w|^2 = d^+(v) \, |\lambda_v|^2$. Therefore $|\lambda_z| = |\lambda_v|$ for all $z \in N^+(v)$.

(iii) This follows by repeating the argument of (ii) for each $u \in N^+(v^*)$, and then for $w \in N^+(u)$, and so on. Since $D^+(v^*) = V$, the case of "in particular" holds. □

Recall that $T \in B(\mathcal{H})$ is a *cohyponormal operator* if T^* is a hyponormal operator. So we have the following by changing slightly the conditions in Proposition 5.2.

Remark 5.3. Let $\mathcal{G} = (V, E, \Lambda)$ be a weighted directed graph with $V^\circ \neq \emptyset$ with all weights non-zero. If $A_{\mathcal{G}}$ is cohyponormal, $N^-(u) \neq \emptyset$ for all $u \in V$, i.e., $\mathcal{G}$ is rootless. Moreover, if we replace "+" ["sup", resp.] by "−" ["inf", resp.] in Proposition 5.2 (ii) and (iii), we obtain conclusions like those of Proposition 5.2.

Proposition 5.4. *Let* $\mathcal{G} = (V, E, \Lambda)$ *be a weighted directed graph with* $V^\circ \neq \emptyset$, *with all weights non-zero and such that* $d^-(u) = d^+(u)$ *for all* $u \in V$. *Suppose that* Λ *has uniform positive magnitude. Let* $A = A_{\mathcal{G}} \in B(\ell^2(V))$ *be the weighted adjacency operator associated to* $\mathcal{G}$. *Then the following conditions are equivalent:*

(i) *A is hyponormal,*

(ii) *A is cohyponormal,*

(iii) *A is normal.*

Proof. To show the implication (i)⇒(iii), we first suppose that A is hyponormal. Let δ be the positive uniform magnitude of Λ. Then

$$\langle [A^*,A]e_u, e_u\rangle = \sum_{w\in N^+(u)} |\lambda_w|^2 - d^-(u)|\lambda_u|^2 = d^+(u)\delta^2 - d^-(u)\delta^2 = 0, \quad u \in V. \tag{5.2}$$

For any $u, v \in V$ with $u \neq v$, consider $\mathcal{M} = \vee\{e_u, e_v\}$. By (5.2), we have

$$[A^*,A]|_{\mathcal{M}} \cong \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix},$$

where $a = \langle [A^*,A]|_{\mathcal{M}} e_v, e_u\rangle$. Also, by the positivity of $[A^*,A]|_{\mathcal{M}}$, we have $a = 0$. Since u and v are arbitrary, $[A^*,A] = 0$. Other cases are trivial or routine. Hence the proof is complete. □

Note that the weighted adjacency operator $A_{\mathcal{G}'}$ of Example 4.6 with the second weighting $\Lambda' = \{\lambda_{v_1} = i, \lambda_{v_2} = 1\}$ is not normal by Proposition 4.1. Hence by Proposition 5.4, $A_{\mathcal{G}'}$ is also neither cohyponormal nor hyponormal.

6. Modeling for Weak-Hyponormality

In this section we consider various weighted adjacency operators showing that the properties of p-hyponormality and p-paranormality are distinct.

Let $\mathcal{G} = (V, E)$ be a directed graph such that $d^+(u, v) = 0$ for any distinct vertices u and v in V. Then it is easy to see that $\mathcal{G}$ is either a directed tree or a directed graph consisting of a single circuit (or a single loop) and some attached directed trees as in Figure 6.1. Surely if $\mathcal{G}$ contains no loop or circuit, it is a directed tree. Suppose that $\mathcal{G}$

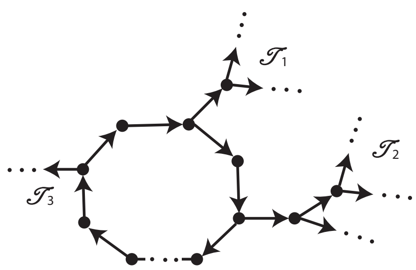

Figure 6.1

contains vertices $u_1, \ldots, u_n = u_1$ such that for all $i = 1, \ldots, n-1$, either $(u_i, u_{i+1}) \in E$ or $(u_{i+1}, u_i) \in E$. Using $d^+(u, v) = 0$ for any distinct vertices u and v in V, it is easy to show that either it is $(u_i, u_{i+1}) \in E$ for all $i = 1, \ldots, n-1$ or $(u_{i+1}, u_i) \in E$ for all $i = 1, \ldots, n-1$ (that is, this collection of vertices forms a circuit). Note that it is also possible that $\mathcal{G}$ contains a loop at a vertex, which we include as a trivial case of a circuit. If $\mathcal{G}$ contains two distinct circuits which share at least one vertex, a contradiction of $d^+(u, v) = 0$ results by walking backwards in each circuit until one finds a vertex w in both but such that $N^-(w)$ includes distinct vertices of C_1 and C_2. If $\mathcal{G}$ contains two circuits which do not intersect, say C_1 and C_2, under our standing assumption that $\mathcal{G}$ is connected for any $u_1 \in C_1$ and $v_1 \in C_2$ there exist $u_2, \ldots, u_{n-1}$ in V such that (with $u_n = v_1$) either $(u_i, u_{i+1}) \in E$ or $(u_{i+1}, u_i) \in E$, $i = 1, \ldots, n-1$. Choose $u_1 \in C_1$ and $v_1 \in C_2$ such that the length n of the finite sequence of vertices $u_1, \ldots, u_n$ is minimal. One shows readily that $(u_2, u_1) \notin E$ (else $N^-(u_1)$ has at least 2 elements), $(u_3, u_2) \notin E$, and so on, so $(u_1, u_2), (u_2, u_3), \ldots$, and (u_{n-1}, v_1) belong to E. Since $u_{n-1} \notin C_2$ from minimality of n, we contradict $d^+(u, v) = 0$ by considering $N^-(v_1)$. Thus $\mathcal{G}$ contains at most one circuit. Finally, suppose $\mathcal{G}$ contains a (single) circuit C. For each $u_1 \in C$, if $N^+(u_1)\backslash C = \emptyset$, there is no tree attached to C at u_1. If $N^+(u_1)\backslash C \neq \emptyset$, consider the subgraph of $\mathcal{G}$ consisting of u_1 and $D^+(N^+(u_1)\backslash C)$. This subgraph is connected (for u and v arbitrary in the subgraph just connect through u_1), satisfies $d^+(u, v) = 0$ for any distinct vertices u and v, and has no circuits. Therefore it is a directed tree attached to C at u_1.

Definition 6.1. A triple $\mathcal{G} = (V, E, \Lambda)$ is a *forested circuit* if $d^+(u, v) = 0$ for any distinct vertices u and v in V. We call an operator T an *FC operator* if it is the weighted adjacency operator associated to a forested circuit $\mathcal{G}$.

Let $\mathcal{G} = (V, E, \Lambda)$ be a forested circuit. Then obviously

$$V^\circ = \bigsqcup_{u\in V'} N^+(u), \tag{6.1}$$

which will be used to prove Theorem 6.4 and occurs in the proof of Lemma 6.2. For each $w \in V^\circ$ there exists a unique vertex $u \in V$ such that $N^-(w) = \{u\}$. In this case, we denote u by w^-. Observe the fact that if $A_\mathcal{G} \in B(\ell^2(V))$ is an FC operator associated to a weighted directed graph $\mathcal{G}$, then it follows from Proposition 3.2 that $\|A_\mathcal{G}\| = \alpha_\mathcal{G} = \beta_\mathcal{G}$, where $\alpha_\mathcal{G}$ and $\beta_\mathcal{G}$ are as in (3.10).

We begin our work with some fundamental properties of FC operators. *Throughout this section $A_\mathcal{G}$ is an FC operator unless otherwise indicated.*

6.1 Some fundamental properties. Let $A_\mathcal{G}$ be an FC operator associated to $\mathcal{G} = (V, E, \Lambda)$ with weights $\Lambda = \{\lambda_v\}_{v\in V^\circ}$ in $\mathbb{C}$.

Lemma 6.2. *Suppose $A = A_\mathcal{G}$ is densely defined. Then the following assertions hold.*
(i) $A^* = B_\mathcal{G}$.
(ii) $(A^*Af)(u) = \|Ae_u\|^2 f(u)$ *for* $u \in V$, $f \in \mathcal{D}(A^*A)$.
(iii) $\mathcal{E}_V \subseteq \mathcal{D}(A^*A)$ *and* $A^*Ae_u = \|Ae_u\|^2 e_u$ *for* $u \in V$.
(iv) $\mathcal{E}_V \subset \mathcal{D}(|A|^\alpha)$, $|A|^\alpha = \overline{|A|^\alpha|_{\mathcal{E}_V}}$ *and* $|A|^\alpha e_u = \|Ae_u\|^\alpha e_u$ *for* $u \in V$, $\alpha > 0$.
(v) $(|A|^\alpha f)(u) = \|Ae_u\|^\alpha f(u)$ *for* $f \in \mathcal{D}(|A|^\alpha)$, $u \in V$, $\alpha > 0$.

Proof. By proofs similar to those for Proposition 3.1.3 of [9], we have that A is densely defined if and only if $e_u \in \mathcal{D}(A)$ for all $u \in V$. It follows from direct computations that

$$A^*f(u) = \sum_{v\in N^+(u)} \overline{\lambda_v} f(v), \quad u \in V, \ f \in \mathcal{D}(A^*)$$

and

$$\mathcal{D}(A^*) = \{f = \sum_{v\in V} f(v)e_v \in \ell^2(V) : \sum_{u\in V'} \left| \sum_{v\in N^+(u)} f(v)\overline{\lambda_v} \right|^2 < \infty\}.$$

Since $\mathcal{D}(A^*) = \mathcal{D}(B_\mathcal{G})$, $A^* = B_\mathcal{G}$. The remaining parts also can be obtained as in [9]. We leave them to the interested readers. □

The following is essentially part of Proposition 3.5.1 of [9] whose proof can be generalized to a forested circuit, and we omit its proof.

Lemma 6.3 (Polar decomposition). *Suppose $A = A_\mathcal{G}$ is densely defined and let $A = U|A|$ be the polar decomposition of A. Then $|A|$ is the diagonal operator subordinated to the orthonormal basis $\{e_u\}_{u\in V}$ with diagonal elements $\{\|Ae_u\|\}_{u\in V}$, and U is the weighted adjacency operator associated to $\mathcal{G}$ with weights $\boldsymbol{\pi} = \{\pi_u\}_{u\in V^\circ}$ in $\mathbb{C}$ given by*

$$\pi_u = \begin{cases} \frac{\lambda_u}{\|Ae_{u^-}\|} & \text{if } u^- \in V_\Lambda^+, \\ 0 & \text{otherwise,} \end{cases} \qquad u \in V^\circ,$$

where $V_\Lambda^+ := \{u \in V : \|Ae_u\| > 0\}$. Moreover, the initial space of U equals $\overline{\mathcal{R}(|A|)} = \ell^2(V_\Lambda^+)$, $\mathcal{N}(U) = \mathcal{N}(A) = \mathcal{N}(|A|) = \ell^2(V\setminus V_\Lambda^+)$ and $\overline{\mathcal{R}(A^*)} = \ell^2(V_\Lambda^+)$.

6.2. Characterizations. We first give a characterization of the p-hyponormality and quasinormality of an FC operator.

Theorem 6.4. *Suppose $A = A_{\mathcal{G}} \in B(\ell^2(V))$ and let $p \in \mathbb{R}_+$. Then the following assertions hold:*
(i) *A is p-hyponormal if and only if it holds that*
1° *for every $u \in V$, if $v \in N^+(u)$ and $\|Ae_v\| = 0$, then $\lambda_v = 0$,*
2° $\|Ae_u\|^{2(p-1)} \sum_{v\in N^+_{\Lambda}(u)} \frac{|\lambda_v|^2}{\|Ae_v\|^{2p}} \leq 1, \quad u \in V^+_{\Lambda},$
(ii) *A is ∞-hyponormal if and only if it holds that*
1° *for every $u \in V$, if $v \in N^+(u)$ and $\|Ae_v\| = 0$, then $\lambda_v = 0$,*
2° $\sup_{p\in\mathbb{R}_+} \sum_{v\in N^+_{\Lambda}(u)} \frac{|\lambda_v|^2}{\|Ae_v\|^{2p}\|Ae_u\|^{2-2p}} \leq 1, \quad u \in V^+_{\Lambda},$
(iii) *A is quasinormal if and only if $\|Ae_u\| = \|Ae_v\|$ for all $u \in V$ and $v \in N^+(u)$ with $\lambda_v \neq 0$.*

Proof. By applying (6.1) and Lemmas 6.2 and 6.3, if we follow proofs of [9, Th. 8.2.1 and Prop. 8.1.7], we may prove (i) and (iii). We omit the details; (ii) is obvious from (i). □

Recall that $T \in B(\mathcal{H})$ is p-paranormal if and only if

$$\Psi_T(s) := |T|^p U^*|T|^{2p}U|T|^p - 2s|T|^{2p} + s^2 I \geq 0, \quad s > 0,$$

where U is the partially isometric part of the polar decomposition of T (cf. [19, Prop. 3]).

Theorem 6.5. *Let $A = A_{\mathcal{G}} \in B(\ell^2(V))$. Suppose $p \in (0, \infty)$. Then A is p-paranormal if and only if*

$$\sum_{v\in N^+(u)} |\lambda_v|^2\|Ae_v\|^{2p} \geq \|Ae_u\|^{2p+2}, \quad u \in V^+_{\Lambda}. \tag{6.2}$$

Proof. It follows from Lemma 6.3 that

$$U^*e_u = \begin{cases} \frac{\overline{\lambda_u}}{\|Ae_{u^-}\|}e_{u^-} & \text{if } u^- \in V^+_{\Lambda}, \\ 0 & \text{otherwise,} \end{cases} \qquad u \in V^\circ. \tag{6.3}$$

By some computations using Lemma 6.2(iv), Lemma 6.3 and (6.3), we obtain that

$$|A|^pU^*|A|^{2p}U|A|^p e_v = \begin{cases} \|Ae_v\|^{2p-2} \sum_{w\in N^+(v)} |\lambda_w|^2\|Ae_w\|^{2p} e_v & \text{if } v \in V^+_{\Lambda}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for $f = \sum_{v\in V} f(v)e_v \in \ell^2(V)$, we have

$$\langle \Psi_A(s)f, f\rangle = \sum_{v\in V^+_{\Lambda}} \Delta_v(s)|f(v)|^2 + \sum_{v\in V\setminus V^+_{\Lambda}} \left(-2s\|Ae_v\|^{2p} + s^2\right)|f(v)|^2, \quad s \in \mathbb{R}_+,$$

where

$$\Delta_v(s) = \|Ae_v\|^{2p-2} \sum_{w\in N^+(v)} |\lambda_w|^2\|Ae_w\|^{2p} - 2s\|Ae_v\|^{2p} + s^2.$$

Since $\|Ae_v\|^{2p} = 0$ for $v \in V\setminus V^+_{\Lambda}$, the inequality

$$\langle \Psi_A(s)f, f\rangle = \sum_{v\in V^+_{\Lambda}} \Delta_v(s)|f(v)|^2 + s^2 \sum_{v\in V\setminus V^+_{\Lambda}} |f(v)|^2 \geq 0, \quad f \in \ell^2(V),$$

is equivalent to $\Delta_v(s) \geq 0$ for all $v \in V^+_{\Lambda}$. Thus A is p-paranormal if and only if $\Delta_v(s) \geq 0$ for all $s > 0$ and all $v \in V^+_{\Lambda}$, which is equivalent to the condition (6.2). □

Note that the associated graph $\mathcal{G}$ of a paranormal FC operator $A_{\mathcal{G}}$ may have *leaves*; see the next subsection.

Remark 6.6. Recall that for each $p > 0$, $T \in B(\mathcal{H})$ is absolutely p-paranormal (i.e., $\left\| |T|^p Tf \right\| \geq \left\| Tf \right\|^{p+1}$ for all unit vectors f in $\mathcal{H}$) if and only if

$$\Omega_T(s) := T^*|T|^{2p}T - (p+1)s^p|T|^2 + ps^{p+1}I \geq 0, \quad s > 0,$$

(cf. [8, p.174]). Let $A = A_{\mathcal{G}} \in B(\ell^2(V))$ be an FC operator associated to $\mathcal{G} = (V, E, \Lambda)$ with weights $\Lambda = \{\lambda_v\}_{v \in V^\circ}$ in $\mathbb{C}$. Observe that for $f = \sum_{u \in V} f(u)e_u \in \ell^2(V)$, we have

$$\langle \Omega_A(s)f, f\rangle = \sum_{u \in V} |f(u)|^2 \left(\sum_{v \in N^+(u)} |\lambda_v|^2 \|Ae_v\|^{2p} - (p+1)s^p \|Ae_u\|^2 + ps^{p+1} \right) \geq 0,$$

which is equivalent to

$$\sum_{v \in N^+(u)} |\lambda_v|^2 \|Ae_v\|^{2p} - (p+1)s^p \|Ae_u\|^2 + ps^{p+1} \geq 0, \text{ for all } u \in V_\Lambda^+.$$

By some direct computations, A is absolutely p-paranormal if and only if A satisfies (6.2), i.e., A is p-paranormal. Some direct computations show as well that A is p-paranormal if and only if A is an $A(p)$-class operator, i.e., $(A^* |A|^{2p} A)^{1/(p+1)} \geq |A|^2$.

6.3. Examples for distinguishing operator classes. Let $\mathcal{G} = (V, E)$ be the directed graph with vertex set

$$V = \{u_0\} \cup \{u_{i,j} : i = 1, 2, \ j \in \mathbb{N}_0\} \cup \{u_{1,2^*}\}$$

and edge set

$$E = \{(u_{1,1}, u_{1,2^*})\} \cup \{(u_0, u_{2,0}), (u_{2,0}, u_{1,0}), (u_{1,0}, u_0)\} \cup \{(u_{i,j}, u_{i,j+1}) : i = 1, 2, \ j \in \mathbb{N}_0\};$$

see Figure 6.2.

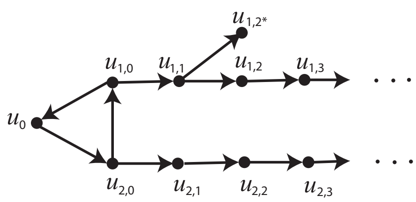

Figure 6.2

Define a weight set Λ on $\mathcal{G}$ by

$$\lambda_{u_0} = 1; \ \lambda_{u_{1,2^*}} = x,$$
$$\lambda_{u_{1,0}} = a, \ \lambda_{u_{1,1}} = b, \ \lambda_{u_{1,2}} = c, \ \lambda_{u_{1,j}} = d, \ \ j \geq 3,$$
$$\lambda_{u_{2,j}} = y, \ \ j \in \mathbb{N}_0,$$

where $a, b, c, d, x, y \in \mathbb{C}$. Let $A = A_{\mathcal{G}}$ be the weighted adjacency operator associated to the graph $\mathcal{G}$ with weights Λ (note that A is an FC operator). As noted following Definition 6.1, $\|A\| = \alpha_{\mathcal{G}} = \beta_{\mathcal{G}} = \sup_{u \in V} \left(\sum_{v \in N^+(u)} |\lambda_v|^2 \right)^{\frac{1}{2}}$. Hence A is bounded with

$$\|A\|^2 = \sup\{|a|^2 + |y|^2, |x|^2 + |c|^2, 1 + |b|^2, |d|^2\}.$$

And we have the following conditions for operator properties:

Normality. By Proposition 4.1, we can see that $\{(x, y, a, b, c, d) : A \text{ is normal}\} = \emptyset$.
p-**Hyponormality.** It follows from Theorem 6.4(i) that A is p-hyponormal for $p \in (0,\infty)$ if and only if the following conditions hold:
(Case 1) if $c = 0$, then
(i) $x = 0, b = 0, \left|y\right| \geq 1$,
(ii) $\left(|a|^2 + \left|y\right|^2\right)^{p-1}\left(|a|^2 + \left|y\right|^{2(1-p)}\right) \leq 1$;
and A is ∞-hyponormal if and only if $x = 0, b = 0, \left|y\right| \geq 1$ and

$$\sup_{p>0}\left(|a|^2 + \left|y\right|^2\right)^{p-1}\left(|a|^2 + \left|y\right|^{2(1-p)}\right) \leq 1;$$

(Case 2) if $c \neq 0$, then
(i) $x = 0,\ y \neq 0, |c| \leq |d|$,
(ii) $\Delta_1 := \left(1 + |b|^2\right)^{p-1}\left(\frac{|b|^2}{|c|^{2p}} + \frac{1}{\left|y\right|^{2p}}\right) \leq 1$,
(iii) $\Delta_2 := \left(|a|^2 + \left|y\right|^2\right)^{p-1}\left(\frac{|a|^2}{\left(1+|b|^2\right)^p} + \left|y\right|^{2(1-p)}\right) \leq 1$;
and A is ∞-hyponormal if and only if $x = 0,\ y \neq 0, |c| \leq |d|$, $\sup_{p>0}\Delta_i \leq 1$ $(i = 1, 2)$.
p-**Paranormality.** By a direct computation with Theorem 6.5, we have that A is p-paranormal if and only if the following conditions hold:
(i) $|b|^2\left(|x|^2 + |c|^2\right)^p + \left|y\right|^{2p} \geq \left(1 + |b|^2\right)^{p+1}$,
(ii) $|a|^2\left(1 + |b|^2\right)^p + \left|y\right|^{2(p+1)} \geq \left(|a|^2 + \left|y\right|^2\right)^{p+1}$,
(iii) $|c|^2|d|^{2p} \geq \left(|x|^2 + |c|^2\right)^{p+1}$.
Quasinormality. It follows from Theorem 6.4(iii) that A is quasinormal if and only if the following conditions hold:
(Case 1) if $b \neq 0$, then $a = 0 = x$ and $\left|y\right|^2 = |c|^2 = |d|^2 = 1 + |b|^2$,
(Case 2) if $b = 0$ and $c = 0$, then $a = 0 = x$ and $\left|y\right| = 1$,
(Case 3) if $b = 0$ and $c \neq 0$, then $a = 0 = x$ and $\left|y\right| = 1$ and $|c| = |d|$.
Examples for distinction. Considering $x = 0 = a, |b| = 1, |d| = \sqrt{2}$ and $c, y \in \mathbb{C} \setminus \{0\}$, we get
(i′) A is quasinormal $\Longleftrightarrow |c| = \left|y\right| = \sqrt{2}$;
(ii′) A is p-hyponormal $(p > 0) \Longleftrightarrow |c| \leq \sqrt{2}$ and

$$\left(\frac{2}{|c|^2}\right)^p + \left(\frac{2}{\left|y\right|^2}\right)^p \leq 2;$$

A is ∞-hyponormal $\Longleftrightarrow |c| \leq \sqrt{2}$ and $\sup_{p>0}\left(\frac{2}{|c|^2}\right)^p + \left(\frac{2}{\left|y\right|^2}\right)^p \leq 2$, which is equivalent to $|c| = \sqrt{2}$ and $\left|y\right| \geq \sqrt{2}$.
(iii′) A is p-paranormal $(p > 0) \Longleftrightarrow |c| \leq \sqrt{2}$ and $\left(\frac{|c|^2}{2}\right)^p + \left(\frac{\left|y\right|^2}{2}\right)^p \geq 2$. Furthermore, we have that

$$\begin{aligned}
\mathcal{H}_0 &:= \cup_{p>0}\{(c, y) : A \text{ is } p\text{-hyponormal}\}\\
&=\{(c, y) : 2 < \left|cy\right|,\ |c| \leq \sqrt{2}\} \cup \{(c, y) : |c| = |y| = \sqrt{2}\},\\
\mathcal{P}_0 &:= \cap_{p>0}\{(c, y) : A \text{ is } p\text{-paranormal}\}\\
&=\{(c, y) : 2 \leq \left|cy\right|,\ |c| \leq \sqrt{2}\},\\
\mathcal{P}_\infty &:= \cup_{p>0}\{(c, y) : A \text{ is } p\text{-paranormal}\}\\
&=\{(c, y) : \sqrt{2} < \left|y\right|,\ |c| \leq \sqrt{2}\} \cup \{(c, y) : |c| = |y| = \sqrt{2}\}.
\end{aligned}$$

Most inclusions above are straightforward; to show $\mathcal{P}_0 \subseteq \{(c,y) : 2 \le |cy|,\ |c| \le \sqrt{2}\}$, set $s := \frac{2}{|c|^2}$ and $t := \frac{2}{|y|^2}$, and insert (to find the boundary) into

$$\left(\frac{|c|^2}{2}\right)^p + \left(\frac{|y|^2}{2}\right)^p = 2.$$

Solve for t, and take the limit as $p \to 0$ to deduce $st = 1$ and therefore $|cy| = 2$. Finally we give Figure 6.3 with $s := \frac{2}{|c|^2}$ and $t := \frac{2}{|y|^2}$, which shows visually gaps among those classes mentioned above with respect to $p \in (0, \infty)$. Observe that

$$\mathcal{P}_0 \backslash \mathcal{H}_0 = \{(s,t) : st = 1, s > 1\} \text{ and } \mathcal{P}_\infty = \{(s,t) : s \ge 1, t < 1\} \cup \{(1,1)\}.$$

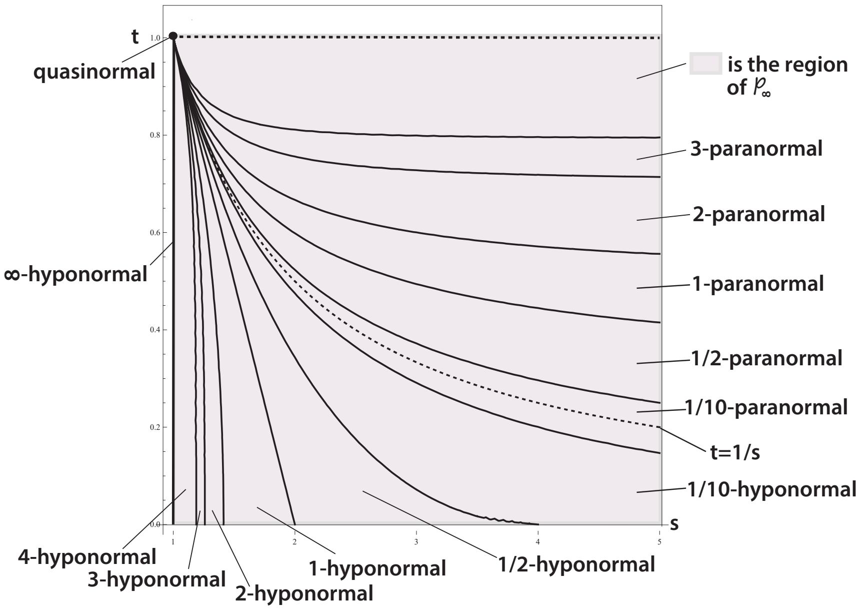

Figure 6.3

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