Filomat 31:13 (2017), 4261–4274
https://doi.org/10.2298/FIL1713261Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Stochastic Impulsive Fractional Differential Evolution Equations with Infinite Delay

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Abstract. In this paper, we investigate a class of stochastic impulsive fractional differential evolution equations with infinite delay in Banach space. Firstly, sufficient conditions of the existence and the uniqueness of the mild solution for this type of equations are derived by means of the successive approximation and the Bihari's inequality. Then we get the stability in mean square of the mild solution. Finally, an example is presented to illustrate the results.

1. Introduction

In resent years, the differential equations of fractional order have been widely studied by many authors (see e.g. [1, 13, 15, 20, 27, 29, 33]) due to their applications in many practical dynamical phenomena arising in engineering, physics, economy and science [4, 5, 9, 19, 23]. When we take environment noise and time delays into account (see. e.g. [2, 6, 8, 11, 12, 14, 18]), it is reasonable to consider the stochastic fractional evolution equations with delays. This paper is concerned with the existence, the uniqueness and the stability of mild solutions of Cauchy problems for the stochastic impulsive fractional evolution equations:

$$\begin{cases} D_t^{\alpha}[x'(t) - g(t, x_t)] = Ax(t) + f(t, x_t) + \sigma(t, x_t)\frac{\mathrm{d}w(t)}{\mathrm{d}t},\ t \in J,\ t \neq t_i, \\ \triangle x(t_i) = I_i(x_{t_i}),\ \triangle x'(t_i) = J_i(x_{t_i}),\ i = 1, 2, \dots, m, \\ x_0 = \varphi \in \mathcal{B},\ x_0' = x_1 \in H, \end{cases} \tag{1}$$

where $J = [0, b]$, $0 < \alpha < 1$, D_t^{α} denotes the Caputo fractional derivative operator of order α. $A : D(A) \subset H \to H$ is a sectorial operator. $g, f : J \times \mathcal{B} \to H$ and $\sigma : J \times \mathcal{B} \to \mathcal{L}(G, H)$ are appropriate mappings. Here $\mathcal{B}$ is an abstract phase space to be defined later. The history $x_t : (-\infty, 0] \to H$, $x_t(s) = x(t + s), s \leq 0$ belongs to the abstract phase space $\mathcal{B}$. Moreover, we denote $\triangle x(t_i) = x(t_i^+) - x(t_i^-)$ for $0 \leq t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ which are fixed numbers. Let $x(t_i^+)$ and $x(t_i^-)$ represent the right and the left limits of $x(t)$ at $t = t_i$, respectively. Similarly, $\triangle x'(t_i) = x'(t_i^+) - x'(t_i^-)$ has the same meaning.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. H, G be

2010 *Mathematics Subject Classification*. Primary 34K30; Secondary 60H30
Keywords. Stochastic impulsive fractional differential evolution equation; Mild solution; The Bihari's inequality; Successive approximation; Banach space.
Received: 19 October 2015; Accepted: 11 March 2016
Communicated by Svetlana Janković
Research supported by the National Science Foundation of China (No.11071050)
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two real separable Hilbert spaces and we denote by $< \cdot, \cdot >_H$, $< \cdot, \cdot >_G$ their inner products and by $|\cdot|_H$, $|\cdot|_G$ their vector norms, respectively. Let $\mathcal{L}(G, H)$ be the collection of all inner bounded operators from G into H, with the usual operator norm $\|\cdot\|$. The symbol $\{w(t), t \geq 0\}$ is a G-valued $\{\mathcal{F}_t\}_{t\geq 0}$ Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with covariance operator Q, i.e.

$$\mathbf{E} < w(t), x >_G < w(s), y >_G = (t \wedge s) < Qx, y >_G, \ \forall x, y \in G$$

where Q is a positive, self-adjoint and trace class operator on G. In particular, we regard $\{w(t), t \geq 0\}$ as a G-valued Q wiener process related to $\{\mathcal{F}_t\}_{t\geq 0}$ (see [2, 14]), and $w(t)$ is defined as

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \ t \geq 0,$$

where $\beta_n(t)\,(= 1, 2, 3, \ldots)$ is a sequence of real-valued standard Brownian motions mutually independent on the probability space $(\Omega, \mathcal{F}, P)$, λ_n, $n = 1, 2, 3, \ldots$ are the eigenvalues of Q and e_n, $n = 1, 2, 3, \ldots$ are the eigenvectors corresponding to λ_n. That is

$$Qe_n = \lambda_n e_n, \ n = 1, 2, 3, \ldots.$$

In order to define stochastic integrals with respect to the Q-wiener process $w(t)$, we introduce the subspace $G_0 = Q^{1/2}(G)$ of G with the inner product,

$$< u, v >_{G_0} = < Q^{1/2}u, Q^{1/2}v >_G \ .$$

It is easy to see that G_0 is a Hilbert space. Let $\mathcal{L}_2^0 = \mathcal{L}_2(G_0, H)$ denote the collection of all Hilbert-Schmidt operators from G_0 into H. It turns out to be a separable Hilbert space equipped with the norm

$$\|\psi\|^2_{\mathcal{L}_2^0} = tr\Big((\psi Q^{1/2})(\psi Q^{1/2})^*\Big), \quad \forall\, \psi \in \mathcal{L}_2^0.$$

Clearly, for any bounded operator $\psi \in \mathcal{L}(G, H)$, this norm reduces to $\|\psi\|^2_{\mathcal{L}_2^0} = tr(\psi Q \psi^*)$.

Let $\Phi : (0, \infty) \to \mathcal{L}_2^0$ be a predictable and $\mathcal{F}_t$-adapted process such that

$$\int_0^t \mathbf{E}\|\Phi(s)\|^2_{\mathcal{L}_2^0} \mathrm{d}s < \infty, \ \forall t > 0.$$

Then we can define the H-valued stochastic integral

$$\int_0^t \Phi(s)\mathrm{d}w(s),$$

which is a continuous square-integrable martingale ([24]). In the following, we assume $\sigma : J \times \mathcal{B} \to \mathcal{L}_2^0$ in (1).

Under the uniform Lipschitz and the linear growth conditions, Øksendal in [21] obtained sufficient conditions for the existence and the uniqueness of solutions of stochastic differential equations. By using the nonlinear alternative of Leray-Schauder type for multivalued maps and properties of the solution operator, Yan and Zhang [31] got sufficient conditions for the existence of solutions for the following impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay.

$$\begin{cases} \mathrm{d}D(t, x_t) \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s, x_s)\mathrm{d}s\mathrm{d}t + F(t, x_{\rho(t,x_t)})\mathrm{d}w(t), t \in J = [0, b], \ t \neq t_k, \ k = 1, 2, \ldots, m. \\ x_0 = \varphi \in \mathcal{B}, \\ \triangle x(t_k) = I_k(x_k), \ k = 1, 2, \ldots, m. \end{cases}$$

where $D(t, x_t) = \varphi(0) + G(t, \varphi)$, $\varphi \in \mathcal{B}$, ρ, I_k $(k = 1, 2, \ldots, m)$ are given functions. Very recently, Xie [30] established the sufficient conditions for the existence and the uniqueness of mild solutions of Cauchy problems for the impulsive fractional integro-differential evolution equations:

$$\begin{cases} D_t^{\alpha}[x'(t) - g(t, x_t)] = Ax(t) + f(t, x_t, Bx(t)),\ t \in J,\ t \neq t_i, \\ \triangle x(t_i) = I_i(x_{t_i}),\ \triangle x'(t_i) = J_i(x_{t_i}),\ i = 1, 2, \ldots, m, \\ x_0 = \varphi \in \mathcal{B},\ x_0' = x_1 \in H, \end{cases} \tag{2}$$

where $Bx(t) = \int_0^t k(t, s)x(s)\mathrm{d}s$, $k \in C(D, \mathbf{R}^+)$, $D = \{(t, s) : 0 \leq s \leq t \leq b\}$. By means of the Kuratowski measure of non-compactness and the progressive estimation method, the conclusions in [30] had improved and generalized many known results. More recently, Chadha and Pandey in [3] obtained sufficient conditions for the existence of mild solutions for an impulsive neutral stochastic fractional differential equation with infinite delays by using resolvent operator and the Krasnoselskii-Schaefer fixed point theorem.

However, so far we have not seen the existence and uniqueness results for the Cauchy problem (1). In this paper, we use the successive approximation and the Bihari's inequality to get the sufficient conditions for the existence and the uniqueness results of the mild solution for the Cauchy problem (1). Our method do not need mappings $g(t, \varphi)$, $f(t, \varphi)$ and $\sigma(t, \varphi)$ satisfy the Lipschitz conditions for the second variable which had been used in [3]. We have to point out that in the process of the existence and uniqueness result, we have mainly adopted the ideas appeared in [16, 17]. Furthermore, we give sufficient conditions that guarantee the stability in mean square of the mild solution.

2. Preliminaries

The collection of all strongly measurable, square-integrable and H-valued random variables, is denoted by $L_2(\Omega, H)$. $L_2(\Omega, H)$ with the norm $\|x(\cdot)\|_{L_2} = (\mathbf{E}|x(\cdot)|_H^2)^{1/2}$ is a Banach space. The expectation $\mathbf{E}$ is defined as $\mathbf{E}x = \int_\Omega x(\omega)\mathrm{d}P$. We introduce the space $\mathcal{PC}$ formed by all H-valued stochastic processes $\{x(t) : t \in [0, b]\}$ such that x is continuous at $t \neq t_k$, $x(t_k^-) = x(t_k)$ and $x(t_k^+)$ exist for all $k = 1, 2, \ldots, m$. When $\mathcal{PC}$ is endowed with the norm $\|x\|_{\mathcal{PC}} = (\sup_{s\in J} \mathbf{E}|x(s)|^2)^{1/2}$, $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space [32]. Next, we present an axiomatic definition of the phase space $\mathcal{B}$ introduced in [7, 8] and [26], where the axioms of the space $\mathcal{B}$ are established for $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into H, with a semi-norm $\|\cdot\|_{\mathcal{B}}$ which satisfies the following axioms.

(A1) If $x : (-\infty, b] \to H$, $b > 0$ is such that $x_0 \in \mathcal{B}$ and $x|_{[0,b]} \in \mathcal{PC}$, then, for every $t \in J$, the following conditions hold:
  (1) $x_t \in \mathcal{B}$,
  (2) $|x(t)| \leq L\|x_t\|_{\mathcal{B}}$,
  (3) $\|x_t\|_{\mathcal{B}} \leq \Gamma(t) \sup_{0\leq s\leq t} |x(s)| + N(t)\|x_0\|_{\mathcal{B}}$,
  where $L > 0$ is a constant; Γ, $N : [0, +\infty) \to [1, +\infty)$ are mappings. Γ is continuous and N is locally bounded. L, Γ, N are independent on $x(\cdot)$.

(A2) The space $\mathcal{B}$ is complete.

Then we have the following useful lemma (see [26]).

Lemma 2.1. *Let $x : (-\infty, b] \to H$ be an $\mathcal{F}_t$ adapted measurable process such that the $\mathcal{F}_0$ adapted process $x_0 = \varphi \in L_2(\Omega, \mathcal{B})$, then*

$$\mathbf{E}\|x_s\|_{\mathcal{B}} \leq N_b E\|\varphi\|_{\mathcal{B}} + \Gamma_b \mathbf{E}(\sup_{0\leq s\leq b} |x(s)|), \tag{3}$$

where $N_b = \sup_{t\in J}\{N(t)\}$ and $\Gamma_b = \sup_{t\in J}\{\Gamma(t)\}$.

Denote by $\mathcal{M}^2((-\infty, b], H)$ be the space of all H-valued càdlàg measurable processes $x = \{x(t)\}_{-\infty<t\leq b}$ such that

(i) $x_0 = \varphi \in \mathcal{B}$ and $x(t)$ is $\mathcal{F}_t$-adapted on $[0, b]$;
(ii) endow the space $\mathcal{M}^2((-\infty, b], H)$ with the norm

$$\|x\|^2_{\mathcal{M}^2} = \mathbf{E}\|\varphi\|^2_{\mathcal{B}} + \mathbf{E}(\sup_{t\in J} |x(t)|^2) < \infty. \tag{4}$$

Then $\mathcal{M}^2((-\infty, b], H)$ with the norm (4) is a Banach space, in the following of this paper, we use $\|\cdot\|$ for this norm.

Definition 2.2. *A stochastic process $x(t) : t \in (-\infty, b] \to H$ is called a mild solution of (1) if*

(i) $x(t)$ is measurable and $x(t)$ is $\mathcal{F}_t$-adapted for $t \in [0, b]$, and x_t $(t \in [0, b])$ is $\mathcal{B}$-valued;
(ii) $\int_0^b \|x(s)\|^2 \mathrm{d}s < \infty$, P-a.s.;
(iii) $x(t)$ has càdlàg path on $t \in [0, b]$ a.s. and $x(t)$ satisfies the following integral equation for each $t \in [0, b]$,

$$\begin{aligned} x(t) &= S_q(t)\varphi(0) + \int_0^t S_q(s)[x_1 - g(0,\varphi)]\mathrm{d}s + \sum_{t_i<t} S_q(t-t_i)I_i(x_{t_i}) \\ &\quad + \sum_{t_i<t}\int_{t_i}^t S_q(t-s)[J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})]\mathrm{d}s \\ &\quad + \int_0^t S_q(t-s)g(s,x_s)\mathrm{d}s + \int_0^t T_q(t-s)f(s,x_s)\mathrm{d}s + \int_0^t T_q(t-s)\sigma(s,x_s)\mathrm{d}w(s), \end{aligned} \tag{5}$$

where $S_q(t)$, $T_q(t) : \mathbf{R}_+ \to \mathcal{L}(H, H)$ $(q = 1 + \alpha)$ are given by

$$S_q(t) = E_{q,1}(At^q) = \frac{1}{2\pi i}\int_{B_r} \frac{e^{\lambda t}\lambda^{q-1}}{\lambda^q - A}\mathrm{d}\lambda, \tag{6}$$

$$T_q(t) = t^{q-1}E_{q,q}(At^q) = \frac{1}{2\pi i}\int_{B_r} \frac{e^{\lambda t}}{\lambda^q - A}\mathrm{d}\lambda, \tag{7}$$

and B_r denotes the Bromwich path [30];
(iv) $x_0 = \varphi \in \mathcal{B}$.

Remark 2.3. *We should mention an important property of $S_\gamma(t)$ and $T_\gamma(t)$, that is there exist positive numbers M and M_b such that $\|S_\gamma(t)\|_{\mathcal{L}(H,H)} \le M$ and $\|T_\gamma(t)\|_{\mathcal{L}(H,H)} \le t^{\gamma-1}M_b$ for $t \in J$, $\gamma \in (0, 2)$ ([15]), which plays an important role in the following discussion.*

Lemma 2.4. *(Bihari's inequality) Assume $T > 0$, $u_0 \ge 0$ and $u(t)$, $v(t)$ be continuous functions on $[0, T]$. Let $\kappa : \mathbf{R}_+ \to \mathbf{R}_+$ be a concave continuous and nondecreasing function such that $\kappa(r) > 0$ for all $r > 0$. If*

$$u(t) \le u_0 + \int_0^t v(s)\kappa(u(s))\mathrm{d}s \quad \text{for all} \quad 0 \le t \le T,$$

then

$$u(t) \le G^{-1}(G(u_0) + \int_0^t v(s)\mathrm{d}s)$$

and for all $t \in [0, T]$, it holds that

$$G(u_0) + \int_0^t v(s)\mathrm{d}s \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{\mathrm{d}s}{\kappa(s)}$, $r \ge 0$ and G^{-1} is the inverse function of G. In particular, if $u_0 = 0$ and $\int_{0^+} \frac{\mathrm{d}s}{\kappa(s)} = \infty$, then $u(t) = 0$ for all $0 \le t \le T$.

Lemma 2.5. *([25]) Let the assumption of Lemma 2.4 hold and $v(t) \ge 0$ for all $t \in [0, T]$. If for all $\epsilon > 0$, there exists $t_1 \ge 0$ for all $0 \le u_0 \le \epsilon$, $\int_{t_1}^T v(s)\mathrm{d}s \le \int_{u_0}^{\epsilon} \frac{1}{\kappa(s)}\mathrm{d}s$ holds. Then for every $t \in [t_1, T]$, the estimates $u(t) \le \epsilon$ holds.*

3. Existence of the Mild Solution

In this section, we first make the following hypotheses.

(H1) $g, f : J \times \mathcal{B} \to H$ and $\sigma : J \times \mathcal{B} \to \mathcal{L}_2^0$ satisfy
$|g(t,\varphi) - g(t,\phi)|^2 \vee |f(t,\varphi) - f(t,\phi)|^2 \vee \|\sigma(t,\varphi) - \sigma(t,\phi)\|^2_{\mathcal{L}_2^0} \le \kappa(\|\varphi - \phi\|^2_{\mathcal{B}})$, for all $t \in J$ and $\varphi, \phi \in \mathcal{B}$, where $\kappa(\cdot)$ is a concave, nondecreasing and continuous function from $\mathbf{R}^+$ to $\mathbf{R}^+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}s}{\kappa(s)} = \infty$.

(H2) $I_k, J_k : \mathcal{B} \to H$ are continuous and there are positive constants p_k, q_k such that for each $\varphi, \phi \in \mathcal{B}$,

$$|I_k(\varphi) - I_k(\phi)|^2 \le p_k\|\varphi - \phi\|^2_{\mathcal{B}},\ |J_k(\varphi) - J_k(\phi)|^2 \le q_k\|\varphi - \phi\|^2_{\mathcal{B}}\ (k = 1, 2, \dots, m).$$

(H3) $|g(t,0)|^2 \vee |f(t,0)|^2 \vee \|\sigma(t,0)\|^2_{\mathcal{L}_2^0} \le K$, K is a positive constant, and $I_k(0) = 0$, $J_k(0) = 0$ $(k = 1, 2, \dots, m)$.

We consider the sequence of successive approximations defined as follows:

$$x^0(t) = S_q(t)\varphi(0) + \int_0^t S_q(s)[x_1 - g(0,\varphi)]\mathrm{d}s,\ t \in J, \tag{8}$$

$$\begin{aligned} x^n(t) \;=\;& S_q(t)\varphi(0) + \int_0^t S_q(s)[x_1 - g(0,\varphi)]\mathrm{d}s + \sum_{t_i<t} S_q(t - t_i)I_i(x^{n-1}_{t_i}) \\ &+ \sum_{t_i<t}\int_{t_i}^t S_q(t-s)[J_i(x^{n-1}_{t_i}) - g(t_i, x^{n-1}_{t_i} + I_i(x^{n-1}_{t_i})) + g(t_i, x^{n-1}_{t_i})]\mathrm{d}s + \int_0^t S_q(t-s)g(s, x^{n-1}_s)\mathrm{d}s \\ &+ \int_0^t T_q(t-s)f(s, x^{n-1}_s)\mathrm{d}s + \int_0^t T_q(t-s)\sigma(s, x^{n-1}_s)\mathrm{d}w(s),\ t \in J, n \ge 1, \end{aligned} \tag{9}$$

$$x^n(t) = \varphi(t),\ -\infty < t \le 0, n \ge 1. \tag{10}$$

Lemma 3.1. *Assume the (H1)-(H3) hold, and* $7mM^2\Gamma_b\sum_{i=1}^m p_i + 14mM^2b^2\Gamma_b\sum_{i=1}^m q_i < 1$, *then* $x^n(t) \in \mathcal{M}^2((-\infty, b]; H)$ *for all* $t \in (-\infty, b]$, $n \ge 0$, *that is*

$$\mathbf{E}\|x^n(t)\|^2 \le \tilde{M},\ n = 1, 2 \dots. \tag{11}$$

where $\tilde{M}$ *is a positive constant.*

Proof. In what follows, we will use K to stand for generic positive real constants and its values may change between occurrences. Obviously, $x^0(t) \in \mathcal{M}^2((-\infty, b], H)$ and

$$\begin{aligned} \mathbf{E}|x^n(t)|^2 \;\le\;& 7\mathbf{E}|S_q(t)\varphi(0)|^2 + 7\mathbf{E}|\int_0^t S_q(s)[x_1 - g(0,\varphi)]\mathrm{d}s|^2 + 7\mathbf{E}|\sum_{t_i<t} S_q(t-t_i)I_i(x^{n-1}_{t_i})|^2 \\ &+7\mathbf{E}|\sum_{t_i<t}\int_{t_i}^t S_q(t-s)[J_i(x^{n-1}_{t_i}) - g(t_i, x^{n-1}_{t_i} + I_i(x^{n-1}_{t_i})) + g(t_i, x^{n-1}_{t_i})]\mathrm{d}s|^2 \\ &+7\mathbf{E}|\int_0^t S_q(t-s)g(s, x^{n-1}_s)\mathrm{d}s|^2 + 7\mathbf{E}|\int_0^t T_q(t-s)f(s, x^{n-1}_s)\mathrm{d}s)|^2 \\ &+7\mathbf{E}|\int_0^t T_q(t-s)\sigma(s, x^{n-1}_s)\mathrm{d}w(s)|^2 \\ =\;& \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5 + \Lambda_6 + \Lambda_7. \end{aligned}$$

It's easy to get the estimations $\Lambda_1 \le 7M^2\mathbf{E}|\varphi(0)|^2$, $\Lambda_2 \le 21M^2b^2(|x_1|^2 + \kappa(\|\varphi\|^2_{\mathcal{B}}) + K)$, and

$$\Lambda_3 \le 7mM^2\sum_{t_i<t}\mathbf{E}\|I_i(x^{n-1}_{t_i})\|^2_{\mathcal{B}} \le 7mM^2\sum_{t_i<t} p_i\mathbf{E}\|x^{n-1}_{t_i}\|_{\mathcal{B}}.$$

By the fact $\|S_q(t)\|_{\mathcal{L}(H,H)} \leq M$ and (H2)-(H3), we have

$$\begin{aligned}
\Lambda_4 &\leq 14\mathbf{E}|\sum_{t_i<t}\int_{t_i}^{t} S_q(t-s)J_i(x^n_{t_i})\mathrm{d}s|^2 + 14\mathbf{E}|\sum_{t_i<t}\int_{t_i}^{t} S_q(t-s)(g(t_i, x^{n-1}_{t_i} + I_i(x^{n-1}_{t_i})) - g(t_i, x^{n-1}_{t_i}))\mathrm{d}s|^2 \\
&\leq 14mM^2b\sum_{t_i<t}\mathbf{E}\int_{t_i}^{t}|J_i(x_{t_i})|^2\mathrm{d}s + 14mM^2b\sum_{t_i<t}\mathbf{E}\int_{t_i}^{t}\kappa(|I(x^{n-1}_{t_i})|^2)\mathrm{d}s \\
&\leq 14mM^2b^2\sum_{t_i<t}q_i\mathbf{E}\|x^{n-1}_{t_i}\|^2_{\mathcal{B}} + 14mM^2b\sum_{t_i<t}\int_{t_i}^{t}\kappa(\mathbf{E}(p_i\|x^{n-1}_{t_i})\|^2_{\mathcal{B}})\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_5 &\leq 7\mathbf{E}|\int_0^t S_q(t-s)g(s, x^{n-1}_s)\mathrm{d}s|^2 \leq 7M^2b\mathbf{E}\int_0^t |g(s, x^{n-1}_s) - g(s, 0) + g(s, 0)|^2\mathrm{d}s \\
&\leq 14M^2b\mathbf{E}\int_0^t [|g(s, x^{n-1}_s) - g(s, 0)|^2 + |g(s, 0)|^2]\mathrm{d}s \\
&\leq 14M^2b\int_0^t \kappa(\mathbf{E}\|x^{n-1}_s\|^2_{\mathcal{B}})\mathrm{d}s + 14M^2b^2K.
\end{aligned}$$

Since $\|T_q(t)\|_{\mathcal{L}(H,H)} \leq t^{\gamma-1}M_b$ and (H1)-(H3), we get the following inequality

$$\begin{aligned}
\Lambda_6 &\leq 7\mathbf{E}|\int_0^t T_q(t-s)f(s, x^{n-1}_s)\mathrm{d}s|^2 \leq 7M_b^2\frac{b^{2q-1}}{2q-1}\mathbf{E}\int_0^t |f(s, x^{n-1}_s) - f(s, 0) + f(s, 0)|^2\mathrm{d}s \\
&\leq 14M_b^2\frac{b^{2q-1}}{2q-1}\int_0^t \kappa(\mathbf{E}\|x^{n-1}_s\|^2_{\mathcal{B}})\mathrm{d}s + 14M_b^2\frac{b^{2q}}{2q-1}K.
\end{aligned}$$

We apply the Hölder inequality and the B-D-G inequality to Λ_7, combining(H1)-(H3), we can obtain

$$\begin{aligned}
\Lambda_7 &\leq 7\mathbf{E}|\int_0^t T_q(t-s)\sigma(s, x^{n-1}_s)\mathrm{d}w(s)|^2 \\
&\leq 7M_b^2b^{2q-2}\mathbf{E}\int_0^t \|\sigma(s, x^{n-1}_s) - \sigma(s, 0) + \sigma(s, 0)\|^2_{\mathcal{L}_2^0}\mathrm{d}s \\
&\leq 14M_b^2b^{2q-2}\int_0^t \kappa(\mathbf{E}\|x^{n-1}_s\|^2_{\mathcal{B}})\mathrm{d}s + 14M_b^2b^{2q-1}K.
\end{aligned}$$

Let

$$c_1 = 7M^2\mathbf{E}|\varphi(0)|^2 + 21M^2b^2(|x_1|^2 + \kappa(\|\varphi\|^2_{\mathcal{B}}) + K) + 14M^2b^2K + 14M_b^2\frac{b^{2q}}{2q-1}K + 14M_b^2b^{2q-1},$$

the estimations for $\Lambda_i,\ i = 1, 2, \ldots, 7$, together yields

$$\begin{aligned}
\mathbf{E}|x^n(t)|^2 \quad &\leq c_1 + 7mM^2\sum_{t_i<t}p_i\mathbf{E}\|x^{n-1}_{t_i}\|_{\mathcal{B}} + 14mM^2b^2\sum_{t_i<t}q_i\mathbf{E}\|x^{n-1}_{t_i}\|^2_{\mathcal{B}} + 14mM^2b\sum_{t_i<t}\int_{t_i}^{t}\kappa(\mathbf{E}(p_i\|x^{n-1}_{t_i})\|^2_{\mathcal{B}})\mathrm{d}s \\
&\quad +(14M^2b + 14M_b^2\frac{b^{2q-1}}{2q-1} + 14M_b^2b^{2q-2})\int_0^t \kappa(\mathbf{E}\|x^{n-1}_s\|^2_{\mathcal{B}})\mathrm{d}s.
\end{aligned}$$

By Lemma 2.1 and the property of $\kappa(\cdot)$, we can find a pair of positive constants α and β, such that $\kappa(u) \leq \alpha+\beta u$, $\forall\ u \geq 0$. Then

$$\begin{aligned}
\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2 \quad &\leq c_1 + 7mM^2N_b\sum_{t_i<t}p_i\mathbf{E}\|\varphi\|_{\mathcal{B}} + 14mM^2b^2\sum_{t_i<t}p_i\mathbf{E}\|\varphi\|_{\mathcal{B}} + 14m^2M^2b^2\alpha \\
&+(14M^2b + 14M_b^2\frac{b^{2q-1}}{2q-1} + 14M_b^2b^{2q-2})b\alpha \\
&+(7mM^2\Gamma_b\sum_{t_i<t}p_i + 14mM^2b^2\Gamma_b\sum_{t_i<t}q_i)\mathbf{E}\sup_{0\leq s\leq t}|x^{n-1}(s)|^2 \\
&+(14mM^2b\sum_{t_i<t}p_i + 14M^2b + 14M_b^2\frac{b^{2q-1}}{2q-1} + 14M_b^2b^{2q-2})\beta\mathbf{E}\int_0^t\sup_{0\leq\theta\leq s}|x^{n-1}(\theta)|ds,
\end{aligned}$$

and

$$\begin{aligned}
\max_{1\leq n\leq\tilde{k}}\{\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2\} \quad \leq \quad & c_1 + 7mM^2N_b\sum_{t_i<t}p_i\mathbf{E}\|\varphi\|_{\mathcal{B}} + 14mM^2b^2\sum_{t_i<t}p_i\mathbf{E}\|\varphi\|_{\mathcal{B}} + 14m^2M^2b\alpha \\
&+(14M^2b + 14M_b^2\frac{b^{2q-1}}{2q-1} + 14M_b^2\frac{b^{2q-1}}{2q-1})b\alpha \\
&+(7mM^2\Gamma_b\sum_{t_i<t}p_i + 14mM^2b^2\Gamma_b\sum_{t_i<t}q_i)\max_{1\leq n\leq\tilde{k}}\{\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2\} \\
&+(14mM^2b\sum_{t_i<t}p_i + 14M^2b + 14M_b^2\frac{b^{2q-1}}{2q-1} + 14M_b^2\frac{b^{2q-1}}{2q-1})\beta\int_0^t\max_{1\leq n\leq\tilde{k}}\{\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2\}ds,
\end{aligned}$$

where $\tilde{k}$ is an arbitrary positive integer. If we let

$$\begin{aligned}
c_2 \quad = \quad & \frac{c_1 + 7mM^2N_b\sum_{t_i<t}p_i\mathbf{E}\|\varphi\|_{\mathcal{B}} + 14mM^2b^2\sum_{t_i<t}p_i\mathbf{E}\|\varphi\|_{\mathcal{B}}}{1 - 7mM^2\Gamma_b\sum_{t_i<t}p_i - 14mM^2b^2\Gamma_b\sum_{t_i<t}q_i} \\
&+\frac{14m^2M^2b\alpha + (14M^2b^2 + 14M_b^2b^{2q} + 14M_b^2b^{2q})\alpha}{1 - 7mM^2\Gamma_b\sum_{t_i<t}p_i - 14mM^2b^2\Gamma_b\sum_{t_i<t}q_i}
\end{aligned}$$

$$c_3 = \frac{(14mM^2b\sum_{t_i<t}p_i + 14M^2b + 14M_b^2\frac{b^{2q-1}}{2q-1} + 14M_b^2b^{2q-2})\beta}{1 - 7mM^2\Gamma_b\sum_{t_i<t}p_i - 14mM^2b^2\Gamma_b\sum_{t_i<t}q_i},$$

then

$$\max_{1\leq n\leq\tilde{k}}\{\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2\} \leq c_2 + c_3\int_0^t\max_{1\leq n\leq\tilde{k}}\{\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2\}ds. \tag{12}$$

By the Gronwall inequality, we have

$$\max_{1\leq n\leq\tilde{k}}\{\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2\} \leq c_2e^{c_3}.$$

Due to the arbitrary of $\tilde{k}$, we have

$$\mathbf{E}\sup_{0\leq s\leq t}|x^n(s)|^2 \leq c_2e^{c_3} = M', \text{ for all } 0 \leq t \leq b,\ n \geq 1.$$

Consequently,

$$\|x^n(t)\|^2 \leq \mathbf{E}\|\varphi\|_{\mathcal{B}}^2 + \mathbf{E}(\sup_{0\leq s\leq b}|x^n(s)|^2) \leq \mathbf{E}\|\varphi\|_{\mathcal{B}}^2 + M' < \infty, \tag{13}$$

so we can take $\tilde{M} = \mathbf{E}\|\varphi\|_{\mathcal{B}}^2 + M'$. This completes the proof of Lemma 3.1.

Theorem 3.2. *If (H1)-(H3) and*

$$\max\{7mM^2\Gamma_b\sum_{i=1}^{m}p_i+14mM^2b^2\Gamma_b\sum_{i=1}^{m}q_i,\ 7mM^2\sum_{i=1}^{m}p_i+7mM^2b\sum_{i=1}^{m}q_i\}<1 \tag{14}$$

hold, then the Cauchy problem (1) has a unique mild solution on $(-\infty,b]$.

Proof. Since

$$\begin{aligned}
|x^{n+m}(t)-x^n(t)|^2 \;=\;& |\sum_{t_i<t}S_q(t-t_i)[I_i(x_{t_i}^{m+n-1})-I_i(x_{t_i}^{n-1})]+\sum_{t_i<t}\int_{t_i}^{t}S_q(t-s)[J(x_{t_i}^{m+n-1})-J(x_{t_i}^{n-1})]\mathrm{d}s\\
&-\sum_{t_i<t}\int_{t_i}^{t}S_q(t-s)[g(t_i,x_{t_i}^{m+n-1}+I_i(x_{t_i}^{m+n-1}))-g(t_i,x_{t_i}^{n-1}+I_i(x_{t_i}^{n-1}))\\
&+g(t_i,x_{t_i}^{m+n-1})-g(t_i,x_{t_i}^{n-1})]\mathrm{d}s\\
&+\int_0^t S_q(t-s)[g(s,x_s^{n+m-1})-g(s,x_s^{n-1})]\mathrm{d}s+\int_0^t T_q(t-s)[f(s,x_s^{n+m-1})-f(s,x_s^{n-1})]\mathrm{d}s\\
&+\int_0^t T_q(t-s)[\sigma(s,x_s^{n+m-1})-\sigma(s,x_s^{n-1})]\mathrm{d}w(s)|^2.
\end{aligned}$$

By the fact $\|S_q(t)\|_{\mathcal{L}(H,H)}\le M$, $\|T_q(t)\|_{\mathcal{L}(H,H)}\le t^{q-1}M_b$ for $t\in J$ and (H1)-(H3), we get

$$\begin{aligned}
\mathbf{E}|x^{n+m}(t)-x^n(t)|^2 \;\le\;& 7\mathbf{E}|\sum_{t_i<t}S_q(t-t_i)[I_i(x_{t_i}^{m+n-1})-I_i(x_{t_i}^{n-1})]|^2+7\mathbf{E}|\sum_{t_i<t}\int_{t_i}^{t}S_q(t-s)[J(x_{t_i}^{m+n-1})-J(x_{t_i}^{n-1})]\mathrm{d}s|^2\\
&+7\mathbf{E}|\sum_{t_i<t}\int_{t_i}^{t}S_q(t-s)[g(t_i,x_{t_i}^{m+n-1}+I_i(x_{t_i}^{m+n-1}))-g(t_i,x_{t_i}^{n-1}+I_i(x_{t_i}^{n-1}))]\mathrm{d}s|^2\\
&+7\mathbf{E}|\sum_{t_i<t}\int_{t_i}^{t}S_q(t-s)[g(t_i,x_{t_i}^{m+n-1})-g(t_i,x_{t_i}^{n-1})]\mathrm{d}s|^2\\
&+7\mathbf{E}|\int_0^t S_q(t-s)[g(s,x_s^{n+m-1})-g(s,x_s^{n-1})]\mathrm{d}s|^2\\
&+7\mathbf{E}|\int_0^t T_q(t-s)[f(s,x_s^{n+m-1})-f(s,x_s^{n-1})]\mathrm{d}s|^2\\
&+7\mathbf{E}|\int_0^t T_q(t-s)[\sigma(s,x_s^{n+m-1})-\sigma(s,x_s^{n-1})]\mathrm{d}w(s)|^2\\
\le\;& 7mM^2\sum_{t_i<t}p_i\mathbf{E}\sup_{0\le s\le t}|x^{m+n-1}(s)-x^{n-1}(s)|^2+7mM^2b\sum_{t_i<t}q_i\mathbf{E}\sup_{0\le s\le t}|x^{m+n-1}(s)-x^{n-1}(s)|^2\\
&+7mM^2\sum_{t_i<t}\int_{t_i}^{t}\kappa((1+p_i)\mathbf{E}\sup_{0\le s\le t_i}|x^{m+n-1}(s)-x^{n-1}(s)|^2)\mathrm{d}s\\
&+7mM^2\sum_{t_i<t}\int_{t_i}^{t}\kappa(\mathbf{E}\sup_{0\le s\le t_i}|x^{m+n-1}(s)-x^{n-1}(s)|^2)\mathrm{d}s\\
&+7M^2b\int_0^t\kappa(\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s\\
&+7M_b^2\frac{b^{2q-1}}{2q-1}\int_0^t\kappa(\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s\\
&+7M_b^2b^{2q-2}\int_0^t\kappa(\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= (7mM^2\sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i)\mathbf{E}\sup_{0\le s\le t}|x^{m+n-1}(s)-x^{n-1}(s)|^2\\
&\quad +7mM^2\sum_{t_i<t}\int_{t_i}^t \kappa((1+p_i)\mathbf{E}\sup_{0\le r\le t_i}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s\\
&\quad +(7m^2M^2+7M^2b+7M_b^2\frac{b^{2q-1}}{2q-1}+7M_b^2b^{2q})\int_0^t \kappa(\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s.
\end{aligned}
$$

Let $\bar{p} = \max_{1\le i\le m} p_i$. Since

$$
\begin{aligned}
&\int_{t_i}^t \kappa\Big((1+p_i)\mathbf{E}\sup_{0\le r\le t_i}|x^{m+n-1}(r)-x^{n-1}(r)|^2\Big)\mathrm{d}s\\
&\le \int_{t_i}^t \kappa\Big((1+p_i)\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2\Big)\mathrm{d}s\\
&\le \int_0^t \kappa\Big((1+\bar{p})\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2\Big)\mathrm{d}s
\end{aligned}
\tag{15}
$$

and $\tilde{\kappa}\circ a(\cdot) = \kappa(a(\cdot))$ is also a concave function, we get

$$
\begin{aligned}
&\mathbf{E}\sup_{0\le s\le t}|x^{n+m}(s)-x^{n}(s)|^2\\
&\le (7mM^2\sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i)\mathbf{E}\sup_{0\le s\le t}|x^{m+n-1}(s)-x^{n-1}(s)|^2 \qquad (16)\\
&\quad +(14m^2M^2+7M^2b+7M_b^2\frac{b^{2q-1}}{2q-1}+7M_b^2b^{2q-2})\int_0^t \tilde{\kappa}(\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s.
\end{aligned}
$$

It is easy to see

$$
\begin{aligned}
&\mathbf{E}\sup_{0\le s\le t}|x^{n+m}(s)-x^{n}(s)|^2 - (7mM^2\sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i)\mathbf{E}\sup_{0\le s\le t}|x^{m+n-1}(s)-x^{n-1}(s)|^2 \qquad (17)\\
&\le (14m^2M^2+7M^2b+7M_b^2\frac{b^{2q-1}}{2q-1}+7M_b^2b^{2q-2})\int_0^t \tilde{\kappa}(\mathbf{E}\sup_{0\le r\le s}|x^{m+n-1}(r)-x^{n-1}(r)|^2)\mathrm{d}s.
\end{aligned}
$$

From Lemma 3.1, we get

$$
\begin{aligned}
&\mathbf{E}\sup_{0\le s\le t}|x^{n+m}(s)-x^{n}(s)|^2 - (7mM^2\sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i)\mathbf{E}\sup_{0\le s\le t}|x^{m+n-1}(s)-x^{n-1}(s)|^2\\
&\le (14m^2M^2+7M^2b+7M_b^2\frac{b^{2q-1}}{2q-1}+7M_b^2b^{2q-2})\int_0^t \tilde{\kappa}(2M')\mathrm{d}s\\
&\le c_4\tilde{\kappa}(2M')t = c_5 t.
\end{aligned}
\tag{18}
$$

Define

$$
\varphi_1(t) = c_5 t,\ \varphi_{n+1}(t) = c_4\int_0^t \tilde{\kappa}(\varphi_n(s))\mathrm{d}s,\ n\ge 1. \tag{19}
$$

Choose $b_1 \in [0,b)$ such that $c_4\tilde{\kappa}(c_5 t)\le c_5$, for all $0\le t\le b_1$.

We give the statement that for any $t\in[0,b_1)$, $\{\varphi_n(t)\}$ is a decreasing sequence. In fact

$$
\varphi_2(t) = c_4\int_0^t \tilde{\kappa}(\varphi_1(s))\mathrm{d}s = c_4\int_0^t \tilde{\kappa}(c_5 s)\mathrm{d}s \le \int_0^t c_5\mathrm{d}s = \varphi_1(t).
$$

By induction, we get

$$\varphi_{n+1}(t) = c_4 \int_0^t \tilde{\kappa}(\varphi_n(s))\mathrm{d}s \le c_4 \int_0^t \tilde{\kappa}(\varphi_{n-1}(s))\mathrm{d}s = \varphi_n(t),\ \forall\, 0 \le t \le b_1. \tag{20}$$

Therefore, the statement is true and we can define the function $\phi(t)$ as

$$\phi(t) = \lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} c_4 \int_0^t \tilde{\kappa}(\varphi_{n-1}(s))\mathrm{d}s = \lim_{n\to\infty} c_4 \int_0^t \tilde{\kappa}(\phi(s))\mathrm{d}s,\ 0 \le t \le b_1. \tag{21}$$

By the Bihari's inequality, we get $\phi(t) = 0$ for all $0 \le t \le b_1$. It means that for all $0 \le t \le b_1$,

$$\lim_{n\to\infty}[\mathbf{E} \sup_{0\le s\le t} |x^{n+m}(s) - x^n(s)|^2 - (7mM^2 \sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i)\mathbf{E} \sup_{0\le s\le t} |x^{m+n-1}(s) - x^{n-1}(s)|^2] = 0. \tag{22}$$

By using the condition $7mM^2 \sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i < 1$ and (22), we get

$$\lim_{n\to\infty} \mathbf{E}|x^{n+m}(t) - x^n(t)|^2 = 0,\ 0 \le t \le b_1, \tag{23}$$

which means that $\{x^n(t)\}$ is a Cauchy sequence in $L_2(\Omega, H)$. Let $\lim_{n\to\infty} x^n(t) = x(t)$, obviously,

$$\|x(t)\|^2 \le \tilde{M},\ 0 \le t \le b_1.$$

Taking limits on both side of equation (9), for all $t \in [0, b_1]$, we have

$$\begin{aligned} x(t) \ &= \ S_q(t)\varphi(0) + \int_0^t S_q(s)[x_1 - g(0,\varphi)]\mathrm{d}s + \sum_{t_i<t} S_q(t-t_i)I_i(x_{t_i}) \\ &\quad + \sum_{t_i<t} \int_{t_i}^t S_q(t-s)[J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})]\mathrm{d}s \\ &\quad + \int_0^t S_q(t-s)g(s,x_s)\mathrm{d}s + \int_0^t T_q(t-s)f(s,x_s)\mathrm{d}s + \int_0^t T_q(t-s)\sigma(s,x_s)\mathrm{d}w(s). \end{aligned} \tag{24}$$

So we have presented the existence of the mild solution of problem (1) on $[0, b_1]$. By iteration we can get the existence of the mild solution of problem (1) on $[0, b]$.

Suppose that $x(t)$ and $\bar{x}(t)$ are two solutions of (1). Using the similar discussion as (22), we get

$$\begin{aligned} &[1 - (7mM^2 \sum_{t_i<t} p_i + 7mM^2 b \sum_{t_i<t} q_i)]\mathbf{E} \sup_{0\le s\le t} |x(s) - \bar{x}(s)|^2 \\ &\le (14m^2M^2 + 7M^2 b + 7M_b^2 \frac{b^{2q-1}}{2q-1} + 7M_b^2 b^{2q-2}) \int_0^t \tilde{\kappa}(\mathbf{E} \sup_{0\le r\le s} |x(r) - \bar{x}(r)|^2)\mathrm{d}s, \end{aligned} \tag{25}$$

the Bihari inequality implies $\mathbf{E}|x(t) - \bar{x}(t)|^2 = 0$, and we have shown the existence and the uniqueness of the mild solution of (1).

4. Stability of Solutions

In this section, we give the continuous dependence of solutions on the initial values by means of the Bihari's inequality. We first propose the following assumption on g instead of (H1),

(H4) $g : J \times \mathcal{B} \to H$ satisfies $|g(t,\varphi) - g(t,\phi)|^2 \le K_1\|\varphi - \phi\|^2_{\mathcal{B}}$, where K_1 is a positive constant.

Definition 4.1. *[26] A mild solution $x^{\varphi,x_1}(t)$ of Cauchy problem (1) with initial value (φ, x_1) is said to be stable in square if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\mathbf{E} \sup_{0\le s\le b} |x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)| \le \epsilon, \quad \text{when} \quad \mathbf{E}\|\varphi - \phi\|^2_{\mathcal{B}} + \mathbf{E}|x_1 - y_1|^2 < \delta, \tag{26}$$

where $y^{\phi,y_1}(t)$ is another solution of (1) with initial value (ϕ, y_1).

Theorem 4.2. *Assume* $21mM^2\sum_{i=1}^m p_i + 21mM^2b\sum_{i=1}^m q_i < 1$, *the conditions of Theorem 3.2 are satisfied and* g *satisfied (H4), then the mild solution of (1) is stable in mean square.*

Proof. Since the proof is similar to the Theorem 18 of [26], we here give only the sketch of the proof. By using the same argument as that in Theorem 3.2, we get that for all $0 \le t \le b$,

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le t}|x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)|^2 \quad\le\quad & 3\mathbf{E}|S_q(t)\varphi(0) - S_q(t)\phi(0)|^2 + 3\mathbf{E}|\int_0^t S_q(s)(|x_1 - y_1|_H + \|\varphi-\phi\|_{\mathcal{B}})\mathrm{d}s|^2 \\ & +3(7mM^2\sum_{t_i<t} p_i + 7mM^2b\sum_{t_i<t} q_i)\mathbf{E}\sup_{0\le s\le t}|x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)|^2 \\ & +3(14m^2M^2 + 7M^2b + 7M_b^2\frac{b^{2q-1}}{2q-1} + 7M_b^2b^{2q-2})\int_0^t \tilde{\kappa}(\mathbf{E}\sup_{0\le r\le s}|x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)|^2)\mathrm{d}s.\end{aligned}$$

Then we get

$$\mathbf{E}\sup_{0\le s\le t}|x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)| \le \frac{\nu}{\Lambda}(|x_1 - y_1|^2 + \|\varphi-\phi\|_{\mathcal{B}}) + \frac{\tilde{\nu}}{\Lambda}\int_0^t \tilde{\kappa}(\mathbf{E}\sup_{0\le r\le s}|x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)|^2)\mathrm{d}s,$$

where $\nu = \max\{6M^2b^2, 6M^2b^2K_1 + 3M^2L^2\}$, $\tilde{\nu} = 3(14m^2M^2 + 7M^2b + 7M_b^2\frac{b^{2q-1}}{2q-1} + 7M_b^2b^{2q-2})$, and $\Lambda = 1 - 3(7mM^2\sum_{t_i<t} p_i + 7mM^2b\sum_{t_i<t} q_i)$. The function $\tilde{\kappa}(u)$ is defined in (16) which has the property as in Lemma 2.4. For any $\epsilon > 0$, letting $\epsilon_1 = \frac{1}{2}\epsilon$, we have $\lim_{s\to 0}\int_s^{\epsilon_1}\frac{1}{\tilde{\kappa}(u)}\mathrm{d}u = \infty$. There exists a positive constant δ and $\delta < \epsilon_1$ such that $\int_\delta^{\epsilon_1}\frac{1}{\tilde{\kappa}(u)}\mathrm{d}u \ge T$. Let $u_0 = \frac{\nu}{\Lambda}(|x_1 - y_1|^2 + \|\varphi-\phi\|_{\mathcal{B}})$, $u(t) = \mathbf{E}\sup_{0\le s\le t}|x^{\varphi,x_1}(s) - y^{\phi,y_1}(s)|$, $v(t) = 1$. If $u_0 \le \delta \le \epsilon_1$, then Lemma 2.5 shows that $\int_{u_0}^{\epsilon_1}\frac{1}{\tilde{\kappa}(u)}\mathrm{d}u \ge \int_\delta^{\epsilon_1}\frac{1}{\tilde{\kappa}(u)}\mathrm{d}u \ge T = \int_0^b v(s)\mathrm{d}s$. So for any $t \in [0, b]$, the estimate $u(t) \le \epsilon_1 \le \epsilon$ holds. This completes the proof of the theorem.

5. Application

In this section, an example is provided to illustrate the results obtained in previous two sections. In the following, we assume $H = L^2([0,\pi])$, $\mathcal{B} = PC_0 \times L^2(\rho, H)$ (see [10]) be the space formed of all functions $\varphi : (-\infty, 0] \to H$ such that $\varphi|_{[0,1]} \in \mathcal{PC}([0,1], H)$, $\varphi(\cdot)$ is Lebesgue-measurable on $(-\infty, 0]$ and $\int_{-\infty}^0 \rho(s)\sup_{s\le\theta\le 0}(E|\psi(\theta)|^2)^{1/2}\mathrm{d}s < \infty$, where $\rho : (-\infty, 0] \to (0,\infty)$ is a continuous function with $1 = \int_{-\infty}^0 \rho(t)\mathrm{d}t < \infty$. The norm $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := [\int_{-\infty}^0 \rho(s)\sup_{s\le\theta\le 0}(\mathbf{E}|\psi(\theta)|^2)\mathrm{d}s]^{1/2} < \infty,$$

then $L = 1$, $\Gamma_1 = 1 + \int_{-\infty}^0 \rho(t)\mathrm{d}t = 2$. We consider the following initial problem.

$$\begin{cases} D_t^\alpha[u_t'(t,x) - \int_{-\infty}^t\int_0^\pi h(s-t,\eta,x)u(s,\eta)\mathrm{d}\eta\mathrm{d}s] = \frac{\partial^2}{\partial x^2}u(t,x) + f(t, \int_{-\infty}^0 p_0(t,s-t)u(s,x)\mathrm{d}s) \\ \qquad\qquad +\sigma(t, \int_{-\infty}^0 q_0(t,s-t)u(s,x)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \ t \in J,\ t \ne t_i; \\ \triangle u(t_i,x) = \int_{-\infty}^{t_i}\bar{p}_i(s-t_i)u(s,x)\mathrm{d}s, \\ \triangle u'(t_i,x) = \int_{-\infty}^{t_i}\bar{q}_i(s-t_i)\frac{u(s,x)}{1+|u(s,x)|}\mathrm{d}s,\ i = 1,2,\dots,m; \\ u(t,0) = u(t,\pi) = 0,\ t\in[0,1]; \quad u(\theta,x) = \varphi(\theta,x),\ \theta\in(-\infty,0], x\in[0,\pi]; \\ \frac{\partial}{\partial t}u(0,x) = z(x), x \in [0,\pi]. \end{cases} \tag{27}$$

where $w(t)$ is an H-valued Winer process and $J = [0, 1]$.

Obviously, the operator $A : H \to H$ by $A = \frac{\partial^2}{\partial x^2}$ with domain $D(A) = \{z \in H :\ z'' \in H,\ z(0) = z(\pi) = 0\}$ is the infinitesimal generator of a strongly continuously cosine family[22]. According to the estimates on the norms of the operators of Theorems 3.3 and 3.4 in [28], we know the operators $S_q(t)$ and $T_q(t)$ in the mild solution of (27) satisfy $\|S_q(t)\|_{\mathcal{L}(H,H)} \le 1$, $\|T_q(t)\|_{\mathcal{L}(H,H)} \le 3$. We suppose that

(a) $h(s,\eta,x)$, $\frac{\partial h(s,\eta,x)}{\partial x}$ are measurable, $h(s,\eta,0)=h(s,\eta,\pi)=0$,

$$L_0=\max\{[\int_0^\pi\int_{-\infty}^0\int_0^\pi\frac{1}{\rho(s)}(\frac{\partial^k h(s,\eta,x)}{\partial x^k})^2\mathrm{d}\eta\mathrm{d}s\mathrm{d}x]^{\frac{1}{2}}:k=0,1\}<\infty,$$

(b) p_0, $q_0\in C(\mathbf{R}^2,\mathbf{R}^+)$, moreover, $(\int_{-\infty}^0 p_0^2(t,\theta)\rho^{-1}(\theta)\mathrm{d}\theta)^{\frac{1}{2}}=d_1(t)\in C(J,\mathbf{R}^+)$ and $(\int_{-\infty}^0 q_0^2(t,\theta)\rho^{-1}(\theta)\mathrm{d}\theta)^{\frac{1}{2}}=d_2(t)\in C(J,\mathbf{R}^+)$.

(c) $\bar{p}_i(\theta),\bar{q}_i(\theta)\in C(\mathbf{R},\mathbf{R}^+)$, moreover, $p_i=(\int_{-\infty}^0\frac{\bar{p}_i^2(\theta)}{\rho(\theta)}\mathrm{d}\theta)^{\frac{1}{2}}<\infty$, and $q_i=(\int_{-\infty}^0\frac{\bar{q}_i^2(\theta)}{\rho(\theta)}\mathrm{d}\theta)^{\frac{1}{2}}<\infty$, $i=1,2,\ldots,m$.

Let $L=\max\{L_0,\max_{t\in J}\{d_1(t)\},\max_{t\in J}\{d_2(t)\}\}$, since we can take $\kappa(\vartheta)=L\cdot\vartheta$ which is a concave function as in (H1), so (H1)-(H3) hold.

According to Theorem 3.2, we get the conclusion that when

$$\max\{14m\sum_{i=1}^m p_i+28m\sum_{i=1}^m q_i,7m\sum_{i=1}^m p_i+7m\sum_{i=1}^m q_i\}<1,\tag{28}$$

problem (27) has a unique mild solution on $(-\infty,1]$.

Obviously, in the Cauchy problem (27), $g(t,\cdot)$, $I_i(\cdot)$ are bounded linear operators and $\|J_i(\phi)\|\le q_i\|\phi\|_{\mathcal{B}}$. More precisely, let $m=2$ and we define $\rho(t)=2e^{2t}$, $\bar{p}_i(t)=\sqrt{2}e^{86t}$, $\bar{q}_i(t)=\sqrt{2}e^{86t}$ for $t\in(-\infty,0]$. By computation, $p_i=\frac{1}{170}$, $q_i=\frac{1}{170}$, $i=1,2$. We set the operators g, f and σ as following,

$$\begin{aligned}
h(s,\eta,x)&=e^{\frac{3}{2}s}\sin\eta,\\
f(t,\int_{-\infty}^0 p_0(t,s-t)u(s,x)\mathrm{d}s)&=\frac{e^{-t}|\int_{-\infty}^t p_0(s-t)u(s,x)\mathrm{d}s|}{(12+e^t)(1+|\int_{-\infty}^t p_0(s-t)u(s,x)\mathrm{d}s|)}\\
&=\frac{e^{-t}|\int_{-\infty}^t e^{2(s-t)}u(s,x)\mathrm{d}s|}{(12+e^t)(1+|\int_{-\infty}^t e^{2(s-t)}u(s,x)\mathrm{d}s|)},\\
\sigma(t,\int_{-\infty}^0 q_0(t,s-t)u(s,x)\mathrm{d}s)&=\frac{e^t}{12+e^t}\frac{|\int_{-\infty}^t q_0(t,s-t)u(s,x)\mathrm{d}s|}{1+|\int_{-\infty}^t q_0(s-t)u(s,x)\mathrm{d}s|}\\
&=\frac{e^t}{12+e^t}\frac{|\int_{-\infty}^t e^{3(s-t)}u(s,x)\mathrm{d}s|}{1+|\int_{-\infty}^t e^{3(s-t)}u(s,x)\mathrm{d}s|},
\end{aligned}$$

then

$$\begin{aligned}
&\|\int_{-\infty}^0\int_0^\pi h(s-t,\eta,x)u(s,\eta)\mathrm{d}\eta\mathrm{d}s-\int_{-\infty}^0\int_0^\pi h(s-t,\eta,x)v(s,\eta)\mathrm{d}\eta\mathrm{d}s\|_H\\
&=\int_{-\infty}^0\frac{e^{3(s-t)}}{2e^s}\int_0^\pi\sin\eta(u(s,\eta)-v(s,\eta))\mathrm{d}\eta\mathrm{d}s\\
&\le\frac{e^{-3t}}{2}\|u_t(\theta,x)-v_t(\theta,x)\|_{\mathcal{B}},
\end{aligned}\tag{29}$$

$$\begin{aligned}
&\|f(t,\int_{-\infty}^0 p_0(s-t)u(s,x)\mathrm{d}s)-f(t,\int_{-\infty}^0 p_0(s-t)v(s,x)\mathrm{d}s)\|_H\\
&\le\frac{1}{12}\Big(|\int_0^\pi(\int_{-\infty}^0 p_0(s-t)[u(s,x)-v(s,x)]\mathrm{d}s)^2\mathrm{d}x\Big)^{\frac{1}{2}}\\
&\le\frac{1}{12}\int_{-\infty}^0\frac{p_0^2(s-t)}{\rho(s)}\mathrm{d}s\int_\pi^0\rho(s)\sup_{s\le\theta\le0}\mathbf{E}|u(\theta,\cdot)-v(\theta,\cdot)|_H^2\mathrm{d}s\\
&\le\frac{e^{-4t}}{48}\|u_t(\theta,x)-v_t(\theta,x)\|_{\mathcal{B}},
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
&\|\sigma(t,\int_{-\infty}^{0}q_0(s-t)u(s,x)\mathrm{d}s)-\sigma(t,\int_{-\infty}^{0}q_0(s-t)v(s,x)\mathrm{d}s)\|_{\mathcal{L}_2^0}\\
&\leq \frac{1}{24}\int_{-\infty}^{0}\frac{q_0^2(s-t)}{\rho(s)}\int_{-\infty}^{0}\rho(s)\sup_{s\leq\theta\leq 0}\mathbf{E}|u(\theta,\cdot)-v(\theta,\cdot)|_H^2\mathrm{d}s\\
&\leq \frac{e^{-6t}}{96}\|u_t(\theta,x)-v_t(\theta,x)\|_{\mathcal{B}}.
\end{aligned} \tag{31}$$

So the inequality (28) holds, the initial problem (27) has a unique mild solution on $(-\infty,1]$. From inequality (29), we know the assumption (H4) holds and $21mM^2\sum_{i=1}^m p_i + 21mM^2\sum_{i=1}^m q_i = \frac{84}{85} < 1$, according to the Theorem 4.2, the mild solution of the Cauchy problem (27) is stable in the mean square.

6. Conclusion

In this paper, we established the existence and uniqueness of mild solution of a class of stochastic impulsive fractional differential evolution equations. The methods and techniques used in this paper are under non-Lipschitz conditions with the Lipschitz conditions being regarded as a special case. We also obtained sufficient conditions for the mild solution to be stable in the mean square. It would be of great interest to extend these results to the stochastic fractional evolutions with delay and Piosson jumps. The regularity of the solution of problem (1) is deserve to study and we will report our research in future work.

Acknowledgments

The authors would like to thank the reviewers for their constructive suggestions and helpful comments.

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