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The Generalized Orthogonal Solutions of the Matrix Inverse Problem $AX = B$ and Optimal Approximation

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Abstract. In this paper, the generalized orthogonal solutions of the matrix inverse problem $AX = B$ and associated optimal approximation problem are considered. The properties and structure of generalized orthogonal matrices are given, the relationship between the generalized orthogonal matrices and the orthogonal matrices are discussed. Necessary and sufficient conditions for the solvability of the matrix inverse problem $AX = B$, the general expression of solutions, and its procrustes problem are discussed. In addition, the corresponding optimal approximation solutions are presented. Finally, the algorithms and corresponding computational examples are given.

1. Introduction

In this paper, I_n is the identity matrix of dimension n. $R(A), A^T$ and $\|A\|$ denote the column space, the transpose and the Frobenius norm of a matrix A, respectively. $R^{n\times m}$, $OR^{n\times n}$, $OR^{p\times n}$, $GOR^{p\times n}_{R(X)}$ and $R^{n\times m}_r$ denote the sets of real $n \times m$ matrices, real $n \times n$ orthogonal matrices, real $p \times n$ column orthogonal matrices, real $p \times n$ generalized orthogonal matrices over space $R(X)$ and real $n \times m$ matrices with rank r, respectively.

Matrix inverse problem has many applications in Hopfield neural networks[1], structural design[2,3], parameter identification[4], vibrating theory[5,6], and so on. The matrix inverse problem $AX = B$ associated with many kinds of matrix sets have been disscussed, such as symmetric matrices, positive semidefinite symmetric matrices, orthogonal matrices, centrosymmetric matrices, Hermitian reflexive matrices, re-nonnegative definite matrices, re-positive definite matrices, symmetric ortho-symmetric matrices, symmetric positive semidefinite matrices, (bi)symmetric nonnegative definite matrices and symmetric orthogonal matrices, see $[7-19]$.

In this paper, we first consider the matrix inverse problem $AX = B$ for generalized orthogonal matrices.

Problem I(Procrustes problem) Given $X \in R^{n\times m}, B \in R^{p\times m}$, find $A \in GOR^{p\times n}_{R(X)}$ such that

$$\|AX - B\| = \min_{C\in GOR^{p\times n}_{R(X)}} \|CX - B\|. \tag{1.1}$$

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Denote the set of all the solutions of problem I by S_{E_1}.

Problem II(Matrix inverse problem) Given $X \in R^{n\times m}, B \in R^{p\times m}$, find $A \in GOR^{p\times n}_{R(X)}$ such that

$$AX = B. \tag{1.2}$$

Denote the set of all the solutions of problem II by S_{E_2}.

Problem III(Approximation problem) Given $A^* \in R^{p\times n}$, find $\widetilde{A} \in S_{E_1}$, $\widehat{A} \in S_{E_2}$ such that

(a) $\|A^* - \widetilde{A}\| = \min\limits_{A\in S_{E_1}} \|A^* - A\|$,

(b) $\|A^* - \widehat{A}\| = \min\limits_{A\in S_{E_2}} \|A^* - A\|$,

where $\|\cdot\|$ is the Frobenius norm.

2. The Properties of Generalized Orthogonal Matrices

In this section, we give the definition and corresponding properties of generalized orthogonal matrices.

Definition 2.1 Given $X \in R^{n\times m}, A \in R^{p\times n}$, A is said to be generalized orthogonal matrix over subspace $R(X)$ if $(Ax, Ax) = (x, x)$ for any $x \in R(X)$.

Let $GOR^{p\times n}_{R(X)}$ denote all the $p\times n$ generalized orthogonal matrices over subspace $R(X)$. If $m = n$, $r(X) = n$, then $GOR^{p\times n}_{R(X)} = OR^{p\times n}$, if $m = n = p$, $r(X) = n$, then $GOR^{p\times n}_{R(X)} = OR^{n\times n}$.

Lemma 2.1(Singular Value Decomposition) For any $X \in R^{n\times m}_r$, then

$$X = U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T = U_1\Sigma V_1^T, \tag{2.1}$$

where $U = (U_1, U_2) \in OR^{n\times n}$, $V = (V_1, V_2) \in OR^{m\times m}$, $U_1 \in OR^{n\times r}$, $V_1 \in OR^{m\times r}$, $\Sigma = diag(\sigma_1, \cdots, \sigma_r), \sigma_1 \geq \cdots \geq \sigma_r > 0$.

Lemma 2.2 Let $X_i \in R^{n\times m_i}, i = 1, 2, \cdots, s$, if $R(X_1) \subset R(X_2) \subset \cdots \subset R(X_s)$, then $GOR^{p\times n}_{R(X_1)} \supset GOR^{p\times n}_{R(X_2)} \supset \cdots \supset GOR^{p\times n}_{R(X_s)}$.

Lemma 2.3 Let $X \in R^{n\times m}, A \in R^{p\times n}$, then $A \in GOR^{p\times n}_{R(X)}$ if and only if $X^TA^TAX = X^TX$.

Proof. (Sufficiency)For any $x \in R(X)$, then there exists $y \in R^m$ such that $x = Xy$, since $X^TA^TAX = X^TX$, so we have $y^TX^TA^TAXy = y^TX^TXy$, it follows that $x^TA^TAx = x^Tx$, that is, $(Ax, Ax) = (x, x)$, by definition 2.1, we get $A \in GOR^{p\times n}_{R(X)}$.

(Necessity)For any $y \in R^n$, let $x = Xy$, then $x \in R(X)$ and $(Ax, Ax) = (x, x)$, this implies that $y^TX^TA^TAXy = y^TX^TXy$, then $X^TA^TAX = X^TX$. □

Theorem 2.1 Suppose that $X \in R^{n\times m}_r$ has the singular value decomposition as in Lemma 2.1, and

$$S = \{A \in R^{p\times n} \mid A = (F, G)U^T, F \in OR^{p\times r}, G \in R^{p\times(n-r)}\}, \tag{2.2}$$

then $S = GOR^{p\times n}_{R(X)}$.

Proof. If $A \in GOR^{p\times n}_{R(X)}$, by Lemma 2.3, we have $X^TA^TAX = X^TX$. Combining with the Lemma 2.1, and let $\overline{A} = AU$, then

$$\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}\overline{A}^T\overline{A}\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\Sigma^2 & 0\\ 0 & 0\end{pmatrix}.$$

Partition $\overline{A}$ conformally with (2.1) as $\overline{A} = \begin{pmatrix}\overline{A}_1 & \overline{A}_2\\ \overline{A}_3 & \overline{A}_4\end{pmatrix}$, then

$$\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\overline{A}_1^T\overline{A}_1 + \overline{A}_3^T\overline{A}_3 & \overline{A}_1^T\overline{A}_2 + \overline{A}_3^T\overline{A}_4\\ \overline{A}_2^T\overline{A}_1 + \overline{A}_4^T\overline{A}_3 & \overline{A}_2^T\overline{A}_2 + \overline{A}_4^T\overline{A}_4\end{pmatrix}\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\Sigma^2 & 0\\ 0 & 0\end{pmatrix},$$

so we have $\overline{A}_1^T\overline{A}_1 + \overline{A}_3^T\overline{A}_3 = I_r$, that is, $\begin{pmatrix} \overline{A}_1 \\ \overline{A}_3 \end{pmatrix} \in OR^{p\times r}$, then $A \in S$.

Conversely, for any $A \in S$, this implies that $A = (F, G)U^T, F \in OR^{p\times r}$, we have

$$X^T A^T AX = V_1\Sigma U_1^T U(F,G)^T(F,G)U^T U_1\Sigma V_1^T = V_1\Sigma F^T F\Sigma V_1^T = V_1\Sigma^2 V_1^T = X^T X,$$

then $A \in GOR^{p\times n}_{R(X)}$. □

Corollary 2.1 If $A \in GOR^{p\times n}_{R(X)}$, then $QA \in GOR^{p\times n}_{R(X)}$ for any $Q \in OR^{p\times n}$.
Corollary 2.2 There exists $A \in GOR^{n\times n}_{R(X)}$ and $|A| \neq 0$ such that $A \notin OR^{n\times n}$.

3. The General Solutions of Problem I and II

In this section, we use the singular value decomposition to give the necessary and sufficient conditions for the existence of generalized orthogonal solutions to the matrix inverse $AX = B$, and the general solutions of problem I and II.

Theorem 3.1 Let $X \in R^{n\times m}, B \in R^{p\times m}$, then the solution set S_{E_1} of problem I can be expressed as

$$S_{E_1} = \left\{ A = \left(Q\begin{pmatrix} I_t & 0 \\ 0 & S \end{pmatrix} P^T, G \right) U^T \mid \forall S \in OR^{(p-t)\times(r-t)}, G \in R^{p\times(n-r)} \right\}, \tag{3.1}$$

where Q, P, U are determined by X, B.

Proof. Because $A \in GOR^{p\times n}_{R(X)}$, by Lemma 2.3, we have

$$\|AX - B\|^2 = \|AX\|^2 + \|B\|^2 - 2tr(B^T AX) = \|X\|^2 + \|B\|^2 - 2tr(B^T AX),$$

so $\|AX - B\| = \min$ is equivalent to $tr(B^T AX) = \max$, by (2.1) and Theorem 2.1, we have

$$\begin{aligned} tr(B^T AX) &= tr(AXB^T) \\ &= tr((F,G)U^T U_1\Sigma V_1^T B^T) \\ &= tr((F,G)\left(\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right)\Sigma V_1^T B^T) \\ &= tr(F\Sigma V_1^T B^T), \end{aligned} \tag{3.2}$$

Assume that the singular value decomposition of $\Sigma V_1^T B^T$ can be written as

$$\Sigma V_1^T B^T = P\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} Q^T, \tag{3.3}$$

where $P \in OR^{r\times r}, Q \in OR^{p\times p}, \Gamma = diag(\gamma_1, \cdots, \gamma_t), \gamma_1 \geq \cdots \geq \gamma_t > 0$.

Let $\overline{F} = Q^T FP = \left(\overline{f}_{ij}\right)_{p\times r}$, because $F \in OR^{p\times r}$, so $\overline{F} \in OR^{p\times r}$, it follows that $|\overline{f}_{ij}| \leq 1$.

By (3.2) and (3.3), we have

$$tr(B^T AX) = tr\left(FP\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} Q^T\right) = tr\left(Q^T FP\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}\right) = \sum_{i=1}^{t} \gamma_i \overline{f}_{ii} \leq \sum_{i=1}^{t} \gamma_i, \tag{3.4}$$

note that $tr(B^T AX) = \sum\limits_{i=1}^{t} \gamma_i$ if and only if $\overline{f}_{ii} = 1$.

So let $\overline{F} = \left(\begin{smallmatrix} I_t & 0 \\ 0 & S \end{smallmatrix}\right), \forall S \in OR^{(p-t)\times(r-t)}$, we have $tr(B^T AX) = \max$, then let

$$A = \left(Q\begin{pmatrix} I_t & 0 \\ 0 & S \end{pmatrix} P^T, G \right) U^T, \forall S \in OR^{(p-t)\times(r-t)},$$

we have $\|AX - B\| = \min\limits_{C \in GOR^{p\times n}_{R(X)}} \|CX - B\|$. □

Next, we give the solvability conditions for problem II and give an expression of the general solution of problem II.

Lemma 3.1[10] Suppose that $X \in R_r^{n\times m}$ has the singular value decomposition as in Lemma 2.1, if $B \in R^{p\times m}$ and $X^TX = B^TB$, then

$$B = W\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T,$$

where $W = (W_1, W_2) \in OR^{p\times p}, W_1 \in OR^{p\times r}$.

Theorem 3.2 Given $X \in R_r^{n\times m}, B \in R^{p\times m}$. Then the solution set S_{E_2} of problem II is nonempty if and only if

$$X^TX = B^TB. \tag{3.5}$$

Moreover, if $A \in S_{E_2}$, then

$$A = (W_1, G)U^T, \forall G \in R^{p\times(n-r)}, \tag{3.6}$$

where U, W_1 are defined as in Lemma 3.1.

Proof. (Necessity) Suppose that there exists $A \in GOR_{R(X)}^{p\times n}$ such that $AX = B$, then

$$X^TA^TAX = B^TB,$$

by lemma 2.3, we have $X^TX = B^TB$.

(Sufficiency)If $X^TX = B^TB$, by Lemma 3.1, we have

$$X = U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T, B = W\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T,$$

where $U \in OR^{n\times n}, V = (V_1, V_2) \in OR^{m\times m}, V_1 \in OR^{m\times r}, W = (W_1, W_2) \in OR^{p\times p}, W_1 \in OR^{p\times r}$. Let $A_0 = (W_1, 0)U^T$, we have $A_0 \in GOR_{R(X)}^{p\times n}$ and $A_0X = B$, then the solution set S_{E_2} of problem II is nonempty.

Moreover, if $A \in S_{E_2}$, then $AX = B$, that is,

$$(F, G)U^TU\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T = W\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T,$$

so $F\Sigma V_1^T = W_1\Sigma V_1^T$, it follows that $F = W_1$, then $A = (W_1, G)U^T, \forall G \in R^{p\times(n-r)}$. □

4. The Solution of Problem III

In this section, for a given matrix, we give the optimal approximation solutions in solutions sets of problem I and II.

Lemma 4.1 Suppose that $X \in R_r^{n\times m}$ has the singular value decomposition as in Lemma 2.1, then there exists $\widetilde{Y} \in OR^{n\times m}$ such that

$$\|\widetilde{Y} - X\| = \min_{Y\in OR^{n\times m}} \|Y - X\|,$$

where $\widetilde{Y} = U\left(\begin{smallmatrix}I_r & 0\\ 0 & M\end{smallmatrix}\right)V^T, \forall M \in OR^{(n-r)\times(m-r)}$.

For convenience, we denote $O(X) = \widetilde{Y}$ in Lemma 4.1.

Theorem 4.1 Given $A^* \in R^{p\times n}$, partition A^*U conformally with (2.2) as

$$A^*U = (A_1, A_2),$$

and partition Q^TA_1P conformally with (3.3) as

$$Q^TA_1P = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}. \tag{4.1}$$

Then there exists $\widetilde{A} \in S_{E_1}$ such that

$$\|A^* - \widetilde{A}\| = \min_{A \in S_{E_1}} \|A^* - A\|.$$

Moreover, $\widetilde{A}$ can be represented as

$$\widetilde{A} = \left(Q \begin{pmatrix} I_t & 0 \\ 0 & O(A_{22}) \end{pmatrix} P^T, A_2 \right) U^T.$$

Proof. Let $A \in S_{E_1}$, from the orthogonal invariance of the Frobenius norm, we have

$$\begin{aligned} \|A^* - A\|^2 &= \left\| A^* - \left(Q \begin{pmatrix} I_t & 0 \\ 0 & S \end{pmatrix} P^T, G \right) U^T \right\|^2 \\ &= \left\| A^* U - \left(Q \begin{pmatrix} I_t & 0 \\ 0 & S \end{pmatrix} P^T, G \right) \right\|^2 \\ &= \left\| A_1 - Q \begin{pmatrix} I_t & 0 \\ 0 & S \end{pmatrix} P^T \right\|^2 + \|A_2 - G\|^2 \\ &= \left\| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - \begin{pmatrix} I_t & 0 \\ 0 & S \end{pmatrix} \right\|^2 + \|A_2 - G\|^2 \\ &= \|I_t - A_{11}\|^2 + \|A_{12}\|^2 + \|A_{21}\|^2 + \|S - A_{22}\|^2 + \|A_2 - G\|^2, \end{aligned}$$

clearly,

$$\|A^* - A\| = \min_{A \in S_{E_1}}$$

is equivalent to

$$G = A_2$$

and

$$\|S - A_{22}\| = \min_{S \in OR^{(p-t)\times(r-t)}}, \tag{4.2}$$

by Lemma 4.1, we know that (4.2) holds if and only if

$$S = O(A_{22}).$$

Let

$$\widetilde{A} = \left(Q \begin{pmatrix} I_t & 0 \\ 0 & O(A_{22}) \end{pmatrix} P^T, A_2 \right) U^T,$$

then $\|A^* - \widetilde{A}\| = \min_{A \in S_{E_1}} \|A^* - A\|$. □

Theorem 4.2 Given $A^* \in R^{p\times n}$, partition A^*U conformally with (3.6) as

$$A^*U = (A_3, A_4).$$

Then there exists $\widehat{A} \in S_{E_2}$ such that

$$\|A^* - \widehat{A}\| = \min_{A \in S_{E_2}} \|A^* - A\|.$$

Moreover, $\widehat{A}$ can be represented as

$$\widehat{A} = (W_1, A_4)U^T.$$

Proof. Let $A \in S_{E_2}$, from the orthogonal invariance of the Frobenius norm, we have

$$\begin{aligned} \|A^* - A\|^2 &= \left\|A^* - (W_1, G)U^T\right\|^2 \\ &= \|A^*U - (W_1, G)\|^2 \\ &= \|A_3 - W_1\|^2 + \|A_4 - G\|^2 , \end{aligned}$$

clearly,

$$\|A^* - A\| = \min_{A \in S_{E_2}}$$

is equivalent to $G = A_4$.

Let

$$\widehat{A} = (W_1, A_4)U^T,$$

then $\|A^* - \widehat{A}\| = \min_{A \in S_{E_2}} \|A^* - A\|$. □

Remark 4.1 The problem III(a) has a unique solution if and only if the matrix A_{22} is full column rank or full row rank matrix.

The problem III(b) has always a unique solution.

5. Algorithms and Computational Examples

In this section, we give algorithms and corresponding computational examples to illustrate the theoretical results.

Algorithm 5.1 The optimal approximation solution of procrustes problem.
Step 1. Input A^*, B, X;
Step 2. Calculate the singular value decomposition of X and $\Sigma V_1^T B^T$;
Step 3. Calculate $O(A_{22})$ by Lemma 4.1;
Step 4. Calculate $\widetilde{A}$ by Theorem 4.1.

Example 5.1 Consider the procrustes problem

$$\|AX - B\| = \min, \;\; s.t. \;\; A \in GOR_{R(X)}^{p \times n}$$

with

$$X = \begin{pmatrix} 0.3908 & 0.3431 & 0.2165 & 0.3802 & 0.3193 & 0.2855 & 0.2322 \\ 0.1091 & 0.1338 & 0.0783 & 0.1494 & 0.1316 & 0.1042 & 0.1002 \\ 0.1720 & 0.2964 & 0.1723 & 0.3060 & 0.3014 & 0.2630 & 0.2638 \\ 0.3441 & 0.2832 & 0.1678 & 0.2872 & 0.2989 & 0.3158 & 0.2421 \\ 0.2535 & 0.2085 & 0.0849 & 0.2475 & 0.2509 & 0.2325 & 0.1902 \\ 0.3599 & 0.3080 & 0.1691 & 0.3455 & 0.2922 & 0.3296 & 0.2631 \\ 0.4201 & 0.3809 & 0.2096 & 0.4223 & 0.3980 & 0.3683 & 0.3053 \\ 0.3214 & 0.1845 & 0.1079 & 0.1844 & 0.2111 & 0.2388 & 0.1518 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0.2200 & 0.9632 & 0.3559 & 0.8710 & 0.3334 & 0.6339 & 0.6574 \\ 0.1602 & 0.5269 & 0.2055 & 0.4688 & 0.1834 & 0.3769 & 0.3769 \\ 0.5948 & 0.8281 & 0.4160 & 0.6705 & 0.2968 & 0.8523 & 0.7419 \\ 0.1040 & 0.8900 & 0.3020 & 0.8240 & 0.3056 & 0.5106 & 0.5642 \\ 0.8222 & 0.7923 & 0.4667 & 0.5926 & 0.2903 & 1.0071 & 0.8200 \\ 0.7158 & 0.5104 & 0.3511 & 0.3458 & 0.1917 & 0.7898 & 0.6094 \end{pmatrix}.$$

Given

$$A^* = \begin{pmatrix} 0.3164 & 0.5017 & 0.1232 & 0.6723 & 0.2794 & 0.8372 & 0.7194 & 0.0577 \\ 0.6996 & 0.7615 & 0.5044 & 0.4315 & 0.9462 & 0.9715 & 0.6500 & 0.9798 \\ 0.6253 & 0.7624 & 0.3473 & 0.6944 & 0.9064 & 0.0569 & 0.7269 & 0.2848 \\ 0.5431 & 0.5761 & 0.0921 & 0.2568 & 0.3927 & 0.4503 & 0.3738 & 0.5950 \\ 0.4390 & 0.7477 & 0.1478 & 0.0098 & 0.0249 & 0.5825 & 0.5816 & 0.9622 \\ 0.2874 & 0.6455 & 0.1982 & 0.5323 & 0.6714 & 0.6866 & 0.1161 & 0.1858 \end{pmatrix}.$$

Note that A_{22} in (4.1) is full column rank, then the optimal approximation problem of procrustes problem has the unique solution. By Algorithm 5.1, the unique optimal approximation solution $\widetilde{A}$ of problem III(a) is

$$\widetilde{A} = \begin{pmatrix} -0.3551 & 0.3849 & 0.5625 & 0.5746 & -0.2329 & 0.4353 & 0.1708 & -0.3132 \\ 0.1172 & 0.6392 & 0.0019 & -0.5649 & 0.6449 & 0.3554 & -0.0695 & 0.0543 \\ -0.0431 & 0.5142 & 0.1487 & 0.6159 & 0.5493 & -0.4917 & 0.1696 & 0.3195 \\ 0.5231 & 0.6925 & 0.2888 & -0.2761 & 0.0455 & 0.0751 & 0.0971 & -0.1444 \\ 0.4173 & 0.6455 & -0.1276 & -0.1543 & -0.5034 & 0.3962 & 0.1963 & 0.8482 \\ -0.1436 & 0.3947 & -0.3355 & 0.3822 & 0.5368 & 0.6589 & -0.2799 & 0.2339 \end{pmatrix},$$

and $\|\widetilde{A} - A^*\| = 2.9685$, $\|\widetilde{A}X - B\| = 2.0617$, $\|X^T\widetilde{A}^T\widetilde{A}X - X^TX\| = 2.7178 \times 10^{-15}$.

Given

$$A^* = \begin{pmatrix} 0.3616 & 0.5099 & 0.0972 & 0.6541 & 0.2639 & 0.8392 & 0.7239 & 0.0472 \\ 0.8323 & 0.7752 & 0.5356 & 0.4934 & 0.7376 & 0.9063 & 0.6126 & 1.0259 \\ 0.6999 & 0.7734 & 0.3306 & 0.6926 & 0.8409 & 0.0428 & 0.7219 & 0.2863 \\ 0.4471 & 0.5623 & 0.1094 & 0.2547 & 0.4832 & 0.4712 & 0.3821 & 0.5902 \\ 0.2302 & 0.7210 & 0.1515 & -0.0312 & 0.2732 & 0.6503 & 0.6156 & 0.9274 \\ 0.4356 & 0.6634 & 0.2064 & 0.5729 & 0.4789 & 0.6314 & 0.0869 & 0.2182 \end{pmatrix}.$$

Note that A_{22} in (4.1) is 4×3 matrix with rank 1, by Algorithm 5.1, the

$$\widetilde{A}_1 = \begin{pmatrix} 0.1154 & 0.4974 & 0.0207 & 0.0956 & 0.0155 & 0.6339 & 0.3448 & -0.6165 \\ -0.4277 & 0.4941 & 0.7804 & 0.1516 & 0.1288 & 0.0260 & -0.3418 & 0.5136 \\ 0.4604 & 0.6125 & -0.2065 & 0.3437 & 0.4751 & -0.4271 & 0.2502 & 0.1556 \\ 0.2990 & 0.6353 & 0.5832 & -0.0090 & -0.1278 & -0.0435 & -0.0028 & 0.0260 \\ -0.4323 & 0.5294 & -0.0358 & -0.2387 & 0.3905 & 0.6215 & 0.2987 & 0.7615 \\ 0.4237 & 0.4525 & -0.1961 & 0.6535 & -0.3639 & 0.3763 & -0.4425 & 0.4354 \end{pmatrix}$$

and

$$\widetilde{A}_2 = \begin{pmatrix} 0.2503 & 0.5080 & 0.0856 & 0.1941 & -0.2467 & 0.5458 & 0.2912 & -0.5459 \\ -0.5351 & 0.4837 & 0.7483 & 0.0941 & 0.3079 & 0.0833 & -0.3083 & 0.4713 \\ 0.4992 & 0.6090 & -0.1207 & 0.4440 & 0.2980 & -0.4966 & 0.2033 & 0.2240 \\ 0.1613 & 0.6222 & 0.5399 & -0.0850 & 0.1051 & 0.0314 & 0.0411 & -0.0297 \\ -0.2760 & 0.5674 & -0.2221 & -0.4046 & 0.4826 & 0.6916 & 0.3594 & 0.6562 \\ 0.2145 & 0.4128 & -0.0599 & 0.7544 & -0.3158 & 0.3569 & -0.4707 & 0.4953 \end{pmatrix}$$

are both the optimal approximation solutions of Problem III(a), and $\|\widetilde{A}_1 - A^*\| = \|\widetilde{A}_2 - A^*\| = 3.1233$, $\|\widetilde{A}_1X - B\| = \|\widetilde{A}_2X - B\| = 2.0617$, $\|X^T\widetilde{A}_1^T\widetilde{A}_1X - X^TX\| = 2.3244 \times 10^{-15}$, $\|X^T\widetilde{A}_2^T\widetilde{A}_2X - X^TX\| = 3.5710 \times 10^{-15}$.

Algorithm 5.2 The optimal approximation solution of matrix inverse problem.
Step 1. Input A^*, B, X;
Step 2. Calculate the singular value decomposition of X;
Step 3. Decompose B by Lemma 3.1;
Step 4. Calculate $\widehat{A}$ by Theorem 4.2.

Example 5.2 Consider the matrix inverse problem

$$AX = B, \ \ s.t. \ \ A \in GOR^{p\times n}_{R(X)}$$

with

$$X = \begin{pmatrix} 0.0646 & 0.0513 & 0.1215 & 0.1897 & 0.1217 \\ 0.1211 & 0.1298 & 0.1796 & 0.2012 & 0.1515 \\ 0.0694 & 0.0536 & 0.1295 & 0.2129 & 0.1196 \\ 0.1284 & 0.1236 & 0.2081 & 0.2802 & 0.1807 \\ 0.1186 & 0.1140 & 0.1964 & 0.2562 & 0.1859 \\ 0.1249 & 0.1486 & 0.1604 & 0.1438 & 0.1046 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0.1359 & 0.1343 & 0.2146 & 0.2808 & 0.1808 \\ 0.1199 & 0.1369 & 0.1673 & 0.1594 & 0.1379 \\ 0.0899 & 0.0762 & 0.1606 & 0.2430 & 0.1512 \\ 0.0786 & 0.0572 & 0.1570 & 0.2526 & 0.1685 \\ 0.1508 & 0.1639 & 0.2166 & 0.2438 & 0.1653 \end{pmatrix}.$$

Given

$$A^* = \begin{pmatrix} 0.9993 & 0.3337 & 0.4380 & 0.2330 & 0.7926 & 0.8299 \\ 0.3554 & 0.2296 & 0.9403 & 0.9325 & 0.3290 & 0.2905 \\ 0.0471 & 0.9361 & 0.0058 & 0.7633 & 0.2235 & 0.4026 \\ 0.2137 & 0.6832 & 0.6103 & 0.8264 & 0.3124 & 0.8621 \\ 0.3978 & 0.9621 & 0.8011 & 0.5735 & 0.5845 & 0.6147 \end{pmatrix}.$$

By calculating, $\|X^T X - B^T B\| = 3.4248 \times 10^{-16}$, which satisfies the condition of Theorem 3.2. By Algorithm 5.2, the unique optimal approximation solution $\widehat{A}$ of problem III(b) is

$$\widehat{A} = \begin{pmatrix} 0.7067 & -0.1765 & 0.3819 & -0.1057 & 0.1422 & 0.6550 \\ -0.1831 & 0.5701 & -0.4966 & 0.0552 & 0.5521 & 0.1964 \\ 0.1379 & 0.3338 & 0.3193 & 0.6014 & -0.1594 & -0.3194 \\ 0.3346 & 0.2844 & 0.1953 & 0.1841 & 0.4654 & -0.5594 \\ -0.0900 & 0.5173 & 0.4288 & 0.0496 & -0.1249 & 0.5816 \end{pmatrix},$$

and $\|\widehat{A} - A^*\| = 3.0880$, $\|\widehat{A}X - B\| = 5.7409 \times 10^{-16}$, $\|X^T \widehat{A}^T \widehat{A} X - X^T X\| = 7.7119 \times 10^{-16}$.

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