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Hypertopologies on $\omega_\mu-$Metric Spaces

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Abstract. The $\omega_\mu-$metric spaces, with ω_μ a regular ordinal number, are sets equipped with a distance valued in a totally ordered abelian group having as character ω_μ, but satisfying the usual formal properties of a real metric. The $\omega_\mu-$metric spaces fill a large and attractive class of peculiar uniform spaces, those with a linearly ordered base. In this paper we investigate hypertopologies associated with $\omega_\mu-$metric spaces, in particular the Hausdorff topology induced by the Bourbaki-Hausdorff uniformity associated with their natural underlying uniformity. We show that two $\omega_\mu-$metrics on a same topological space X induce on the hyperspace $CL(X)$, the set of all non-empty closed sets of X, the same Hausdorff topology if and only if they are uniformly equivalent. Moreover, we explore, again in the $\omega_\mu-$metric setting, the relationship between the Kuratowski and Hausdorff convergences on $CL(X)$ and prove that an $\omega_\mu-$sequence $\{A_\alpha\}_{\alpha<\omega_\mu}$ which admits A as Kuratowski limit converges to A in the Hausdorff topology if and only if the join of A with all A_α is $\omega_\mu-$compact.

1. Introduction

The hyperspace $CL(X)$ of all closed nonempty subsets of a bounded metric space (X, d) can be metrized with the Hausdorff metric d_H, defined as:

$$d_H(A, B) := \max\{\sup\{\rho(x, A) : x \in B\}, \sup\{\rho(x, B) : x \in A\}\},$$

which in turn induces on $CL(X)$ the Hausdorff topology $\tau_H(d)$, [4]. It is very well-known that two compatible metrics on a same topological space X generate the same Hausdorff topology on $CL(X)$ if and only if they are uniformly equivalent and that the proof of this result is based on the Efremović Lemma. It is also very well-known that any sequence in $CL(X)$ which converges in the Hausdorff topology to a limit A has the same A as its Kuratowski limit. The converse is not generally true even in the hyperspace of the non-empty compact subsets of X, as Kuratowski proved in [13]. In this paper we investigate the cited properties in the $\omega_\mu-$metric case. The $\omega_\mu-$metric spaces fill a large and attractive class of peculiar uniform spaces containing the usual metric ones. In an extensive work [20] Sikorski introduced the concept of $\omega_\mu-$metric space as a set X equipped with a distance $\rho : X \times X \to G$, valued in a totally ordered abelian

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additive group G, admitting a decreasing $\omega_\mu-$sequence convergent to 0 in the order topology, satisfying the usual formal properties of a real metric, i.e. positiveness, simmetry and triangle inequality. It is worth noting that in 1934 D. Kurepa had already introduced the pseudo-distancial spaces which are equivalent to $\omega_\mu-$metric spaces,[12, 14]. When the range of an $\omega_\mu-$metric ρ is a complete lattice, the existence of infima and suprema is guaranteed and so, consequently, the introduction of the distance between points and sets and of the Hausdorff distance between sets in the usual way. The Hausdorff distance ρ_H on $CL(X)$, when it makes sense, is in its turn an $\omega_\mu-$metric, which induces on $CL(X)$ a topology $\tau_H(\rho)$, which has been named again as the Hausdorff topology associated with ρ, [21]. Of course, the Hausdorff topology $\tau_H(\rho)$ is at the same time the topology induced by the Bourbaki-Hausdorff uniformity on $CL(X)$ associated with the underlying uniformity of ρ, [4, 10]. By using essentially a generalization of the Efremović Lemma due to Alfsen-Njastad, [1], related to uniform spaces with a linearly ordered base of diagonal nhbds, actually the underlying uniform spaces of $\omega_\mu-$metric spaces are of this type, we show that two "$\omega-$metrics" on a same space X generate the same Hausdorff topology on $CL(X)$ iff they are uniformly equivalent. We formulate also a uniform version as follows: Two uniformities with linearly ordered bases generate the same Hausdorff topology iff they coincide. Then, again in the $\omega_\mu-$metric framework, we focus on the relationship between the Hausdorff convergence and the Kuratowski convergence for $\omega_\mu-$sequences. To end we give a necessary and sufficient condition for the Kuratowski convergence to be as strong as the Hausdorff convergence involving $\omega_\mu-$compactness.

2. Background

First, we need to review some relevant definitions, notations and results that we draw from [4, 6, 7, 22].

In an extensive work [20], Sikorski introduced the concept of $\omega_\mu-$metric space as a set X equipped with a distance $\rho : X \times X \to G$, valued in a totally ordered abelian additive group G, admitting a decreasing $\omega_\mu-$sequence convergent to 0 in the order topology, satisfying the usual formal properties of a real metric, i.e. positiveness, symmetry and triangle inequality. Recall that if $(G, +, <)$ is a totally ordered abelian group, whose neutral element 0 is not isolated in the order topology, the *character* of G is the minimal ordinal number ω_μ for which there is a strictly decreasing $\omega_\mu-$sequence convergent to 0.

Any $\omega_\mu-$metric is naturally associated with the topology τ_ρ having as a base the collection of all balls defined as usual. The topology τ_ρ carries peculiar properties. In particular, it is T_2 and paracompact, $\omega_\mu-$additive [3, 20, 21], that is every $\alpha-$intersection of open sets is in its turn open for any $\alpha < \omega_\mu$, and $0-$dimensional in the uncountable case.

Moreover, naturally attached to ρ there is the diagonal uniformity $\mathscr{U}_\rho$ admitting as a base the collection of diagonal neighborhods

$$\{U_\alpha := \{(x, y) \in X \times X : \rho(x, y) < \epsilon_\alpha\} : \alpha < \omega_\mu\},$$

where $\{\epsilon_\alpha\}_{\alpha<\omega_\mu}$ is a strictly decreasing $\omega_\mu-$ sequence convergent to 0 in G. Since $U_\alpha \subset U_\beta$ when $\beta < \alpha$, the uniformity $\mathscr{U}_\rho$ has a linearly ordered base.

Remind that a (diagonal) uniformity has a linearly ordered base when it admits a base $\{U_\alpha : \alpha \in A\}$ of diagonal neighborhoods (entourages), α running over an ordered set $(\Lambda, <)$, and U_α contains U_β whenever $\alpha < \beta$, [6, 7, 22].

The two concepts of $\omega_\mu-$metric and uniformity with a linearly ordered base are dual of each other. Actually in 1934 D. Kurepa had already introduced the pseudo-distancial spaces which later revealed equivalent to uniform spaces with a linearly ordered base,[12, 14]. In [21], Stevenson and Thron proved :

Theorem 2.1. *A separated uniform space $(X, \mathscr{U})$ is $\omega_\mu-$ metrizable if and only if it has a linearly ordered base and $\aleph_\mu$ is the least power of such a base.*

More precisely, they constructed for any uniformity $\mathscr{U}$ on X with a linearly ordered base $\{U_\alpha : \alpha < \omega_\mu\}$ with minimal power $\aleph_\mu$, an $\omega_\mu-$metric ρ on X having $\mathscr{U}$ as natural associated uniformity, taking its values in J_μ, the group of of all functions $x : \{\alpha < \omega_\mu\} \to \mathbb{Z}$ with the pointwise addition and lexicographic order, with range the complete, as proved by Sierpinski [19], lattice D_μ, of all functions $x : \{\alpha < \omega_\mu\} \to \{0, 1\}$. In J_μ the ω_μ−sequence $\{1_\alpha\}_{\alpha<\omega_\mu}$ with 1_α so defined:

$$1_\alpha(\beta) = 0, \text{ when } \beta \neq \alpha \text{ and } 1_\alpha(\alpha) = 1$$

is a minimal ω_μ−sequence decreasing and convergent to the zero of J_μ.
As it is possible to notice the theory of non-metrizable linearly uniformizable spaces appears as a generalization of the metrizable case but there are also various particular features which don't have analogous for metrizable spaces. For example, an ω_μ−metric space which is ω_μ−totally bounded and ω_μ−complete, just complete, not necessarily is ω_μ−compact and the Bourbaki-Hausdorff uniformity associated with a complete ω_μ−metric is generally not complete, [2].

We conclude with the following essential tool due to Alfsen and Njastad, [1]:

Lemma 2.1. Generalized Efremovič Lemma in the uniform version *: Let $(X, \mathscr{U})$ be a uniform space. Let $\{x_\alpha\}, \{y_\alpha\}$, be two nets with α running in a totally ordered set Λ, and U, V two diagonal neighborhoods with $V^4 \subset U$. If $(x_\alpha, y_\alpha) \notin U$ for each $\alpha \in \Lambda$, then there exists a cofinal subset Γ in Λ so that $(x_\alpha, y_\beta) \notin V$ for each $\alpha, \beta \in \Gamma$.*
Generalized Efremovič Lemma in the ω_μ−metric version *: Let (X, ρ) be an ω_μ−metric space. Let $\{x_\alpha\}, \{y_\alpha\}$ be two nets with α running in a totally ordered set Λ, and ϵ, η two elements in the basic group with $0 < 4\eta < \epsilon$. If $\rho(x_\alpha, y_\alpha) \geq \epsilon$ for each $\alpha \in \Lambda$, then there exists a cofinal subset Γ in Λ so that $\rho(x_\alpha, y_\beta) > \eta$ for each $\alpha, \beta \in \Gamma$.*

3. Hausdorff Hypertopology on $CL(X)$

When the range of an ω_μ−metric ρ is a complete lattice, as in the construction of Stevenson and Thron (see section 2), the existence of infima and suprema is guaranteed and so, consequently, the introduction of the distance between points x and sets A as:

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\},$$

of the *Hausdorff distance* between sets A, B, as:

$$\rho_H(A, B) := \max\{\sup\{\rho(x, A) : x \in B\}, \sup\{\rho(x, B) : x \in A\}\}.$$

The Hausdorff distance ρ_H, when it makes sense, is an ω_μ−metric [21]. The topology $\tau_H(\rho)$ associated with the Hausdorff distance has been named the *Hausdorff hypertopology* associated with ρ.

It follows that, if $\mathscr{U}$ is a uniformity on X with a linearly ordered base $\{U_\alpha : \alpha < \omega_\mu\}$, the Bourbaki-Hausdorff uniformity on $CL(X)$, the set of all closed nonempty subsets of X, having as a base:

$$\{\ H(U_\alpha) := \{(A, B) \in CL(X) \times CL(X) : A \subseteq U_\alpha[B] \text{ and } B \subseteq U_\alpha[A]\ \} : \alpha < \omega_\mu\ \}$$

associated with $\mathscr{U}$ is ω_μ−metrizable in its turn. Of course, the topology associated with the Bourbaki-Hausdorff uniformity relative to the underlying uniformity is just the Hausdorff topology.

Observe that a same topological space can have compatible "ω−metrics" with values in groups with different characters. In other words: given a topological space X, it can happen that there are an ω_μ−metric and an ω_ν−metric both on X with ω_μ different from ω_ν, as the following example illustrates. But, that is possible only if X is discrete.

Example 3.1. Take the set Ω_μ, of all ordinals less than ω_μ, $\omega_\mu \neq \omega_0$. Obviously, Ω_μ endowed with the discrete metric is a uniformly discrete ω_0−metric space. On the other hand, Ω_μ equipped with the discrete topology admits as compatible ω_μ−metric taking its values in J_μ, (see section 2), the following one : $d(\alpha, \beta) = 1_\alpha$ if $\alpha < \beta$ and $d(\alpha, \alpha) = \mathbf{0}$. Indeed, for each $\alpha \in \Omega_\mu$ the ball centered at α, $S_{1_\beta}[\alpha] = \{\alpha\}$ when $\alpha < \beta$. But, Ω_μ is not d−uniformly discrete because for each 1_β, $S_{1_\beta}[\alpha] \neq \{\alpha\}$ when $\alpha > \beta$. So definitively, there is an ω_0−metric and at the same time an ω_μ−metric, $\omega_\mu \neq \omega_0$, both generating the same topology on Ω_μ.

But, in the case X admits an ω_μ−metric and an ω_ν−metric with, for example, $\omega_\nu < \omega_\mu$, then X has to be discrete. In fact, if $\mathscr{B}_{\mathscr{U}} = \{U_\alpha : \alpha < \omega_\mu\}$ and $\mathscr{B}_{\mathscr{V}} = \{V_\beta : \beta < \omega_\nu\}$ are bases for the underlying uniformities, for any fixed point $x \in X$ it happens that each $\beta < \omega_\nu$ has a correspondent ordinal $\alpha(\beta)$, $\omega_\mu > \alpha(\beta) > \beta$ so that $U_{\alpha(\beta))}[x] \subseteq V_\beta[x]$. Since the set $\{\alpha(\beta) : \beta < \omega_\nu\}$ has a cardinality less than or equal to $\aleph_\nu (< \aleph_\mu)$, then, by the ω_μ−additivity of X, it follows that $\cap\{U_{\alpha(\beta)} : \beta < \omega_\nu\}$ is a nhbd of x. Finally:

$$\cap \{U_{\alpha(\beta)}[x] : \beta < \omega_\nu\} \subseteq \cap \{V_\beta[x] : \beta < \omega_\nu\} = \{x\}.$$

Now, we are ready to show that the Hausdorff hypertopology associated with an ω_μ−metric is a uniform character.

Theorem 3.1. *Let X be a topological space.If d_μ is an ω_μ−metric on X with base group G and ρ_ν an ω_ν−metric again on X with base group F, then $\tau_H(d_\mu) = \tau_H(\rho_\nu)$ on $CL(X)$ if and only if d_μ and ρ_ν are uniformly equivalent.*

Proof. It is trivial that two uniformly equivalent "ω−metrics" determine the same Hausdorff topology. Conversely. Suppose that, for example, $id : (X, d_\mu) \to (X, \rho_\nu)$ is not uniformly continuous. For simplicity, choose an ω_μ−sequence $\{\epsilon_\alpha\}_{\alpha<\omega_\mu}$ decreasing to zero in G. Then, a positive σ can be identified in F so that, for all $\alpha < \omega_\mu$, there are in X two points x_α, y_α for which $d_\mu(x_\alpha, y_\alpha) < \epsilon_\alpha$ but $\rho_\nu(x_\alpha, y_\alpha) \geq \sigma$. By the ω_μ−metric version of lemma 2.1 there exist a cofinal subset Λ in $\{\alpha < \omega_\mu\}$ and a positive η in F such that $\rho_\nu(x_\beta, y_\gamma) > \eta$ for all $\beta, \gamma \in \Lambda$. Introduce then $A_\gamma = \{x_\delta : \delta \in \Lambda\} \cup \{y_\delta : \delta > \gamma, \delta \in \Lambda\}$, $\gamma \in \Lambda$, and $A = \{x_\delta : \delta \in \Lambda\}$. Now, any accumulation point a for A but not in A is a cluster point for the ω_μ−sequence $\{x_\delta\}_{\delta\in\Lambda}$ as well. In fact, for each $\alpha < \omega_\mu$ there is a point $x_{\beta(\alpha)}$ in A so that $d_\mu(x_{\beta(\alpha)}, a) < \epsilon_\alpha$. Since the net $\{d_\mu(x_{\beta(\alpha)}, a)\}_{\alpha<\omega_\mu}$ is convergent to 0 in the base group G, whose character is ω_μ, it has to be an ω_μ−sequence. Hence, the set $\{\beta(\alpha) : \alpha < \omega_\mu\}$ cannot be bounded above. Consequently, since d_μ and ρ_ν induce the same topology of X, the two adjacent ω_μ−sequences $\{x_\delta\}_{\delta\in\Lambda}, \{y_\delta\}_{\delta\in\Lambda}$ cannot have cluster points. Thus, the sets A and A_γ are all closed. Finally, it happens that $d_H(A_\gamma, A) \to 0$ while $\rho_H(A_\gamma, A) \nrightarrow 0$ and, consequently, $\tau_H(d_\mu) \neq \tau_H(\rho_\nu)$. From $A \subset A_\gamma$ it follows trivially $A \subset S_{\epsilon_\alpha, d_\mu}[A_\gamma]$ for each α and $\gamma \in \Lambda$, and, furthermore, for each $\delta > \alpha$ it happens that $y_\delta \in S_{\epsilon_\delta, d_\mu}[x_\delta] \subseteq S_{\epsilon_\delta, d_\mu}[A] \subseteq S_{\epsilon_\alpha, d_\mu}[A]$. Thus, $\{A_\gamma\}_{\gamma\in\Lambda}$ converges to A in $\tau_H(d_\mu)$, while no A_γ can be contained in $S_{\eta,\rho_\nu}[A]$. In fact, any y_δ in A_γ has a ρ_ν−distance from any point in A greater than η. □

The uniform formulation of the previous result is the following one.

Theorem 3.2. *Two uniformities with linearly ordered base give rise to the same Hausdorff topology on $CL(X)$ if and only if they coincide.*

4. Hausdorff Convergence vs Kuratowski Convergence

We now compare in the ω_μ−metric setting two generally different modes of convergence, the Hausdorff convergence and Kuratowski convergence. It is known that the Kuratowski convergence is weaker than the Hausdorff convergence. In the metric classical framework, in [13], Kuratowski gave a necessary and sufficient condition involving compactness for them to agree. We perform in the ω_μ−metric setting an achievement comprehensive of the Kuratowski one by replacing compactness with ω_μ−compactness which is weaker than compactness in the uncountable case.

We start with the definition and some observations on ω_μ−compactness.

Definition 4.1. An ω_μ−metric space is ω_μ−*compact* if and only if any ω_μ−sequence admits a cluster point.

We recall that any union of less than $\aleph_\mu$ ω_μ−compact subsets is itself ω_μ−compact. And, a closed subset of an ω_μ−compact space is ω_μ−compact in its turn and, vice versa, any ω_μ−compact subset is closed. The ω_0−compactness is just the usual compactness. But, in the uncountable case the ω_μ−compactness is weaker than compactness, as we illustrate by exhibiting the following example.

Example 4.1. The space D^0_μ of all functions $f : \{\alpha < \omega_\mu\} \to \{0, 1\}$ taking the value 1 only on a finite number of coordinates carries as $\omega_\mu-$metric the $J_\mu-$valued distance $\rho : D^0_\mu \times D^0_\mu \to J_\mu$ defined as : $\rho(f, g) = 0$ if $f = g$ and $\rho(f, g) = 1_\alpha$ where α is the first coordinate in which f, g differ, otherwise. D^0_μ is $\omega_\mu-$compact,[21], but not compact. That's why D^0_μ is not totally bounded. Namely, D^0_μ admits an infinite uniformly discrete subset done by all f_n, n a positive integer, defined as $f_n(n) = 1$ while $f_n(\alpha) = 0,\ \alpha \neq n$. When $n < m$, then $\rho(f_n, f_m) = 1_n$. But, for each integer $n,\ 1_n > 1_{\omega_0}$. Consequently, any two distinct f_n, f_m have a $\rho-$distance greater than 1_{ω_0}.

Definition 4.2. Let (X, τ) be a Hausdorff space, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net of subsets of X. A point x_0 is said a *limit point* of $\{A_\lambda\}_{\lambda\in\Lambda}$ if each neighborhood of x_0 intersects A_λ for all λ in some residual subset of Λ. A point x_1 is said a *cluster point* of $\{A_\lambda\}_{\lambda\in\Lambda}$ if each neighborhood of x_1 intersects A_λ for all λ in some cofinal subset of Λ.

The set of all limit points of the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is denoted by LiA_λ and it is called *lower (closed) limit*, while the set of all cluster points of $\{A_\lambda\}_{\lambda\in\Lambda}$ is denoted by LsA_λ and it is called *upper (closed) limit*. The lower limit is the smaller set and the upper limit the larger one. Moreover, a net $\{A_\lambda\}_{\lambda\in\Lambda}$ is said *Kuratowski convergent* to A, more synthetically $\{A_\lambda\}_{\lambda\in\Lambda} \xrightarrow{K} A$, if and only if $LsA_\lambda \subseteq A \subseteq LiA_\lambda$.

From now on, (X, d) stands for an $\omega_\mu-$metric space with $d : X \times X \to G$, and G a totally ordered *Dedekind complete* abelian group with character ω_μ. Next, the sign $\xrightarrow{H}$ is used to mean Hausdorff convergence.

By joining together the following two steps, we can show that the Kuratowski convergence on the hyperspace $CL(X)$ of an $\omega_\mu-$metric space X forces the Hausdorff convergence if and only if X is $\omega_\mu-$compact.

Theorem 4.1. *Let (X, d) be an $\omega_\mu-$metric space. If $\{A_\alpha\}_{\alpha<\omega_\mu}$ is a net of $\omega_\mu-$compact subsets of X having as Hausdorff limit A in its turn $\omega_\mu-$compact, then $\bigcup_{\alpha<\omega_\mu}(A \cup A_\alpha)$ is ω_μ − compact.*

Proof. Of course, any $\omega_\mu-$sequence $\{x_\alpha\}_{\alpha<\omega_\mu}$ when contained in A clusters. If not in A, but contained in a not cofinal union of A_α again clusters. That's why any not cofinal union of $\omega_\mu-$compact sets is $\omega_\mu-$compact. So, for simplicity, suppose x_α extracted by A_α for each α. Since the hypothesis assumes that $d_H(A_\alpha, A) \to 0$, an $\omega_\mu-$sequence $\{a_\alpha\}_{\alpha<\omega_\mu}$ can be identified in A in such a way that $\{d(x_\alpha, a_\alpha)\}_{\alpha<\omega_\mu} \to 0$. By the $\omega_\mu-$compactness of A, it follows that $\{a_\alpha\}_{\alpha<\omega_\mu}$clusters in A. Consequently, by the adjacency with $\{a_\alpha\}_{\alpha<\omega_\mu}$, also $\{x_\alpha\}_{\alpha<\omega_\mu}$ clusters. □

Theorem 4.2. *Let (X, d) be an $\omega_\mu-$metric space. If $\{A_\alpha\}_{\alpha<\omega_\mu}$ is a net in $CL(X)$ having as Kuratowski limit A and $\bigcup_{\alpha<\omega_\mu}(A \cup A_\alpha)$ is $\omega_\mu-$compact, then $\{A_\alpha\}_{\alpha<\omega_\mu}$ converges in the Hausdorff hypertopology to the same A.*

Proof. Being $A \subseteq Li\{A_\alpha\}$, for every $a \in A$ and every positive ϵ in G there exists $a_\alpha \in A_\alpha$ such that $d(a, a_\alpha) < \epsilon$, eventually. Hence, $A \subseteq S_\epsilon[A_\alpha]$, eventually. To acquire the final result by the way of contradiction, suppose that there exists a positive ϵ in G such that $A_\alpha \nsubseteq S_\epsilon[A]$, cofinally. In that case a point a_α can be extracted by A_α so that $d(a_\alpha, a) \geq \epsilon$ for all $a \in A$, cofinally, i.e. α running in a cofinal subset Λ. So, the net $\{a_\alpha\}_{\alpha\in\Lambda}$ is an $\omega_\mu-$sequence, which by the $\omega_\mu-$compactness of $\bigcup_{\alpha<\omega_\mu}(A \cup A_\alpha)$ admits a cluster point that, of course, is at the same time, a cluster point of the $\omega_\mu-$sequence $\{A_\alpha\}_{\alpha<\omega_\mu}$, then belonging to A, but having a positive d-distance from A, a contradiction. □

By joining the previous results we show that:

Theorem 4.3. *Let (X, d) an $\omega_\mu-$metric space. For $\omega_\mu-$sequences in $CL(X)$ the Kuratowski convergence forces the Hausdorff convergence if and only if the space X is $\omega_\mu-$compact.*

Proof. One way is due to Theorem 4.2. Vice versa, in the case X is not $\omega_\mu-$compact there is in $CL(X)$ an $\omega_\mu-$sequence which is Kuratowski convergent but not convergent in the Hausdorff hypertopology. In that case there is in X an $\omega_\mu-$sequence $\{x_\alpha\}_{\alpha<\omega_\mu}$ with no cluster point. Let F be a nonempty $\omega_\mu-$compact subset of X and let denote as $F_\alpha = F \cup \{x_\alpha\}$, with $\alpha < \omega_\mu$. Since $\{F_\alpha\}_{\alpha<\omega_\mu}$ has no cluster points outside of F, it admits as its Kuratowski limit just F. Nevertheless, $\{d_H(F_\alpha, F)\}_{\alpha<\omega_\mu} \nrightarrow 0$. If it were not so, after choosing in G a decreasing $\omega_\mu-$sequence $\{\epsilon_\alpha\}_{\alpha<\omega_\mu}$ convergent to zero, for each $\alpha < \omega_\mu$ an index $\beta(\alpha) > \alpha$ and two points $x_{\beta(\alpha)}$ in $F_{\beta(\alpha)}$, $a_{\beta(\alpha)}$ in F could be identified so that $d(x_{\beta(\alpha)}, a_{\beta(\alpha)}) < \epsilon_\alpha$. By the cofinality of $\{\beta(\alpha) : \alpha < \omega_\mu\}$, the nets $\{x_{\beta(\alpha)}\}_{\alpha<\omega_\mu}$, $\{a_{\beta(\alpha)}\}_{\alpha<\omega_\mu}$ should be two $\omega_\mu-$sequences adjacent to each other. But, $\{a_{\beta(\alpha)}\}_{\alpha<\omega_\mu}$ should cluster in F, then so the $\omega_\mu-$subsequence $\{x_{\beta(\alpha)}\}_{\alpha<\omega_\mu}$ of the starting one. A violation. □

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