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Optimality and Duality for Nonsmooth Minimax Programming Problems Using Convexifactor

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Abstract. The aim of this work is to study optimality conditions for nonsmooth minimax programming problems involving locally Lipschitz functions by means of the idea of convexifactors that has been used in [J. Dutta, S. Chandra, Convexifactors , generalized convexity and vector optimization, Optimization, 53 (2004) 77-94]. Further, using the concept of optimality conditions, Mond-Weir and Wolfe type duality theory has been developed for such a minimax programming problem. The results in this paper extend the corresponding results obtained using the generalized Clarke subdifferential in the literature.

1. Introduction

Minimax programming problems have been the subject of immense interest in the past few years. Some of the basic results of minimax programming problems can be found in books by Danskin [9] and Demyanov and Molozemov [10]. It is well known that optimality and duality lay down the foundation of algorithms for a solution of an optimization problem and hence constitute an important portion in the study of mathematical programming. The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [23]. After the work of Schmitendorf [23], many researchers have worked in this direction, like Ahmad and Husain [1], Ahmad et al. [2], Antczak [5], Lai et al. [16], Jayswal et al. [14], Yang and Hou [25], etc.

As is well known, the notion of subdifferentiability plays a fundamental role in the study of nonsmooth optimization. In recent years, an incredible arrangement of research in nonsmooth analysis has focused on the growth of generalized subdifferentials that give sharp extremality conditions and good calculus rules for nonsmooth functions. A fairly extensive list of references pertaining to several aspects of these generalized subdifferentials and their importance in nonsmooth analysis and optimization is given in [7, 11, 13, 22, 24].

Very recently, the notion of convexifactor was introduced for extended real-valued functions by Jeyakumar and Luc [15] and was further explored by Dutta and Chandra [12, 13] and Li and Zhang [18] to extend various results in nonsmooth analysis and optimization. Convexifactors are important tools of nonsmooth

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analysis, as they are subsets of many well known subdifferentials such as the subdifferentials of Clarke [7], Michel-Penot [21], Ioffe-Morduchovich [22], and Treiman [24]. For more information on convexifactors and their application in optimization, consult [11, 17, 19].

It is clear from Dutta and Chandra [12, 13], Jeyakumar and Luc [15] that, the convexifactors are not necessarily convex or compact. These relaxations allow applications to a large class of nonsmooth continuous functions. Therefore, it should be useful and interesting to study optimality conditions for various nonsmooth optimization problems. In this aspect, we consider the following nonsmooth minimax programming problem to discuss optimality conditions and duality theory in terms of convexifactors:

$$\text{(P)} \qquad \min_{x\in R^n} \max_{1\le i\le k} f_i(x)$$

$$\text{subject to } g_j(x) \le 0,\ j = 1, 2, ..., m,$$

where $f_i : R^n \to R, i \in I = \{1, 2, ..., k\}, g_j : R^n \to R, j \in J = \{1, 2, ..., m\}$, are locally Lipschitz functions on R^n. The region where the constraints are satisfied (feasibility region) is given by $D = \{x \in R^n : g_j(x) \le 0, j = 1, ..., m\}$.

The rest of the paper is written as follows: Section 2 contains the preliminaries and basic definitions which are used in the sequel. Section 3 is devoted to the optimality conditions. In Section 4 and 5, we associate two duals, namely Mond-Weir type dual and Wolfe type dual, to the problem (P) and derive duality results. Finally, conclusion and further developments are given in Section 6.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Throughout this paper, we shall be concerned with finite-dimensional spaces and for any $A \subset R^n$, the convex hull of A is denoted by co(A).

In this section, we recall some basic definition and lemmas, and present some auxiliary results which will be helpful in proving our mains results in the sequel of the paper.
Let $f : R^n \to R \cup \{+\infty\}$, be an extended real-valued function, then

$$f^-(x, d) = \lim_{t\to 0+} \inf \frac{f(x + td) - f(x)}{t},$$

$$f^+(x, d) = \lim_{t\to 0+} \sup \frac{f(x + td) - f(x)}{t}$$

denote, respectively, the lower and upper Dini directional derivatives of f at $x \in R^n$ in the direction $d \in R^n$.

Now, we begin with the definitions of convexifactor given by Dutta and Chandra [13].

Definition 2.1. *A function $f : R^n \to R \cup \{+\infty\}$ is said to admit a lower convexifactor $\partial_L f(x)$ at $x \in R^n$ if $\partial_L f(x) \subset R^n$ is closed and*

$$f^+(x, d) \ge \inf_{x^*\in\partial_L f(x)} \langle x^*, d\rangle,\ \forall d \in R^n,$$

where $\langle ., .\rangle$ denotes the inner product of the vectors.

Definition 2.2. *A function $f : R^n \to R \cup \{+\infty\}$ is said to admit an upper convexifactor $\partial^U f(x)$ at $x \in R^n$ if $\partial^U f(x) \subset R^n$ is closed and*

$$f^-(x, d) \le \sup_{x^*\in\partial^U f(x)} \langle x^*, d\rangle,\ \forall d \in R^n.$$

A closed set $\partial^C f(x)$ is said to be a convexifactor of f at x if it is both an upper and lower convexifactor of f at x.

Definition 2.3. *A function $f : R^n \to R \cup \{+\infty\}$ is said to admit a lower regular convexifactor $\partial_L f(x)$ at $x \in R^n$ if $\partial_L f(x) \subset R^n$ is closed and*

$$f^-(x, d) = \inf_{x^*\in\partial_L f(x)} \langle x^*, d\rangle,\ \forall d \in R^n.$$

Definition 2.4. *A function $f : R^n \to R \cup \{+\infty\}$ is said to admit an upper regular convexifactor $\partial^U f(x)$ at $x \in R^n$ if $\partial^U f(x) \subset R^n$ is closed and*

$$f^+(x, d) = \sup_{x^* \in \partial^U f(x)} \langle x^*, d \rangle, \ \forall d \in R^n.$$

Remark 2.5. *Since $f^-(x, d) \leq f^+(x, d)$, for all $d \in R^n$, an upper (lower) regular convexifactor is a convexifactor of f at x. But the converse is not true (see, Dutta and Chandra [12]).*

Remark 2.6. [18] (i) *Let $\partial^C f(x)$ be a convexifactor of f at x. Then for all $\lambda \in R$, $\lambda\partial^C f(x)$ is a convexifactor of λf at x.*
(ii) *Let $\partial^U f(x)$ be an upper regular convexifactor of f at x. Then for all $\lambda > 0$, $\lambda\partial^U f(x)$ is a upper regular convexifactor of λf at x.*
(iii) *Let $\partial_L f(x)$ be a lower regular convexifactor of f at x. Then for all $\lambda > 0$, $\lambda\partial_L f(x)$ is a lower regular convexifactor of λf at x.*

Lemma 2.7. [15] *Assume that the functions $f_1, f_2 : R^n \to R$ admit upper convexifactor $\partial^U f_1(x)$ and $\partial^U f_2(x)$ at x, respectively, and that one of the convexifactor is upper regular at x. Then, $\partial^U f_1(x) + \partial^U f_2(x)$ is an upper convexifactor of $f_1 + f_2$ at x.*

Similarly, if one of the convexifactors is lower regular at x. Then, $\partial_L f_1(x) + \partial_L f_2(x)$ is a lower convexifactor of $f_1 + f_2$ at x.

From now on, whenever we say that f admits a convexifactor at x, we shall always denote it by $\partial^C f(x)$. Along the lines of Dutta and Chandra [12] and Li and Zhang [18], we now give the definitions of ∂^C-convex, strict ∂^C-convex, ∂^C-pseudoconvex, strict ∂^C-pseudoconvex and ∂^C-quasiconvex functions by using the concept of convexifactors. Assume that $f : R^n \to R$ admits a convexifactor $\partial^C f(\bar{x})$ at $\bar{x} \in R^n$.

Definition 2.8. *A function $f : R^n \to R$ is said to be ∂^C-convex at $\bar{x} \in R^n$ if, for all $x \in R^n$,*

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall \xi \in \partial^C f(\bar{x}).$$

If strict inequality holds in above definition for $x \neq \bar{x}$, then f is said to be strict ∂^C-convex at $\bar{x}$.

Definition 2.9. *A function $f : R^n \to R$ is said to be ∂^C-pseudoconvex at $\bar{x} \in R^n$ if, for all $x \in R^n$,*

$$f(x) < f(\bar{x}) \Rightarrow \langle \xi, x - \bar{x} \rangle < 0, \ \forall \xi \in \partial^C f(\bar{x}),$$

equivalently

$$\langle \xi, x - \bar{x} \rangle \geq 0 \Rightarrow f(x) \geq f(\bar{x}), \ \forall \xi \in \partial^C f(\bar{x}).$$

Definition 2.10. *A function $f : R^n \to R$ is said to be strict ∂^C-pseudoconvex at $\bar{x} \in R^n$ if, for all $x \in R^n$, $x \neq \bar{x}$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \langle \xi, x - \bar{x} \rangle < 0, \ \forall \xi \in \partial^C f(\bar{x}),$$

equivalently

$$\langle \xi, x - \bar{x} \rangle \geq 0 \Rightarrow f(x) > f(\bar{x}), \ \forall \xi \in \partial^C f(\bar{x}).$$

Definition 2.11. *A function $f : R^n \to R$ is said to be ∂^C-quasiconvex at $\bar{x} \in R^n$ if, for all $x \in R^n$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \langle \xi, x - \bar{x} \rangle \leq 0, \ \forall \xi \in \partial^C f(\bar{x}),$$

equivalently

$$\langle \xi, x - \bar{x} \rangle > 0 \Rightarrow f(x) > f(\bar{x}), \ \forall \xi \in \partial^C f(\bar{x}).$$

It is well known that the problem (P) is equivalent (see, for example, [8]) to the following parametric optimization problem:

$$(\text{EP}) \qquad \min \quad v$$

$$\text{subject to } f_i(x) \leq v, i = 1, 2, ..., k, \tag{1}$$

$$g_j(x) \leq 0, j = 1, 2, ..., m, \tag{2}$$

$$(x, v) \in R^n \times R. \tag{3}$$

Let $A = \{(x, v) \in R^n \times R : f_i(x) \leq v, i = 1, 2, ..., k, g_j(x) \leq 0, j = 1, ..., m\}$ be the set of all feasible solutions of (EP). For problem (EP), we define the Lagrange function as follows

$$L(x, v, \lambda, \mu) = v + \sum_{i=1}^{k} \lambda_i (f_i(x) - v) + \sum_{j=1}^{m} \mu_j g_j(x).$$

The following results gives the relationship between the minimax programming problem (P) and the corresponding parametric optimization problem (EP).

Lemma 2.12. [8] *If a point (x, v) is feasible in the parametric optimization problem* (EP)*, then x is a feasible point in the considered minimax programming problem* (P)*. And so, if a point x is feasible in* (P)*, then there exists $v \in R$ such that (x, v) is a feasible point in* (EP)*.*

Lemma 2.13. [8] *A point $\bar{x}$ is an optimal solution in the considered minimax programming problem* (P) *with the corresponding optimal value of the objective function of* (P) *equal to $\bar{v}$ if and only if a point $(\bar{x}, \bar{v})$ is an optimal solution in its associated parametric optimization problem* (EP) *with the corresponding optimal value of the objective function of* (EP) *equal to $\bar{v}$.*

3. Optimality Conditions

In this section we give necessary and some sufficient optimality conditions for minimax programming problem (P) in terms of convexifactors. To obtain necessary optimality conditions for minimax programming problem (P), we use the following Slater-type weak constraint qualification which is defined as follows on the lines of Mangasarian [20].

Definition 3.1. *The minimax programming problem* (P) *is said to satisfy the Slater-type weak constraint qualification at $\bar{x} \in D$, if g_j is ∂^C-pseudoconvex at $\bar{x}$, and there exists an $x_0 \in R^n$ such that $g_j(x_0) < 0$ where $j \in J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\}$.*

Note that, If g is a differentiable function at $\bar{x}$ and admits an upper regular convexifactor $\partial^U g(\bar{x})$ at $\bar{x}$, then the above Slater-type weak constraint qualification reduces to Slater's weak constraint qualification given by Mangasarian [20].

Theorem 3.2 (Parametric Necessary Optimality Conditions). *Let $\bar{x} \in D$ be an optimal solution to the considered minimax programming problem* (P) *with the corresponding optimal value for* (P) *equal to $\bar{v}$ and the Slater-type weak constraint qualification be satisfied at $\bar{x}$. Assume that $f_i, i \in I, g_j, j \in J$ are continuous and admit bounded convexifactors $\partial^C f_i(\bar{x}), i \in I, \partial^C g_j(\bar{x}), j \in J$, respectively and that $\partial^C f_i, i \in I, \partial^C g_j, j \in J$, are upper semicontinuous at $\bar{x}$. Then, there exist $\bar{\lambda} \in R^k_+, \bar{\lambda} \neq 0$ and $\bar{\mu} \in R^m_+$ with $\sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j=1}^{m} \bar{\mu}_j = 1$, such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ satisfies the following* *conditions*

$$0 \in \sum_{i=1}^{k} \bar{\lambda}_i \text{co}(\partial^C f_i(\bar{x})) + \sum_{j=1}^{m} \bar{\mu}_j \text{co}(\partial^C g_j(\bar{x})), \tag{4}$$

$$\bar{\lambda}_i(f_i(\bar{x}) - \bar{v}) = 0, i = 1, 2, ..., k, \tag{5}$$

$$f_i(\bar{x}) \leq \bar{v}, i = 1, 2, ..., k, \tag{6}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, j = 1, 2, ..., m, \tag{7}$$

$$g_j(\bar{x}) \leq 0, j = 1, 2, ..., m. \tag{8}$$

Proof. Let $\bar{x} \in D$ be an optimal solution to the considered minimax programming problem (P) with the corresponding optimal value of the objective function equal to $\bar{v}$. Then, by Lemma 2.13, $(\bar{x}, \bar{v})$ is an optimal solution in its associated parametric optimization problem (EP) with the corresponding optimal value of the objective function equal to $\bar{v}$. Therefore, if we apply Theorem 3.3 [13] to problem (EP), then there exist $\bar{\lambda} \in R^k_+$ and $\bar{\mu} \in R^m_+$ with $\sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j=1}^{m} \bar{\mu}_j = 1$, such that the above relations (4)-(8) are satisfied.

It is remaining to show $\bar{\lambda} \neq 0$. Suppose to contrary that $\bar{\lambda} = 0$, then by $\sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j=1}^{m} \bar{\mu}_j = 1$, we have that $\sum_{j=1}^{m} \bar{\mu}_j = 1$.

Also by (4), it is clear that there exists $\zeta_j \in \text{co}(\partial^C g_j(\bar{x})), j = 1, 2, ..., m$ such that

$$\sum_{j=1}^{m} \bar{\mu}_j \zeta_j = 0. \tag{9}$$

On the other hand, by assumption, the Slater-type weak constraint qualification is satisfied at $\bar{x}$, therefore, taking into account the structure of the constraint in problem (EP), the same constraint qualification is satisfied at $(\bar{x}, \bar{v})$ in problem (EP), and hence, we have that g_j is ∂^C-pseudoconvex at $\bar{x}$, and there exists an $x_0 \in R^n$ such that

$$g_j(x_0) < 0, j \in J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\},$$

which intern implies that $g_j(x_0) < 0 = g_j(\bar{x}), j \in J(\bar{x})$.

From using ∂^C- pseudoconvexity of $g_j, j \in J(\bar{x})$, we get

$$\left\langle \zeta_j, x_0 - \bar{x} \right\rangle < 0, \ \zeta_j \in \partial^C g_j(\bar{x}), \ j \in J(\bar{x}).$$

Now, equation (7) gives $\bar{\mu}_j = 0$ for $j \notin J(\bar{x})$ and from $\bar{\mu} \in R^m_+$, $\sum_{j=1}^{m} \bar{\mu}_j = 1$, we obtain

$$\left\langle \sum_{j=1}^{m} \bar{\mu}_j \zeta_j, x_0 - \bar{x} \right\rangle < 0, \zeta_j \in \partial^C g_j(\bar{x}).$$

This clearly shows that

$$\left\langle \sum_{j=1}^{m} \bar{\mu}_j \zeta_j, x_0 - \bar{x} \right\rangle < 0, \zeta_j \in \text{co}(\partial^C g_j(\bar{x})),$$

which contradicts (9). Hence, $\bar{\lambda} \neq 0$. □

Remark 3.3. *Since the convex hull of a convexifactor can be properly contained in the Clarke sub-differential (see [15]), therefore the above optimality conditions expressed in terms of convexifactors are generally sharper than those expressed in terms of Clarke sub-differential. Also, note that the convex hull appearing before the convexifactors in Theorem 3.2 cannot be removed in general (see Example 3.1 in Dutta and Chandra [13]).*

Remark 3.4. *On compare the above results those in [4], we observe that our parametric necessary optimality conditions are close to those of Theorem 13 [4] and also generalizes them in view of Remark 3.3.*

In the following example we give a nonsmooth minimax programming problem and investigate the parametric necessary optimality conditions in terms of convexifactors.

Example 3.5. *Consider the following nonsmooth minimax optimization problem:*

$$\text{(P)} \qquad \min_{x\in R} \max_{1\le i\le 2} f_i(x) = (x^2 - i\,|x-1|)$$

$$\text{subject to } g(x) = 1 - x \le 0.$$

Note that the set of feasible solutions of (P) *is* $D = [1,\infty)$, *and the optimal objective value is achieved at* $\bar{x} = 1$, *where the corresponding optimal value of the objective function equal to* $\bar{v}$.
Consider the bounded convexifactors of $f_i, i = 1,2$ *and* g *at* $\bar{x} = 1$ *are* $\partial^C f_1(1) = \{0,3\}, \partial^C f_2(1) = \{0,4\}$ *and* $\partial^C g(1) = \{-1\}$. *It is easy to observe that* $\partial^C f_i, i = 1,2$ *and* $\partial^C g$ *are upper semicontinuous at* $\bar{x} = 1$.
Now, by simple calculation, for all $x \in R$, *we have*

$$g(x) < g(\bar{x}) \Rightarrow \langle \zeta, x - \bar{x}\rangle < 0,\ \forall \zeta \in \partial^C g(\bar{x}),$$

or,

$$\langle \zeta, x - \bar{x}\rangle \ge 0 \Rightarrow g(x) \ge g(\bar{x}),\ \forall \zeta \in \partial^C g(\bar{x}).$$

It means that g *is* ∂^C*-pseudoconvex at* $\bar{x} = 1$. *Thus, clearly, we can observe that the Slater-type weak constraint qualification holds at* $\bar{x} = 1$.
Since all assumptions of Theorem 3.2 are fulfilled, then, it is easy to see that there exist $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2) \in R^2_+, \bar{\lambda} \ne 0$ *and* $\bar{\mu} \ge 0$ *with* $\sum_{i=1}^{2} \bar{\lambda}_i + \bar{\mu} = 1$ *such that the parametric necessary optimality conditions (4)-(8) are satisfied.*

In a subsequent part, we can see that parametric necessary optimality conditions are sufficient under generalized convexity.

Theorem 3.6 (Sufficient Optimality Conditions). *Let* $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ *with* $\bar{x} \in D, \bar{v} \in R, \bar{\lambda} \in R^k_+, \bar{\lambda} \ne 0$ *and* $\bar{\mu}_j \in R^m_+$, *satisfying the conditions (4)-(8). Further, assume that the functions* $f_i(.), i \in I(\bar{x}) = \{i \in I : \bar{\lambda}_i > 0\}$, *and* $g_j(.), j \in J(\bar{x})$ *are* ∂^C*-convex at* $\bar{x}$ *on* D. *Then* $\bar{x}$ *is an optimal point in* (P) *with the corresponding optimal objective value equal to* $\bar{v}$.

Proof. We proceed by contradiction. Suppose that $\bar{x}$ is not an optimal solution for (P), then there exists a feasible point $\tilde{x}$ for (P), such that

$$f_i(\tilde{x}) < f_i(\bar{x}), i = 1,2,...,k. \tag{10}$$

Since $f_i(.), i \in I(\bar{x})$ and $g_j(.), j \in J(\bar{x})$ are ∂^C-convex at $\bar{x}$ on D, then, by Definition 2.8, the following inequalities

$$f_i(x) - f_i(\bar{x}) \ge \langle \zeta_i, x - \bar{x}\rangle,\ i \in I(\bar{x}), \tag{11}$$

$$g_j(x) - g_j(\bar{x}) \ge \left\langle \xi_j, x - \bar{x}\right\rangle,\ j \in J(\bar{x}), \tag{12}$$

hold for any $\zeta_i \in \partial^C f_i(\bar{x}), i \in I(\bar{x})$, any $\xi_j \in \partial^C g_j(\bar{x}), j \in J(\bar{x})$, and all $x \in D$. Therefore, they are also satisfied for $x = \tilde{x}$. Multiplying (11) by $\bar{\lambda}_i > 0, i \in I(\bar{x})$, (12) by $\bar{\mu}_j \ge 0, j \in J(\bar{x})$, we get

$$\bar{\lambda}_i f_i(\tilde{x}) - \bar{\lambda}_i f_i(\bar{x}) \ge \langle \bar{\lambda}_i \zeta_i, \tilde{x} - \bar{x}\rangle,\ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I(\bar{x}),$$

$$\bar{\mu}_j g_j(\tilde{x}) - \bar{\mu}_j g_j(\bar{x}) \ge \left\langle \bar{\mu}_j \xi_j, \tilde{x} - \bar{x}\right\rangle,\ \forall \xi_j \in \partial^C g_j(\bar{x}), j \in J(\bar{x}).$$

From (7), (10) and from the feasibility of $\tilde{x}$ in problem (P), we get

$$\langle \bar{\lambda}_i \zeta_i, \tilde{x} - \bar{x}\rangle < 0,\ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I(\bar{x}),$$

$\left\langle \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle \leq 0, \ \forall \xi_j \in \partial^C g_j(\bar{x}), j \in J(\bar{x}).$

On adding both sides of the above inequalities, we obtain

$$\left\langle \sum_{i \in I(\bar{x})} \bar{\lambda}_i \zeta_i + \sum_{j \in J(\bar{x})} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I(\bar{x}), \xi_j \in \partial^C g_j(\bar{x}), j \in J(\bar{x}).$$

This clearly shows that

$$\left\langle \sum_{i \in I(\bar{x})} \bar{\lambda}_i \zeta_i + \sum_{j \in J(\bar{x})} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0, \ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(\bar{x})), i \in I(\bar{x}), \xi_j \in \mathrm{co}(\partial^C g_j(\bar{x})), j \in J(\bar{x}),$$

which contradicts (4). This completes the proof. □

Theorem 3.7 (Sufficient Optimality Conditions). *Let $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ with $\bar{x} \in D, \bar{v} \in R, \bar{\lambda} \in R^k_+, \bar{\lambda} \neq 0$ and $\bar{\mu}_j \in R^m_+$, satisfying the conditions (4)-(8). Further, assume that the functions $f_i(.), i \in I$ are ∂^C-pseudoconvex and $\bar{\mu}_j g_j(.), j \in J$ are ∂^C-quasiconvex at $\bar{x}$ on D. Then $\bar{x}$ is an optimal point in* (P) *with the corresponding optimal objective value equal to $\bar{v}$.*

Proof. We proceed by contradiction. Suppose that $\bar{x}$ is not an optimal solution for (P), then there exists a feasible point $\tilde{x}$ for (P), such that

$f_i(\tilde{x}) < f_i(\bar{x}), i = 1, 2, ..., k,$

which by ∂^C-pseudoconvex of $f_i(.), i \in I$ at $\bar{x}$ on D, gives

$\langle \zeta_i, \tilde{x} - \bar{x} \rangle < 0, \ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I.$

Since $0 \neq \bar{\lambda} \in R^k_+$, then we obtain

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i, \tilde{x} - \bar{x} \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I. \tag{13}$$

On the other hand, from the feasibility of $\tilde{x}$ to (P), $\bar{\mu} \in R^m_+$ and equality (7), we have

$\bar{\mu}_j g_j(\tilde{x}) \leq \bar{\mu}_j g_j(\bar{x}), j \in J,$

which by ∂^C-quasiconvexity of $\bar{\mu}_j g_j(.), j \in J$ at $\bar{x}$ on D, gives

$\left\langle \xi'_j, \tilde{x} - \bar{x} \right\rangle \leq 0, \ \forall \xi'_j \in \partial^C (\bar{\mu}_j g_j)(\bar{x}), j \in J.$

The above inequality together with Remark 2.6 yields

$\left\langle \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle \leq 0, \ \forall \xi_j \in \partial^C g_j(\bar{x}), j \in J.$

Thus,

$$\left\langle \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle \leq 0, \ \forall \xi_j \in \partial^C g_j(\bar{x}), j \in J. \tag{14}$$

On adding the inequalities (13) and (14), we obtain

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i + \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I, \xi_j \in \partial^C g_j(\bar{x}), j \in J.$$

This clearly shows that

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i + \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0,\ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(\bar{x})), i \in I, \xi_j \in \mathrm{co}(\partial^C g_j(\bar{x})), j \in J,$$

which contradicts (4). This completes the proof. □

Theorem 3.8 (Sufficient Optimality Conditions). *Let $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ with $\bar{x} \in D, \bar{v} \in R, \bar{\lambda} \in R^k_+, \bar{\lambda} \neq 0$ and $\bar{\mu}_j \in R^m_+$, satisfying the conditions (4)-(8). Further, assume that the functions $f_i(.), i \in I$ are strict ∂^C-pseudoconvex and $\bar{\mu}_j g_j(.), j \in J$ are ∂^C-quasiconvex at $\bar{x}$ on D. Then $\bar{x}$ is an optimal point in* (P) *with the corresponding optimal objective value equal to $\bar{v}$.*

Proof. The proof follows along similar lines as the proof of Theorem 3.7 and hence is omitted. □

Theorem 3.9 (Sufficient Optimality Conditions). *Let $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ with $\bar{x} \in D, \bar{v} \in R, \bar{\lambda} \in R^k_+, \bar{\lambda} \neq 0$ and $\bar{\mu}_j \in R^m_+$, satisfying the conditions (4)-(8). Further, assume that the functions $f_i(.), i \in I$ are ∂^C-quasiconvex and the functions $g_j(.), j \in J(\bar{x}), j \neq l$ are ∂^C-quasiconvex at $\bar{x}$ on D and each $g_l(.)$ is strict ∂^C-pseudoconvex at $\bar{x}$ on D with $\bar{\mu}_l > 0$. Then $\bar{x}$ is an optimal point in* (P) *with the corresponding optimal objective value equal to $\bar{v}$.*

Proof. We proceed by contradiction. Suppose that $\bar{x}$ is not an optimal solution for (P), then there exists a feasible point $\tilde{x}$ for (P), such that

$$f_i(\tilde{x}) < f_i(\bar{x}), i = 1, 2, ..., k,$$

which by ∂^C-quasiconvex of $f_i(.), i \in I$ at $\bar{x}$ on D, gives

$$\langle \zeta_i, \tilde{x} - \bar{x} \rangle \leq 0,\ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I.$$

Since $0 \neq \bar{\lambda} \in R^k_+$, then we obtain

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i, \tilde{x} - \bar{x} \right\rangle \leq 0,\ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I. \tag{15}$$

On the other hand, from the feasibility of $\tilde{x}$ to (P) and $j \in J(\bar{x})$, we have

$$g_j(\tilde{x}) \leq 0 = g_j(\bar{x}),$$

which by ∂^C-quasiconvex of $g_j(.), j \in J(\bar{x}), j \neq l$ and strict ∂^C-pseudoconvex of $g_l(.)$ at $\bar{x}$ on D, we have, respectively

$$\left\langle \xi_j, \tilde{x} - \bar{x} \right\rangle \leq 0,\ \forall \xi_j \in \partial^C g_j(\bar{x}), j \in J(\bar{x}), j \neq l, \tag{16}$$

$$\langle \xi_l, \tilde{x} - \bar{x} \rangle < 0,\ \forall \xi_l \in \partial^C g_l(\bar{x}). \tag{17}$$

Since $\bar{\mu}_j \geq 0, \forall j \in j(\bar{x}), \bar{\mu}_l > 0$ and $\bar{\mu}_j = 0$ for $j \in J \setminus J(\bar{x})$, therefore, from (16) and (17), we have

$$\left\langle \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0,\ \forall \xi_j \in \partial^C g_j(\bar{x}), j \in J. \tag{18}$$

On adding the inequalities (15) and (18), we obtain

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i + \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0,\ \forall \zeta_i \in \partial^C f_i(\bar{x}), i \in I, \xi_j \in \partial^C g_j(\bar{x}), j \in J.$$

This clearly shows that

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i + \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \tilde{x} - \bar{x} \right\rangle < 0,\ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(\bar{x})), i \in I, \xi_j \in \mathrm{co}(\partial^C g_j(\bar{x})), j \in J,$$

which contradicts (4). This completes the proof. □

4. Mond-Weir Type Duality

Using necessary optimality conditions of Theorem 3.2, we introduce Mond-Weir type dual (MWD) to the problem (EP) in terms of convexifactors as follows:

(MWD) Max q

$$\text{subject to } 0 \in \sum_{i=1}^{k} \lambda_i \mathrm{co}(\partial^C f_i(y)) + \sum_{j=1}^{m} \mu_j \mathrm{co}(\partial^C g_j(y)), \tag{19}$$

$$\lambda_i(f_i(y) - q) \geq 0, i = 1, 2, ..., k, \tag{20}$$

$$\mu_j g_j(y) \geq 0, j = 1, 2, ..., m, \tag{21}$$

where $0 \neq \lambda \in R^k_+, \mu \in R^m_+, y \in R^n, q \in R$.
Let W denote the set of all feasible points of (MWD). Further, we denote by Y the set $Y = \{y \in R^n : (y, q, \lambda, \mu) \in W\}$.
Now, we establish duality theorems relating (EP) and (MWD) under suitable generalized convexity with respect to convexifactor assumptions. Since (EP) is equivalent to (P), this implies the duality theorems relate (P) and (MWD) also.

Theorem 4.1 (Weak Duality). *Let (x, v) and (y, q, λ, μ) be feasible solutions in problems* (EP) *and* (MWD), *respectively. Further, assume that $f_i(.), i \in I(y)$, and $g_j(.), j \in J(y)$ are ∂^C-convex at y on $A \cup Y$. Then $v \geq q$.*

Proof. We proceed by contradiction, suppose $v < q$. Then, by (1) and (20), we get

$$f_i(x) < f_i(y), i = 1, 2, ..., k. \tag{22}$$

Since $f_i(.), i \in I(y)$ and $g_j(.), j \in J(y)$ are ∂^C-convex at y on $A \cup Y$, then, by Definition 2.8, the following inequalities

$$f_i(x) - f_i(y) \geq \langle \zeta_i, x - y \rangle, \ i \in I(y), \tag{23}$$

$$g_j(x) - g_j(y) \geq \left\langle \xi_j, x - y \right\rangle, \ j \in J(y), \tag{24}$$

hold for any $\zeta_i \in \partial^C f_i(y), i \in I(y)$, any $\xi_j \in \partial^C g_j(y), j \in J(y)$, and all $x \in A \cup Y$. Multiplying (23) by $\lambda_i > 0, i \in I(y)$, (24) by $\mu_j \geq 0, j \in J(y)$, we get

$$\lambda_i f_i(x) - \lambda_i f_i(y) \geq \langle \lambda_i \zeta_i, x - y \rangle, \ \forall \zeta_i \in \partial^C f_i(y), i \in I(y),$$

$$\mu_j g_j(x) - \mu_j g_j(y) \geq \left\langle \mu_j \xi_j, x - y \right\rangle, \ \forall \xi_j \in \partial^C g_j(y), j \in J(y).$$

From (21), (22) and from the feasibility of x in problem (P), we get

$$\langle \lambda_i \zeta_i, x - y \rangle < 0, \ \forall \zeta_i \in \partial^C f_i(y), i \in I(y),$$

$$\left\langle \mu_j \xi_j, x - y \right\rangle \leq 0, \ \forall \xi_j \in \partial^C g_j(y), j \in J(y).$$

On adding both sides of the above inequalities, we obtain

$$\left\langle \sum_{i \in I(y)} \lambda_i \zeta_i + \sum_{j \in J(y)} \mu_j \xi_j, x - y \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(y), i \in I(y), \xi_j \in \partial^C g_j(y), j \in J(y).$$

This clearly shows that

$$\left\langle \sum_{i \in I(y)} \lambda_i \zeta_i + \sum_{j \in J(y)} \mu_j \xi_j, x - y \right\rangle < 0, \ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(y)), i \in I(y), \xi_j \in \mathrm{co}(\partial^C g_j(y)), j \in J(y),$$

which contradicts (19). This completes the proof. □

Theorem 4.2 (Weak Duality). *Let* (x, v) *and* (y, q, λ, μ) *be feasible solutions in problems* (EP) *and* (MWD), *respectively. Further, assume that* $f_i(.), i \in I$ *are* ∂^C*-pseudoconvex and* $\mu_j g_j(.), j \in J$ *are* ∂^C*-quasiconvex at* y *on* $A \cup Y$*. Then* $v \geq q$.

Proof. We proceed by contradiction, suppose $v < q$. Then, by (1) and (20), we get

$$f_i(x) < f_i(y), i = 1, 2, ..., k,$$

which by ∂^C-pseudoconvex of $f_i(.), i \in I$ at y on $A \cup Y$, we have

$$\langle \zeta_i, x - y \rangle < 0, \ \forall \zeta_i \in \partial^C f_i(y), i \in I.$$

Since $0 \neq \lambda \in R^k_+$, then we obtain

$$\left\langle \sum_{i=1}^{k} \lambda_i \zeta_i, x - y \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(y), i \in I. \tag{25}$$

On the other hand, from the feasibility of x to (EP), $\mu \in R^m_+$, and inequality (21), we have

$$\mu_j g_j(x) \leq \mu_j g_j(y), j \in J,$$

which by ∂^C-quasiconvexity of $\mu_j g_j(.), j \in J$ at y on $A \cup Y$, gives

$$\left\langle \xi'_j, x - y \right\rangle \leq 0, \ \forall \xi'_j \in \partial^C(\mu_j g_j)(y), j \in J.$$

The above inequality together with Remark 2.6 yields

$$\left\langle \mu_j \xi_j, x - y \right\rangle \leq 0, \ \forall \xi_j \in \partial^C g_j(y), j \in J.$$

Thus,

$$\left\langle \sum_{j=1}^{m} \mu_j \xi_j, x - y \right\rangle \leq 0, \ \forall \xi_j \in \partial^C g_j(y), j \in J. \tag{26}$$

On adding the inequalities (25) and (26), we obtain

$$\left\langle \sum_{i=1}^{k} \lambda_i \zeta_i + \sum_{j=1}^{m} \mu_j \xi_j, x - y \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(y), i \in I, \xi_j \in \partial^C g_j(y), j \in J.$$

This clearly shows that

$$\left\langle \sum_{i=1}^{k} \lambda_i \zeta_i + \sum_{j=1}^{m} \mu_j \xi_j, x - y \right\rangle < 0, \ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(y)), i \in I, \xi_j \in \mathrm{co}(\partial^C g_j(y)), j \in J,$$

which contradicts (19). This completes the proof. □

Theorem 4.3 (Strong Duality). *Let* $(\bar{x}, \bar{v})$ *be an optimal point for* (EP). *Assume that the hypotheses of Theorem 3.2 hold for the problem* (EP). *Then, there exist* $\bar{\lambda} \in R^k_+$ *and* $\bar{\mu} \in R^m_+$ *such that* $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ *is feasible for* (MWD) *and the corresponding objective values of* (EP) *and* (MWD) *are equal. Further, if the hypotheses of the weak duality theorem (Theorem 4.1 or Theorem 4.2) hold for all feasible solutions of* (MWD), *then* $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ *is optimal for* (MWD).

Proof. Since $(\bar{x}, \bar{v})$ is optimal for (EP) and all the assumptions of Theorem 3.2 are satisfied for the problem (EP), therefore, there exist $0 \neq \bar{\lambda} \in R^k_+$ and $\bar{\mu} \in R^m_+$, such that the conditions (4)-(8) hold, which implies that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (MWD) and the objective values of (EP) and (MWD) are equal. By weak duality theorem (Theorem 4.1 or 4.2), for any feasible solution (y, q, λ, μ) of (MWD), we have $\bar{v} \geq q$. It follows that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is optimal for (MWD). □

Theorem 4.4 (Strict Converse Duality). *Let $(\bar{x}, \bar{v})$ and $(\bar{y}, \bar{q}, \bar{\lambda}, \bar{\mu})$ be optimal in* (EP) *and* (MWD), *respectively. Assume that the hypothesis of Theorem 3.2 is fulfilled. Further, assume that $f_i(.), i \in I(\bar{y})$, and $g_j(.), j \in J(\bar{y})$ are respectively strict ∂^C-convex and ∂^C-convex at $\bar{y}$ on $A \cup Y$. Then $(\bar{x}, \bar{v}) = (\bar{y}, \bar{q})$.*

Proof. We proceed by contradiction. Suppose that $(\bar{x}, \bar{v}) \neq (\bar{y}, \bar{q})$. According to Theorem 4.3, we know that there exist $0 \neq \bar{\lambda} \in R^k_+$ and $\bar{\mu} \in R^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (MWD) and

$$\bar{v} = \bar{q},$$

which by (1) and (20), we have

$$f_i(\bar{x}) \leq f_i(\bar{y}), i = 1, 2, ..., k.$$

Now, proceeding as in Theorem 4.1, we see that the strict ∂^C-convex of $f_i(.), i \in I(\bar{y})$, and ∂^C-convex of $g_j(.), j \in J(\bar{y})$ at $\bar{y}$ on $A \cup Y$, yields the following inequality

$$\left\langle \sum_{i=1}^{k} \bar{\lambda}_i \zeta_i + \sum_{j=1}^{m} \bar{\mu}_j \xi_j, \bar{x} - \bar{y} \right\rangle < 0, \ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(\bar{y})), i \in I, \xi_j \in \mathrm{co}(\partial^C g_j(\bar{y})), j \in J,$$

which contradicts (19). This completes the proof. □

Theorem 4.5 (Strict Converse Duality). *Let $(\bar{x}, \bar{v})$ and $(\bar{y}, \bar{q}, \bar{\lambda}, \bar{\mu})$ be optimal in* (EP) *and* (MWD), *respectively. Assume that the hypothesis of Theorem 3.2 is fulfilled. Further, assume that $f_i(.), i \in I$ are strict ∂^C-pseudoconvex and $\bar{\mu}_j g_j(.), j \in J$ are ∂^C-quasiconvex at $\bar{y}$ on $A \cup Y$. Then $(\bar{x}, \bar{v}) = (\bar{y}, \bar{q})$.*

Proof. The proof follows on the similar lines of Theorem 4.2 and 4.4. □

5. Wolfe Type Duality

Before presenting Wolf type dual model for (P), we first state the following parameter-free necessary optimality conditions. This can be obtained by replacing $\bar{v}$ with $f_i(\bar{x})$ and rewriting the result of Theorem 3.2 as follows.

Theorem 5.1 (Parameter-free Necessary Optimality Conditions). *Let $\bar{x} \in D$ be an optimal solution to problem* (P) *and the Slater-type weak constraint qualification be satisfied at $\bar{x}$. Assume that $f_i, i \in I, g_j, j \in J$ are continuous and admit bounded convexifactors $\partial^C f_i(\bar{x}), i \in I, \partial^C g_j(\bar{x}), j \in J$, respectively and that $\partial^C f_i, i \in I, \partial^C g_j, j \in J$, are upper semicontinuous at $\bar{x}$. Then, there exist $\bar{\lambda} \in R^k_+, \bar{\lambda} \neq 0$ and $\bar{\mu} \in R^m_+$ with $\sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j=1}^{m} \bar{\mu}_j = 1$, such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfies the following conditions*

$$0 \in \sum_{i=1}^{k} \bar{\lambda}_i \mathrm{co}(\partial^C f_i(\bar{x})) + \sum_{j=1}^{m} \bar{\mu}_j \mathrm{co}(\partial^C g_j(\bar{x})), \tag{27}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, j = 1, 2, ..., m, \tag{28}$$

$$g_j(\bar{x}) \leq 0, j = 1, 2, ..., m. \tag{29}$$

Now, we formulate a parametric-free Wolfe type dual (WD) to (P) in terms of convexifactors as follows:

$$\text{(WD)} \qquad \text{Max} \quad \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y)$$

$$\text{subject to } 0 \in \sum_{i=1}^{k} \lambda_i \mathrm{co}(\partial^C f_i(y)) + \sum_{j=1}^{m} \mu_j \mathrm{co}(\partial^C g_j(y)), \tag{30}$$

$$\lambda \in R^k_+, \sum_{i=1}^{k} \lambda_i = 1, \mu \in R^m_+, y \in R^n. \tag{31}$$

Remark 5.2. *It can be observed easily, by the definition of the Lagrange function for problem* (EP)*, that a parametric-free Wolfe type dual for* (P) *and a parametric Wolfe type dual for* (EP) *have the same form.*

In the proofs of duality results in the sense of Wolfe, we need the following lemma.

Lemma 5.3. [3] *For each $x \in R^n$, one has*

$$\max_{1\le i\le k} f_i(x) = \max_{\lambda\in\Lambda} \sum_{i=1}^{k} \lambda_i f_i(x),$$

where $\Lambda = \{\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \in R_+^k : \sum_{i=1}^{k} \lambda_i = 1\}$.

Let $\tilde{W}$ denote the set of all feasible points of (WD). Further, we denote by $\tilde{Y}$ the set $\tilde{Y} = \{y \in R^n : (y, \lambda, \mu) \in \tilde{W}\}$. We shall now prove the weak, strong and strict converse duality results.

Theorem 5.4 (Weak Duality). *Let x and (y, λ, μ) be feasible solutions in problems* (P) *and* (WD)*, respectively. Further, assume that $f_i(.), i \in I(y)$, and $g_j(.), j \in J(y)$ are ∂^C-convex at y on $D \cup \tilde{Y}$. Then,*

$$\max_{1\le i\le k} f_i(x) \ge \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y).$$

Proof. We proceed by contradiction, suppose

$$\max_{1\le i\le k} f_i(x) < \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y).$$

Thus, by Lemma 5.3, we have

$$\sum_{i=1}^{k} \lambda_i f_i(x) < \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y). \tag{32}$$

On the other hand, since $f_i(.), i \in I(y)$ and $g_j(.), j \in J(y)$ are ∂^C-convex at y on $D \cup \tilde{Y}$, then, by Definition 2.8, the following inequalities

$$f_i(x) - f_i(y) \ge \langle \zeta_i, x - y \rangle, \ i \in I(y), \tag{33}$$

$$g_j(x) - g_j(y) \ge \left\langle \xi_j, x - y \right\rangle, \ j \in J(y), \tag{34}$$

hold for any $\zeta_i \in \partial^C f_i(y), i \in I(y)$, any $\xi_j \in \partial^C g_j(y), j \in J(y)$, and all $x \in D \cup \tilde{Y}$. Multiplying (33) by $\lambda_i > 0, i \in I(y)$, (34) by $\mu_j \ge 0, j \in J(y)$, we get

$$\lambda_i f_i(x) - \lambda_i f_i(y) \ge \langle \lambda_i \zeta_i, x - y \rangle, \ \forall \zeta_i \in \partial^C f_i(y), i \in I(y), \tag{35}$$

$$\mu_j g_j(x) - \mu_j g_j(y) \ge \left\langle \mu_j \xi_j, x - y \right\rangle, \ \forall \xi_j \in \partial^C g_j(y), j \in J(y). \tag{36}$$

On adding both sides of the above inequalities, we obtain

$$\sum_{i\in I(y)} [\lambda_i f_i(x) - \lambda_i f_i(y)] + \sum_{j\in J(y)} [\mu_j g_j(x) - \mu_j g_j(y)] \ge \left\langle \sum_{i\in I(y)} \lambda_i \zeta_i + \sum_{j\in J(y)} \mu_j \xi_j, x - y \right\rangle,$$

$$\forall \zeta_i \in \partial^C f_i(y), i \in I(y), \xi_j \in \partial^C g_j(y), j \in J(y).$$

This clearly shows that

$$\sum_{i\in I(y)}[\lambda_i f_i(x)-\lambda_i f_i(y)]+\sum_{j\in J(y)}[\mu_j g_j(x)-\mu_j g_j(y)]\geq\left\langle\sum_{i\in I(y)}\lambda_i\zeta_i+\sum_{j\in J(y)}\mu_j\xi_j,x-y\right\rangle,$$

$$\forall\zeta_i\in \mathrm{co}(\partial^C f_i(y)), i\in I(y), \xi_j\in \mathrm{co}(\partial^C g_j(y)), j\in J(y).$$

By (30), it follows that

$$\sum_{i\in I(y)}\lambda_i f_i(x)-\sum_{i\in I(y)}\lambda_i f_i(y)+\sum_{j\in J(y)}\mu_j g_j(x)-\sum_{j\in J(y)}\mu_j g_j(y)\geq 0.$$

Thus,

$$\sum_{i\in I(y)}\lambda_i f_i(x)+\sum_{j\in J(y)}\mu_j g_j(x)\geq\sum_{i\in I(y)}\lambda_i f_i(y)+\sum_{j\in J(y)}\mu_j g_j(y).$$

From the feasibility of x to (P), it follows $\sum\limits_{j\in J(y)}\mu_j g_j(x)\leq 0$. Hence, we obtain

$$\sum_{i\in I(y)}\lambda_i f_i(x)\geq\sum_{i\in I(y)}\lambda_i f_i(y)+\sum_{j\in J(y)}\mu_j g_j(y),$$

which contradicts (32). This completes the proof. □

Theorem 5.5 (Weak Duality). *Let x and (y,λ,μ) be feasible solutions in problems* (P) *and* (WD)*, respectively. Assume that for some i, and some j, $\partial^C f_i(y)$, $\partial^C g_j(y)$ are respectively upper regular convexifactors of $f_i(.)$, $i=1,2,...,k$ and $g_j(.)$, $j=1,2,...,m$ at y on $D\cup\tilde{Y}$, and for some $i_0\neq i$, $j_0\neq j$, $\partial^C f_{i_0}(y)$, $\partial^C g_{j_0}(y)$ are respectively lower regular convexifactors of $f_{i_0}(.)$ and $g_{j_0}(.)$ at y on $D\cup\tilde{Y}$. Also, assume that $\sum_{i=1}^{k}\lambda_i\partial^C f_i(y)$ is an upper regular convexifactor of $\sum_{i=1}^{k}\lambda_i f_i(.)$ and $\sum_{j=1}^{m}\mu_j\partial^C g_j(y)$ is a lower regular convexifactor of $\sum_{j=1}^{m}\mu_j g_j(.)$ at y on $D\cup\tilde{Y}$. Further, if $\sum_{i=1}^{k}\lambda_i f_i(.)+\sum_{j=1}^{m}\mu_j g_j(.)$ is ∂^C-pseudoconvex at y on $D\cup\tilde{Y}$, then,*

$$\max_{1\leq i\leq k} f_i(x)\geq\sum_{i=1}^{k}\lambda_i f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y).$$

Proof. Since for some i, and some j, $\partial^C f_i(y)$, $\partial^C g_j(y)$ are respectively upper regular convexifactors of $f_i(.)$, $i=1,2,...,k$ and $g_j(.)$, $j=1,2,...,m$ at y on $D\cup\tilde{Y}$, and for some $i_0\neq i$, $j_0\neq j$, $\partial^C f_{i_0}(y)$, $\partial^C g_{j_0}(y)$ are respectively lower regular convexifactors of $f_{i_0}(.)$ and $g_{j_0}(.)$ at y on $D\cup\tilde{Y}$, using Remark 2.6 and Lemma 2.7, we have that $\sum_{i=1}^{k}\lambda_i\partial^C f_i(y)$, $\sum_{j=1}^{m}\mu_j\partial^C g_j(y)$ are convexifactor of $\sum_{i=1}^{k}\lambda_i f_i(.)$ and $\sum_{j=1}^{m}\mu_j g_j(.)$ at y on $D\cup\tilde{Y}$, respectively. Also, since $\sum_{i=1}^{k}\lambda_i\partial^C f_i(y)$ is an upper regular convexifactor of $\sum_{i=1}^{k}\lambda_i f_i(.)$ and $\sum_{j=1}^{m}\mu_j\partial^C g_j(y)$ is a lower regular convexifactor of $\sum_{j=1}^{m}\mu_j g_j(.)$ at y on $D\cup\tilde{Y}$, using Lemma 2.7, we have that $\sum_{i=1}^{k}\lambda_i\partial^C f_i(y)+\sum_{j=1}^{m}\mu_j\partial^C g_j(y)$ is a convexifactor of $\sum_{i=1}^{k}\lambda_i f_i(.)+\sum_{j=1}^{m}\mu_j g_j(.)$ at y on $D\cup\tilde{Y}$.

Now, by means of contradiction, suppose

$$\max_{1\leq i\leq k} f_i(x)<\sum_{i=1}^{k}\lambda_i f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y).$$

Thus, by Lemma 5.3, we have

$$\sum_{i=1}^{k} \lambda_i f_i(x) < \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y). \tag{37}$$

Using the feasibility of x in (P) and $\mu_j \geq 0, j = 1, 2, ..., m$, we have

$$\sum_{j=1}^{m} \mu_j g_j(x) \leq 0. \tag{38}$$

By (37) and (38), it follows that

$$\sum_{i=1}^{k} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x) < \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y),$$

which by ∂^C-pseudoconvex of $\sum_{i=1}^{k} \lambda_i f_i(.) + \sum_{j=1}^{m} \mu_j g_j(.)$ at y on $D \cup \tilde{Y}$, gives

$$\left\langle \sum_{i=1}^{k} \lambda_i \zeta_i + \sum_{j=1}^{m} \mu_j \xi_j, x - y \right\rangle < 0, \ \forall \zeta_i \in \partial^C f_i(y), i \in I, \xi_j \in \partial^C g_j(y), j \in J.$$

This clearly shows that

$$\left\langle \sum_{i=1}^{k} \lambda_i \zeta_i + \sum_{j=1}^{m} \mu_j \xi_j, x - y \right\rangle < 0, \ \forall \zeta_i \in \mathrm{co}(\partial^C f_i(y)), i \in I, \xi_j \in \mathrm{co}(\partial^C g_j(y)), j \in J,$$

which contradicts (30). This completes the proof. □

Theorem 5.6 (Strong Duality). *Let $\bar{x}$ be an optimal point for* (P). *Assume that the hypotheses of Theorem 5.1 hold. Then, there exist $\bar{\lambda} \in R_+^k$ and $\bar{\mu} \in R_+^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for* (WD) *and the corresponding objective values of* (P) *and* (WD) *are equal. Further, if the hypotheses of the weak duality theorem (Theorem 5.4 or Theorem 5.5) hold for all feasible solutions of* (WD)*, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is optimal for* (WD).

Proof. Since $\bar{x}$ is optimal for (P) and all the assumptions of Theorem 5.1 are satisfied for the problem (P), therefore, there exist $0 \neq \bar{\lambda} \in R_+^k$ and $\bar{\mu} \in R_+^m$, such that the conditions (27)-(29) hold, which implies that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and the objective values of (P) and (WD) are equal. Optimality of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ for (WD) follows from weak duality theorem (Theorem 5.4 or 5.5). □

Theorem 5.7 (Strict Converse Duality). *Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be optimal in* (P) *and* (WD)*, respectively. Assume that the hypothesis of Theorem 5.1 is fulfilled. Further, assume that $f_i(.), i \in I(\bar{y})$, and $g_j(.), j \in J(\bar{y})$ are respectively strict ∂^C-convex and ∂^C-convex at $\bar{y}$ on $A \cup \tilde{Y}$. Then $\bar{x} = \bar{y}$.*

Proof. We proceed by contradiction. Suppose that $\bar{x} \neq \bar{y}$. According to Theorem 5.6, we know that there exist $0 \neq \bar{\lambda} \in R_+^k$ and $\bar{\mu} \in R_+^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and

$$\max_{1 \leq i \leq k} f_i(\bar{x}) = \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}).$$

Thus, by Lemma 5.3, we have

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) \leq \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}). \tag{39}$$

Now, proceeding as in Theorem 5.4, we see that the strict ∂^C-convex of $f_i(.), i \in I(\bar{y})$, and ∂^C-convex of $g_j(.), j \in J(\bar{y})$ at $\bar{y}$ on $A \cup \tilde{Y}$, yields the following inequality

$$\sum_{i\in I(\bar{y})} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i\in I(y)} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J(\bar{y})} \bar{\mu}_j g_j(\bar{y}),$$

which contradicts (39). This completes the proof. □

Theorem 5.8 (Strict Converse Duality). *Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be optimal in* (P) *and* (WD), *respectively. Assume that for some i, and some j, $\partial^C f_i(\bar{y})$, $\partial^C g_j(\bar{y})$ are respectively upper regular convexifactors of $f_i(.), i = 1, 2, ..., k$ and $g_j(.), j = 1, 2, ..., m$ at $\bar{y}$ on $D \cup \tilde{Y}$, and for some $i_0 \neq i$, $j_0 \neq j$, $\partial^C f_{i_0}(\bar{y})$, $\partial^C g_{j_0}(\bar{y})$ are respectively lower regular convexifactors of $f_{i_0}(.)$ and $g_{j_0}(.)$ at $\bar{y}$ on $D \cup \tilde{Y}$. Also, assume that $\sum_{i=1}^{k} \bar{\lambda}_i \partial^C f_i(\bar{y})$ is an upper regular convexifactor of $\sum_{i=1}^{k} \bar{\lambda}_i f_i(.)$ and $\sum_{j=1}^{m} \bar{\mu}_j \partial^C g_j(\bar{y})$ is a lower regular convexifactor of $\sum_{j=1}^{m} \bar{\mu}_j g_j(.)$ at $\bar{y}$ on $D \cup \tilde{Y}$. Further, if $\sum_{i=1}^{k} \bar{\lambda}_i f_i(.) + \sum_{j=1}^{m} \bar{\mu}_j g_j(.)$ is strict ∂^C-pseudoconvex at $\bar{y}$ on $D \cup \tilde{Y}$ and the hypothesis of Theorem 5.1 is fulfilled. Then $\bar{x} = \bar{y}$.*

Proof. The proof follows on the similar lines of Theorem 5.5 and 5.7. □

6. Conclusion

In this paper, we have derived optimality conditions for a nonsmooth minimax programming problem by using generalize convex functions in terms of convexifactors. We associate two dual problems, namely Mond-Weir type dual and Wolfe type dual, with nonsmooth minimax programming problem and establish various duality results. These results generalize several results appeared in the literature (see, for instance, [3, 4, 6]). Duality theory by using convexifactor will be studied for nonsmooth variational and nonsmooth control problems, which will orient the future research of the authors.

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