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Some Fixed Point Results for (β-ψ_1-ψ_2)-contractive Conditions in Ordered b-metric-like Spaces

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Abstract. This paper extends and generalizes results of Mukheimer [$(\alpha-\psi-\varphi)$-contractive mappings in ordered partial b-metric spaces, J. Nonlinear Sci. Appl. 7(2014), 168–179]. A new concept of (β-ψ_1-ψ_2)-contractive mapping using two altering distance functions in ordered b-metric-like space is introduced and basic fixed point results have been studied. Useful examples are illustrated to justify the applicability and effectiveness of the results presented herein. As an application, the existence of solution of fourth-order two-point boundary value problems is discussed and rationalized by a numerical example.

1. Introduction and Preliminaries

A number of generalization of metric spaces have been considered by researchers in recent years. For example, b-metric spaces due to Czerwik in [4, 5] and Bakhtin [3], partial metric spaces due to Matthews [11], metric-like spaces due to Amini-Harandi [2], etc. Recently, Shukla [18] combined the two notions of partial metric space and b-metric space to introduce partial b-metric spaces. This concept was further extended by Alghamdi *et al.* [1] as b-metric-like space. They established some existence and uniqueness results in a b-metric-like space and in a partially ordered b-metric-like space.

On the other hand, the concept of α-admissible maps was introduced by Samet *et al.* [15], and an interesting class of α-contraction type mappings was suggested to establish the existence and uniqueness of fixed points. Thereafter a massive growth occurred in fixed point theory using this concept and its variants. Mukheimer [12] used the notion of α-admissible maps in partial b-metric space and discussed basic fixed point results.

The purpose of this paper is to introduce two new concepts, namely (β-ψ_1-ψ_2)-contractive mapping of type-I and type-II using altering distance functions in ordered b-metric-like space and to extend certain results of Mukheimer [12]. Indeed, some new fixed point results for such mappings have been obtained. Some useful examples are discussed to justify the applicability and effectiveness of our results over the contractive condition due to Mukheimer [12] as well the usage of factor β. An application of the derived

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results to fourth-order two-point boundary value problems is presented and justified with a numerical example.

Before going to results, let us recall some basic concepts and notations.

Definition 1.1. *[3–5] Let $\mathcal{F}$ be a nonempty set and $k \geq 1$ be a given real number. A function $d : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ is called a b-metric if $\forall\ u, v, z \in \mathcal{F}$ the following conditions hold:*

(S_1) $d(u, v) = 0$ *if and only if* $u = v$;
(S_2) $d(u, v) = d(v, u)$;
(S_3) $d(u, v) \leq k[d(u, z) + d(z, v)]$.

Then $(\mathcal{F}, d)$ is said to be a b-metric space and k is the coefficient of $(\mathcal{F}, d)$.

The following notion is given in the paper of Shukla [18].

Definition 1.2. *[18] Let $k \geq 1$ be a given real number and $\mathcal{F}$ be a nonempty set. A function $p_b : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ is said to be a partial b-metric if $\forall\ u, v, z \in \mathcal{F}$ the following assertions hold:*

(i) $u = v$ *if and only if* $p_b(u, u) = p_b(u, v) = p_b(v, v)$;
(ii) $p_b(u, u) \leq p_b(u, v)$;
(iii) $p_b(u, v) = p_b(v, u)$;
(iv) $p_b(u, v) \leq k[p_b(u, z) + p_b(z, v)] - p_b(z, z)$.

Then $(\mathcal{F}, p_b)$ is called a partial b-metric space and k is the coefficient of $(\mathcal{F}, p_b)$.

Definition 1.3. *[2] Let $\mathcal{F}$ be a nonempty set and a mapping $\sigma : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+$ is such that $\forall\ u, v, z \in \mathcal{F}$, it satisfies*

(σ_1) $\sigma(u, v) = 0$ *implies* $u = v$
(σ_2) $\sigma(u, v) = \sigma(v, u)$;
(σ_3) $\sigma(u, v) \leq \sigma(u, z) + \sigma(z, v)$.

Then $(\mathcal{F}, \sigma)$ is said to be a metric-like space.

Examples of metric-like spaces are as follows.

Example 1.4. *[16] Let $\mathcal{F} = \mathbb{R}$; then the mappings $\sigma_i : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+ (i \in \{2, 3, 4\})$, defined by*

$$\sigma_2(u, v) = |u| + |v| + a, \qquad \sigma_3(u, v) = |u - b| + |v - b|, \qquad \sigma_4(u, v) = u^2 + v^2, \tag{1}$$

are metric-like on $\mathcal{F}$, where $a \geq 0$ and $b \in \mathbb{R}$.

Definition 1.5. *[1] Let $\mathcal{F}$ be a nonempty set and $k \geq 1$ be a real number. A function $\sigma_b : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+$ is b-metric-like if $\forall\ u, v, z \in \mathcal{F}$, the following assertions hold:*

$(\sigma_b 1)$ $\sigma_b(u, v) = 0$ *implies* $u = v$
$(\sigma_b 2)$ $\sigma_b(u, v) = \sigma_b(v, u)$
$(\sigma_b 3)$ $\sigma_b(u, v) \leq k[\sigma_b(u, z) + \sigma_b(z, v)]$.

The pair $(\mathcal{F}, \sigma_b)$ is called a b-metric-like space with the coefficient k.

In a b-metric-like space $(\mathcal{F}, \sigma_b)$ if $u, v \in \mathcal{F}$ and $\sigma_b(u, v) = 0$, then $u = v$, but the converse may not be true and $\sigma_b(u, u)$ may be positive for $u \in \mathcal{F}$. Clearly, every b-metric and every partial b-metric is a b-metric-like with the same coefficient k. However, the converses of these facts need not hold [18].

Every b-metric-like σ_b on $\mathcal{F}$ generates a topology τ_{σ_b} on $\mathcal{F}$ whose base is the family of all open σ_b-balls $\{B_{\sigma_b}(u, \delta) : u \in \mathcal{F}, \delta > 0\}$, where $B_{\sigma_b}(u, \delta) = \{v \in \mathcal{F} : |\sigma_b(u, v) - \sigma_b(u, u)| < \delta\}$, $\forall\ u \in \mathcal{F}$ and $\delta > 0$.

Definition 1.6. *[1, 18] Let $(\mathcal{F}, \sigma_b)$ be a b-metric-like space with coefficient k, let $\{u_n\}$ be a sequence in $\mathcal{F}$ and $u \in \mathcal{F}$. Then* *(i)* $\{u_n\}$ *is called convergent to* u *w.r.t.* τ_{σ_b}*, if* $\lim_{n\to\infty} \sigma_b(u_n, u) = \sigma_b(u, u)$;

(ii) $\{u_n\}$ *is called a Cauchy sequence in* $(\mathcal{F}, \sigma_b)$ *if* $\lim_{n,m\to\infty} \sigma_b(u_n, u_m)$ *exists (and is finite).*

(iii) $(\mathcal{F}, \sigma_b)$ *is called a complete b-metric-like space if for every Cauchy sequence* $\{u_n\}$ *in* $\mathcal{F}$ *there exists* $u \in \mathcal{F}$ *such that*

$$\lim_{n,m\to\infty} \sigma_b(u_n, u_m) = \lim_{n\to\infty} \sigma_b(u_n, u) = \sigma_b(u, u). \tag{2}$$

It is clear that the limit of a sequence is usually not unique in a b-metric-like space (already partial metric spaces have this property).

Definition 1.7. *[12, 18] A triple* $(\mathcal{F}, \preceq, \sigma_b)$ *is said to be an ordered b-metric-like space if* $(\mathcal{F}, \preceq)$ *is a partially ordered set and* σ_b *is a b-metric-like on* $\mathcal{F}$.

Lemma 1.8. [6] Let $(\mathcal{F}, \sigma_b)$ be a b-metric-like space with coefficient $k > 1$ and assume that $u_n \to u$ and $v_n \to v$. Then we have

$$\begin{aligned}\frac{1}{k^2}\sigma_b(u, v) - \frac{1}{k}\sigma_b(u, u) - \sigma_b(v, v) &\le \liminf_{n\to\infty} \sigma_b(u_n, v_n)\\ &\le \limsup_{n\to\infty} \sigma_b(u_n, v_n)\\ &\le k\sigma_b(u, u) + k^2\sigma_b(v, v) + k^2\sigma_b(u, v).\end{aligned}$$

In the following proposition we give a proof of the completeness of a b-metric-like space. Similar proof can be given for other examples used later in the text.

Proposition 1.9. *Let* $\mathcal{F} = [0, +\infty)$ *and* $\sigma_b(x, y) = (\max\{x, y\})^2$ *for* $x, y \in \mathcal{F}$*. Then* $(\mathcal{F}, \sigma_b)$ *is a complete b-metric-like space.*

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{F}$ such that $\lim_{m,n\to\infty} \sigma_b(x_m, x_n) = t \in \mathbb{R}_+$. Then $\lim_{m,n\to\infty} \max\{x_m, x_n\} = \sqrt{t}$, *i.e.*$\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}\ \forall m, n \ge n_0\ \max\{x_m, x_n\} \in (\sqrt{t} - \epsilon, \sqrt{t} + \epsilon)$.
Putting $m = n$ we obtain

$$\forall\ n \ge n_0, x_n \in (\sqrt{t} - \epsilon, \sqrt{t} + \epsilon),$$

meaning that $\lim_{n\to\infty} x_n = \sqrt{t}$ (in the usual sense). Then

$$\lim_{n\to\infty} \sigma_b(x_n, \sqrt{t}) = \lim_{n\to\infty}(\max\{x_n, \sqrt{t}\})^2 = t,$$

i.e. $\lim_{n\to\infty} \sigma_b(x_m, x_n) = \lim_{n\to\infty} \sigma_b(x_n, \sqrt{t}) = \sigma_b(\sqrt{t}, \sqrt{t})$.
This proves that the space $(\mathcal{F}, \sigma_b)$ is σ_b-complete. □

Definition 1.10. *[12] Let* $\mathcal{F}$ *be a nonempty set and suppose* $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ *and* $\beta : \mathcal{F} \times \mathcal{F} \to [0, 1)$ *are mappings. Then* $\mathcal{P}$ *is called* β*-admissible if for all* $u, v \in \mathcal{F}$,

$$\beta(u, v) \ge 1 \Rightarrow \beta(\mathcal{P}u, \mathcal{P}v) \ge 1.$$

Also we say that $\mathcal{P}$ *is* L_β*-admissible (or* R_β*-admissible) if for* $u, v \in \mathcal{F}$,

$$\beta(u, v) \ge 1 \Rightarrow \beta(\mathcal{P}u, v) \ge 1 \text{ (or } \beta(u, \mathcal{P}v) \ge 1).$$

Example 1.11. Let $\mathcal{F} = (0, \infty)$. If $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ and $\beta\colon \mathcal{F} \times \mathcal{F} \to (0, \infty)$ are defined by $\mathcal{P}x = \frac{3x^2}{2}$, $\forall\, x \in \mathcal{F}$ and

$$\beta(x, y) = \begin{cases} 3, & \text{if } y \geq x \\ 0, & \text{otherwise} \end{cases}$$

then $\mathcal{P}$ is β-admissible.

Definition 1.12. *[9] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it satisfies the following properties*
(i) ψ is continuous and nondecreasing;
(ii) $\psi(t) = 0$ iff $t = 0$.

2. Main Results

In this section, we introduce two new notions in ordered b-metric-like space and derive related fixed point results.

Result-I. In order to prove our first result we introduce a new type of contractive mapping, named as, (β-ψ_1-ψ_2)-contractive mapping of type-I in ordered b-metric-like space.

Definition 2.1. *Let $(\mathcal{F}, \sigma_b)$ be a partially ordered b-metric-like space with the coefficient $k \geq 1$. A mapping $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ is said to be (β-ψ_1-ψ_2)-contractive mapping of type-I, if there exist two altering distance functions ψ_1, ψ_2 and $\beta\colon \mathcal{F} \times \mathcal{F} \to [0, \infty)$ such that*

$$\beta(u, \mathcal{P}u)\beta(v, \mathcal{P}v)\psi_1(k\sigma_b(\mathcal{P}u, \mathcal{P}v)) \leq \psi_1(\Delta_k^{\mathcal{P}}(u, v)) - \psi_2(\Delta_k^{\mathcal{P}}(u, v)), \tag{3}$$

for all comparable $u, v \in \mathcal{F}$, where

$$\Delta_k^{\mathcal{P}}(u, v) = \max \left\{ \begin{array}{l} \sigma_b(u, v), \sigma_b(u, \mathcal{P}u), \sigma_b(v, \mathcal{P}v), \dfrac{\sigma_b(u, \mathcal{P}v) + \sigma_b(v, \mathcal{P}u)}{4k}, \\ \dfrac{\sigma_b(u, \mathcal{P}u)\sigma_b(v, \mathcal{P}v)}{1 + \sigma_b(u, v)}, \dfrac{\sigma_b(u, \mathcal{P}u)\sigma_b(v, \mathcal{P}v)}{1 + \sigma_b(\mathcal{P}u, \mathcal{P}v)} \end{array} \right\}. \tag{4}$$

The first result of the paper is as follows:

Theorem 2.2. *Let $(\mathcal{F}, \preceq, \sigma_b)$ be a σ_b-complete ordered b-metric-like space with the coefficient $k \geq 1$. Let $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ be a (β-ψ_1-ψ_2)-contractive mapping of type-I. Assume that the following assertions hold:*

(1) $\mathcal{P}$ is β-admissible and L_β-admissible (or R_β-admissible);
(2) There exists $u_1 \in \mathcal{F}$ such that $u_1 \preceq \mathcal{P}u_1$ and $\beta(u_1, \mathcal{P}u_1) \geq 1$;
(3) $\mathcal{P}$ is continuous, nondecreasing with respect to $\preceq$ and if $\mathcal{P}^n u_1 \to z$ then $\beta(z, z) \geq 1$.

Then $\mathcal{P}$ has a fixed point.

Proof. Starting with the point u_1, let the sequence $\{u_n\}$ be defined in $\mathcal{F}$ by $u_{n+1} = \mathcal{P}u_n$, $\forall\, n \geq 1$. We have $u_2 = \mathcal{P}u_1 \preceq \mathcal{P}u_2 = u_3$ since $u_1 \preceq \mathcal{P}u_1$ and $\mathcal{P}$ is nondecreasing. Also, $u_3 = \mathcal{P}u_2 \preceq \mathcal{P}u_3 = u_4$ since $u_2 \preceq \mathcal{P}u_2$ and $\mathcal{P}$ is nondecreasing. By induction, we get

$$u_1 \preceq u_2 \preceq u_3 \cdots \preceq u_n \preceq u_{n+1} \preceq \cdots.$$

If $u_n = u_{n+1}$ for some $n \in \mathbb{N}$, then the fixed point of $\mathcal{P}$ is u with $u = u_n$ and the proof is completed. So we may assume $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since $\mathcal{P}$ is β-admissible, we deduce

$$\beta(u_1, \mathcal{P}u_1) = \beta(u_1, u_2) \geq 1 \Rightarrow \beta(\mathcal{P}u_1, \mathcal{P}u_2) = \beta(u_2, u_3) \geq 1.$$

By induction on n we get

$$\beta(u_n, u_{n+1}) \geq 1 \text{ and } \beta(u_{n+1}, u_{n+2}) \geq 1, \quad \forall \; n \in \mathbb{N}. \tag{5}$$

Hence, by applying (3) we get

$$\begin{aligned}\psi_1(\sigma_b(u_{n+1}, u_{n+2})) &\leq \beta(u_n, \mathcal{P}u_{n+1})\beta(u_{n+1}, \mathcal{P}u_{n+2})\psi_1(s\sigma_b(\mathcal{P}u_n, \mathcal{P}u_{n+1})) \\ &\leq \psi_1(\Delta_k^{\mathcal{P}}(u_n, u_{n+1})) - \psi_2(\Delta_k^{\mathcal{P}}(u_n, u_{n+1})),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\Delta_k^{\mathcal{P}}(u_n, u_{n+1}) &= \max\left\{\begin{array}{l}\sigma_b(u_n, u_{n+1}), \sigma_b(u_n, \mathcal{P}u_n), \sigma_b(u_{n+1}, \mathcal{P}u_{n+1}), \\ \dfrac{\sigma_b(u_n, \mathcal{P}u_{n+1}) + \sigma_b(u_{n+1}, \mathcal{P}u_n)}{4k}, \\ \dfrac{\sigma_b(u_n, \mathcal{P}u_n)\sigma_b(u_{n+1}, \mathcal{P}u_{n+1})}{1 + \sigma_b(u_n, u_{n+1})}, \dfrac{\sigma_b(u_n, \mathcal{P}u_n)\sigma_b(u_{n+1}, \mathcal{P}u_{n+1})}{1 + \sigma_b(\mathcal{P}u_n, \mathcal{P}u_{n+1})}\end{array}\right\} \\ &= \max\left\{\begin{array}{l}\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2}), \dfrac{\sigma_b(u_n, u_{n+2}) + \sigma_b(u_{n+1}, u_{n+1})}{4k}, \\ \dfrac{\sigma_b(u_n, u_{n+1})\sigma_b(u_{n+1}, u_{n+2})}{1 + \sigma_b(u_n, u_{n+1})}, \dfrac{\sigma_b(u_n, u_{n+1})\sigma_b(u_{n+1}, u_{n+2})}{1 + \sigma_b(u_{n+1}, u_{n+2})}\end{array}\right\} \\ &= \max\left\{\begin{array}{l}\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2}), \\ \dfrac{k\sigma_b(u_n, u_{n+1}) + k\sigma_b(u_{n+1}, u_{n+2}) + 2k\sigma_b(u_{n+1}, u_{n+2})}{4k} \\ \dfrac{\sigma_b(u_n, u_{n+1})\sigma_b(u_{n+1}, u_{n+2})}{1 + \sigma_b(u_n, u_{n+1})}, \dfrac{\sigma_b(u_n, u_{n+1})\sigma_b(u_{n+1}, u_{n+2})}{1 + \sigma_b(u_{n+1}, u_{n+2})}\end{array}\right\} \\ &< \max\{\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2})\}.\end{aligned} \tag{7}$$

From (6) and (7) we get

$$\begin{aligned}\psi_1(\sigma_b(u_{n+1}, u_{n+2})) &\leq \psi_1(\max\{\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2})\}) \\ &\quad - \psi_2(\max\{\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2})\}).\end{aligned} \tag{8}$$

Suppose that

$$\max\{\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2})\} = \sigma_b(u_{n+1}, u_{n+2}).$$

Then (6) implies that

$$\begin{aligned}\psi_1(\sigma_b(u_{n+1}, u_{n+2})) &\leq \psi_1(\sigma_b(u_{n+1}, u_{n+2})) - \psi_2(\sigma_b(u_{n+1}, u_{n+2})) \\ &< \psi_1(\sigma_b(u_{n+1}, u_{n+2}))\})\end{aligned}$$

which is a contradiction. Therefore we have

$$\max\{\sigma_b(u_n, u_{n+1}), \sigma_b(u_{n+1}, u_{n+2})\} = \sigma_b(u_n, u_{n+1})$$

and so

$$\psi_1(\sigma_b(u_{n+1}, u_{n+2})) \leq \psi_1(\sigma_b(u_n, u_{n+1})) - \psi_2(\sigma_b(u_n, u_{n+1})).$$

Thus the sequence $\{\sigma_b(u_n, u_{n+1})\}$ is nondecreasing. Since it is bounded from below, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} \sigma_b(u_n, u_{n+1}) = \gamma$. Using the properties of functions ψ_1 and ψ_2 we get that

$$\begin{aligned}\psi_1(\gamma) \leq \liminf \psi_1(\sigma_b(u_{n+1}, u_{n+2})) &\leq \limsup \psi_1(\sigma_b(u_{n+1}, u_{n+2})) \\ &\leq \limsup[\psi_1(\sigma_b(u_n, u_{n+1})) - \psi_2(\sigma_b(u_n, u_{n+1}))] \\ &\leq \limsup \psi_1(\sigma_b(u_n, u_{n+1})) - \liminf \psi_2(\sigma_b(u_n, u_{n+1})) \\ &\leq \psi_1(\gamma) - \psi_2(\gamma),\end{aligned}$$

which is not possible for $\gamma > 0$. Thus,

$$\gamma = \lim_{n\to\infty} \sigma_b(u_n, u_{n+1}) = 0. \tag{9}$$

Now, we have to show that $\{u_n\}$ is a σ_b-Cauchy sequence in $(\mathcal{F}, \sigma_b)$. Suppose the contrary; then, there exist $\epsilon > 0$ and two subsequences $\left\{u_{h(r)}\right\}$ and $\left\{u_{l(r)}\right\}$ of $\{u_n\}$ such that $h(r) > l(r) > k$ and

$$\sigma_b(u_{h(r)}, u_{l(r)}) \geq \epsilon. \tag{10}$$

We may also assume

$$\sigma_b(u_{l(r)}, u_{h(r)-1}) < \epsilon. \tag{11}$$

By choosing $h(r)$ to be the smallest index exceeding $l(r)$ for which (10) holds. Then we get

$$\begin{aligned} \epsilon \leq \sigma_b(u_{h(r)}, u_{l(r)}) &\leq k\sigma_b(u_{l(r)}, u_{h(r)-1}) + k\sigma_b(u_{h(r)-1}, u_{h(r)}) \\ &< k\epsilon + k\sigma_b(u_{h(r)-1}, u_{h(r)}). \end{aligned} \tag{12}$$

Taking the upper limit in (11) as $r \to \infty$, obtain

$$\frac{\epsilon}{k} \leq \liminf_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}) \leq \limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}) \leq \epsilon. \tag{13}$$

Also, from (12), (13), we obtain

$$\epsilon \leq \limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}) \leq k\epsilon.$$

By $(\sigma_b 3)$, we deduce

$$\begin{aligned} \sigma_b(u_{l(r)+1}, u_{h(r)}) &\leq k\sigma_b(u_{l(r)+1}, u_{l(r)}) + k\sigma_b(u_{l(r)}, u_{h(r)}) \\ &\leq k\sigma_b(u_{l(r)+1}, u_{l(r)}) + k^2\sigma_b(u_{l(r)}, u_{h(r)-1}) + k^2\sigma_b(u_{h(r)-1}, u_{h(r)}) \\ &\leq k\sigma_b(u_{l(r)+1}, u_{l(r)}) + k^2\epsilon + k^2\sigma_b(u_{h(r)-1}, u_{h(r)}), \end{aligned} \tag{14}$$

and applying the upper limit in (14) as $r \to \infty$, we obtain

$$\limsup_{r\to\infty} \sigma_b(u_{l(r)+1}, u_{h(r)}) \leq k^2\epsilon.$$

Finally,

$$\begin{aligned} \sigma_b(u_{l(r)+1}, u_{h(r)-1}) &\leq k\sigma_b(u_{l(r)+1}, u_{l(r)}) + k\sigma_b(u_{l(r)}, u_{h(r)-1}) \\ &\leq k\sigma_b(u_{l(r)+1}, u_{l(r)}) + k\epsilon. \end{aligned} \tag{15}$$

Also, applying the upper limit as $r \to \infty$ in (15), we get

$$\limsup_{r\to\infty} \sigma_b(u_{l(r)+1}, u_{h(r)-1}) \leq k\epsilon.$$

Hence,

$$\frac{\epsilon}{k} \leq \liminf_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}) \leq \limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}) \leq \epsilon. \tag{16}$$

Similarly,

$$\limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)}) \leq k\epsilon, \tag{17}$$

$$\frac{\epsilon}{k} \leq \limsup_{r\to\infty} \sigma_b(u_{l(r)+1}, u_{h(r)}), \tag{18}$$

and

$$\limsup_{r\to\infty} \sigma_b(u_{l(r)+1}, u_{h(r)-1}) \leq k\epsilon. \tag{19}$$

Since $\mathcal{P}$ is L_β-admissible, using (5), we obtain $\beta(u_{l(r)}, u_{l(r)+1}) \geq 1$ and $\beta(u_{h(r)}, u_{h(r)+1}) \geq 1$.
By using (3) we get

$$\begin{aligned}\psi_1(k\sigma_b(u_{l(r)+1}, u_{h(r)})) &\leq \beta(u_{l(r)}, u_{l(r)+1})\beta(u_{h(r)}, u_{h(r)+1})\psi_1(k\sigma_b(\mathcal{P}u_{l(r)}, \mathcal{P}u_{h(r)-1})) \\ &\leq \psi_1(\Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1})) - \psi_2(\Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1})),\end{aligned} \tag{20}$$

where

$$\begin{aligned}\Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1}) &= \max\left\{\begin{array}{l}\sigma_b(u_{l(r)}, u_{h(r)-1}), \sigma_b(u_{l(r)}, \mathcal{P}u_{l(r)}), \sigma_b(u_{h(r)-1}, \mathcal{P}u_{h(r)-1}), \\ \dfrac{\sigma_b(u_{l(r)}, \mathcal{P}u_{h(r)-1}) + \sigma_b(u_{h(r)-1}, \mathcal{P}u_{l(r)})}{4k}, \\ \dfrac{\sigma_b(u_{l(r)}, \mathcal{P}u_{l(r)})\sigma_b(u_{h(r)-1}, \mathcal{P}u_{h(r)-1})}{1+\sigma(u_{l(r)}, u_{h(r)-1})}, \\ \dfrac{\sigma_b(u_{l(r)}, \mathcal{P}u_{l(r)})\sigma_b(u_{h(r)-1}, \mathcal{P}u_{h(r)-1})}{1+\sigma(\mathcal{P}u_{l(r)}, \mathcal{P}u_{h(r)-1})}\end{array}\right\} \\ &= \max\left\{\begin{array}{l}\sigma_b(u_{l(r)}, u_{h(r)-1}), \sigma_b(u_{l(r)}, u_{l(r)+1}), \sigma_b(u_{h(r)-1}, u_{h(r)}), \\ \dfrac{\sigma_b(u_{l(r)}, u_{h(r)}) + \sigma_b(u_{h(r)-1}, u_{l(r)+1})}{4k}, \\ \dfrac{\sigma_b(u_{l(r)}, u_{l(r)+1})\sigma_b(u_{h(r)-1}, u_{h(r)})}{1+\sigma(u_{l(r)}, u_{h(r)-1})}, \\ \dfrac{\sigma_b(u_{l(r)}, u_{l(r)+1})\sigma_b(u_{h(r)-1}, u_{h(r)})}{1+\sigma(u_{l(r)+1}, u_{h(r)})}\end{array}\right\}.\end{aligned} \tag{21}$$

Applying the upper limit in (21) as $r \to \infty$, and using (9), (16), (17) and (19) we obtain

$$\begin{aligned}\limsup_{r\to\infty} \Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1}) &= \max\left\{\begin{array}{l}\limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}), \limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{l(r)+1}), \\ \limsup\limits_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{h(r)}), \\ \dfrac{\limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)}) + \limsup\limits_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{l(r)+1})}{4k}, \\ \dfrac{\limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{l(r)+1}) \limsup\limits_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{h(r)})}{1+\limsup\limits_{r\to\infty} \sigma(u_{l(r)}, u_{h(r)-1})}, \\ \dfrac{\limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{l(r)+1}) \limsup\limits_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{h(r)})}{1+\limsup\limits_{r\to\infty} \sigma(u_{l(r)+1}, u_{h(r)})}\end{array}\right\} \\ &= \max\left\{\begin{array}{l}\limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}), 0, 0, \\ \dfrac{\limsup\limits_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)}) + \limsup\limits_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{l(r)+1})}{4k}, 0, 0\end{array}\right\} \\ &\leq \max\left\{\epsilon, \frac{\epsilon}{2}\right\} \\ &= \epsilon.\end{aligned} \tag{22}$$

Next, applying the upper limit in (20) as $r \to \infty$, and using (18), (22) we get

$$\begin{aligned}\psi_1\Big(k\frac{\epsilon}{k}\Big) &\le \psi_1\Big(\limsup_{r\to\infty} k\sigma_b(u_{l(r)+1}, u_{h(r)})\Big)\\ &\le \psi_1\Big(\limsup_{r\to\infty} \Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1})\Big) - \psi_2\Big(\liminf_{r\to\infty}\Big(\Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1})\Big)\Big)\\ &\le \psi_1(\epsilon) - \psi_2\Big(\liminf_{r\to\infty}\Big(\Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1})\Big)\Big),\end{aligned}$$

which implies that

$$\psi_2\Big(\liminf_{r\to\infty}\Big(\Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1})\Big)\Big) = 0,$$

i.e.

$$\liminf_{r\to\infty} \Delta_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1}) = 0.$$

Therefore by using (20) we obtain

$$\liminf_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}) = 0,$$

which is a contradiction with (16). Thus, $\{u_n\}$ is a σ_b-Cauchy sequence in the b-metric-like space $(\mathcal{F}, \sigma_b)$. Since $(\mathcal{F}, \sigma_b)$ is σ_b-complete, then there exists $z \in \mathcal{F}$ such that

$$\lim_{n\to\infty} \sigma_b(u_n, z) = 0.$$

Therefore, by using (9), the condition $\sigma_b(u_n, u_n) \le \sigma_b(z, u_n)$ and $\lim_{n\to\infty} \sigma_b(u_n, u_n) = 0$ we get

$$\lim_{n\to\infty} \sigma_b(u_n, z) = \lim_{n\to\infty} \sigma_b(u_n, u_n) = \sigma_b(z, z) = 0.$$

By $(\sigma_b 3)$, we obtain

$$\sigma_b(z, \mathcal{P}z) \le k\sigma_b(z, \mathcal{P}u_n) + k\sigma_b(\mathcal{P}u_n, \mathcal{P}z). \tag{23}$$

So using the continuity of $\mathcal{P}$ and applying the limit in (23) as $n \to \infty$, we get

$$\sigma_b(z, \mathcal{P}z) \le k \lim_{n\to\infty} \sigma_b(z, u_{n+1}) + k \lim_{n\to\infty} \sigma_b(\mathcal{P}u_n, \mathcal{P}z) = k\sigma_b(\mathcal{P}z, \mathcal{P}z). \tag{24}$$

Since $\beta(z, z) \ge 1$ and using (3) we get

$$\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) \le \beta(z, \mathcal{P}z)\beta(z, \mathcal{P}z)\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) \le \psi_1(\Delta_k^{\mathcal{P}}(z, z)) - \psi_2(\Delta_k^{\mathcal{P}}(z, z)),$$

where

$$\begin{aligned}\Delta_k^{\mathcal{P}}(z, z) &= \max\left\{\begin{array}{l} \sigma_b(z, z), \sigma_b(z, \mathcal{P}z), \sigma_b(z, \mathcal{P}z), \dfrac{\sigma_b(z, \mathcal{P}z) + \sigma_b(z, \mathcal{P}z)}{4k},\\ \dfrac{\sigma_b(z, \mathcal{P}z)\sigma_b(z, \mathcal{P}z)}{1 + \sigma_b(z, z)}, \dfrac{\sigma_b(z, \mathcal{P}z)\sigma_b(z, \mathcal{P}z)}{1 + \sigma_b(\mathcal{P}z, \mathcal{P}z)} \end{array}\right\}\\ &< \sigma_b(z, \mathcal{P}z).\end{aligned}$$

Therefore

$$\begin{aligned}\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) &\le \beta(z, \mathcal{P}z)\beta(z, \mathcal{P}z)\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z))\\ &\le \psi_1(\sigma_b(z, \mathcal{P}z)) - \psi_2(\sigma_b(z, \mathcal{P}z))).\end{aligned} \tag{25}$$

Since ψ_1 is nondecreasing, $k\sigma_b(\mathcal{P}z, \mathcal{P}z) \le \sigma_b(z, \mathcal{P}z)$ and $k\sigma_b(\mathcal{P}z, \mathcal{P}z) = \sigma_b(z, \mathcal{P}z)$, which is possible only that $\sigma_b(z, \mathcal{P}z) = 0$ and $\mathcal{P}z = z$. Hence, z is a fixed point of $\mathcal{P}$. □

We note that the previous result may still be valid when $\mathcal{P}$ is not necessarily continuous. We have the following result.

Theorem 2.3. *Let $(\mathcal{F}, \preceq, \sigma_b)$ be a σ_b-complete ordered partial b-metric-like space with the coefficient $k \geq 1$. Let $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ be a (β-ψ_1-ψ_2)-contractive mapping of type-I. Suppose that the following conditions hold:*

(1) $\mathcal{P}$ is β-admissible and L_β-admissible (or R_β-admissible);
(2) There exists $u_1 \in \mathcal{F}$ such that $u_1 \preceq \mathcal{P}u_1$ and $\beta(u_1, \mathcal{P}u_1) \geq 1$;
(3) $\mathcal{P}$ is nondecreasing, with respect to $\preceq$;
(4) If $\{u_n\}$ is a sequence in $\mathcal{F}$ such that $u_n \preceq u_{n+1}$ and $\beta(u_n, u_{n+1}) \geq 1$ $\forall$ $n \in \mathbb{N}$, and $u_n \to u \in \mathcal{F}$, as $n \to \infty$, then $u_n \preceq u$ and $\beta(u_n, u) \geq 1$ $\forall$ $n \in \mathbb{N}$.

Then, $\mathcal{P}$ has a fixed point.

Proof. Following the lines of proof of Theorem 2.2, the sequence $\{u_n\}$ defined by $u_{n+1} = \mathcal{P}u_n, \forall\, n \in \mathbb{N}$ is a nondecreasing σ_b-Cauchy sequence in the σ_b-complete b-metric-like space $(\mathcal{F}, \sigma_b)$. From the completeness of $(\mathcal{F}, \sigma_b)$, it follows that there exists $z \in \mathcal{F}$ such that $\lim\limits_{n\to\infty} u_n = z$. By assumption on $\mathcal{F}$, we deduce $u_n \preceq z$, $\forall$ $n \in \mathbb{N}$. So it is enough to show $\mathcal{P}z = z$. Now, by using (3) and $\beta(u_n, z) \geq 1$, $\forall$ $n \in \mathbb{N}$, we have

$$\begin{aligned}\psi_1(k\sigma_b(u_{n+1}, \mathcal{P}z)) &\leq \beta(u_n, \mathcal{P}u_n)\beta(z, \mathcal{P}z)\psi_1(k\sigma_b(\mathcal{P}u_n, \mathcal{P}z))\\ &\leq \psi_1(\Delta_k^{\mathcal{P}}(u_n, z)) - \psi_2(\Delta_k^{\mathcal{P}}(u_n, z)),\end{aligned} \tag{26}$$

where

$$\begin{aligned}\Delta_k^{\mathcal{P}}(u_n, z) &= \max\left\{\begin{array}{l}\sigma_b(u_n, z), \sigma_b(u_n, \mathcal{P}u_n), \sigma_b(z, \mathcal{P}z), \dfrac{\sigma_b(u_n, \mathcal{P}z) + \sigma_b(\mathcal{P}u_n, z)}{4k},\\ \dfrac{\sigma_b(u_n, \mathcal{P}u_n)\sigma_b(z, \mathcal{P}z)}{1 + \sigma_b(u_n, z)}, \dfrac{\sigma_b(u_n, \mathcal{P}u_n)\sigma_b(z, \mathcal{P}z)}{1 + \sigma_b(\mathcal{P}u_n, \mathcal{P}z)}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{l}\sigma_b(u_n, z), \sigma_b(u_n, u_{n+1}), \sigma_b(z, \mathcal{P}z), \dfrac{\sigma_b(u_n, \mathcal{P}z) + \sigma_b(u_{n+1}, z)}{4k},\\ \dfrac{\sigma_b(u_n, u_{n+1})\sigma_b(z, z)}{1 + \sigma_b(u_n, z)}, \dfrac{\sigma_b(u_n, u_{n+1})\sigma_b(z, z)}{1 + \sigma_b(u_{n+1}, z)}\end{array}\right\}.\end{aligned} \tag{27}$$

Applying the limit as $n \to \infty$ in (27) and using Lemma 1.8, we get

$$\begin{aligned}\frac{\sigma_b(z, \mathcal{P}z)}{4k^2} &= \min\left\{\sigma_b(z, \mathcal{P}z), \frac{\frac{\sigma_b(z, \mathcal{P}z)}{k}}{4k}\right\}\\ &\leq \liminf_{n\to\infty} \Delta_k^{\mathcal{P}}(u_n, z)\\ &\leq \limsup_{n\to\infty} \Delta_k^{\mathcal{P}}(u_n, z)\\ &\leq \max\left\{\sigma_b(z, \mathcal{P}z), \frac{k\sigma_b(z, \mathcal{P}z)}{4k}\right\} = \sigma_b(z, \mathcal{P}z).\end{aligned} \tag{28}$$

Again, by using (26) and taking the upper limit as $n \to \infty$,

$$\begin{aligned}\psi_1(k\sigma_b(u_{n+1}, \mathcal{P}z)) &\leq \beta(u_n, \mathcal{P}u_n)\beta(z, \mathcal{P}z)\psi_1(k\sigma_b(\mathcal{P}u_n, \mathcal{P}z))\\ &\leq \psi_1(\Delta_k^{\mathcal{P}}(u_n, z)) - \psi_2(\Delta_k^{\mathcal{P}}(u_n, z)),\end{aligned}$$

and using Lemma 1.8, we get

$$\begin{aligned}\psi_1(\sigma_b(z,\mathcal{P}z)) &= \psi_1\Big(k\frac{1}{k}\sigma_b(u_{n+1},\mathcal{P}z)\Big)\\ &\le \psi_1\Big(k\limsup_{n\to\infty}\sigma_b(u_{n+1},\mathcal{P}z)\Big)\\ &\le \psi_1\Big(\limsup_{n\to\infty}\Delta_k^{\mathcal{P}}(u_n,z)\Big)-\psi_2\Big(\liminf_{n\to\infty}\Delta_k^{\mathcal{P}}(u_n,z)\Big)\\ &\le \psi_1(\sigma_b(z,\mathcal{P}z))-\psi_2(\sigma_b(z,\mathcal{P}z))\\ &< \psi_1(\sigma_b(z,\mathcal{P}z)),\end{aligned}$$

a contradiction. Therefore $z=\mathcal{P}z$. Hence z is a fixed point of $\mathcal{P}$. □

Result-II. In order to prove our second result we introduce the notion of (β-ψ_1-ψ_2)-contractive mapping of type-II in an ordered b-metric-like space.

Definition 2.4. *Let $(\mathcal{F},\sigma_b)$ be a partially ordered b-metric-like space with coefficient $k\ge 1$. The mapping $\mathcal{P}\colon\mathcal{F}\to\mathcal{F}$ is called a (β-ψ_1-ψ_2)-contractive mapping of type-II, if there exist two altering distance functions ψ_1, ψ_2 and $\beta\colon\mathcal{F}\times\mathcal{F}\to[0,\infty)$ such that*

$$\beta(u,\mathcal{P}u)\beta(v,\mathcal{P}v)\psi_1(k\sigma_b(\mathcal{P}u,\mathcal{P}v))\le\psi_1((\Delta_I)_k^{\mathcal{P}}(u,v))-\psi_2((\Delta_I)_k^{\mathcal{P}}(u,v)),\tag{29}$$

for all comparable $u,v\in\mathcal{F}$, where

$$(\Delta_I)_k^{\mathcal{P}}(u,v)=\max\left\{\begin{array}{l}\sigma_b(u,v),\sigma_b(v,\mathcal{P}v),\sigma_b(u,\mathcal{P}u),\dfrac{\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}u)}{4k},\\ \dfrac{\sigma_b(u,\mathcal{P}u)\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}v)\sigma_b(v,\mathcal{P}u)}{1+k[\sigma_b(u,\mathcal{P}u)+\sigma_b(v,\mathcal{P}v)]},\\ \dfrac{\sigma_b(u,\mathcal{P}u)\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}v)\sigma_b(v,\mathcal{P}u)}{1+\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}u)}\end{array}\right\}.\tag{30}$$

Theorem 2.5. *Instead of the (β-ψ_1-ψ_2)-contractive condition of type-I in Theorem 2.2, assume that (β-ψ_1-ψ_2)-contractive condition of type-II is satisfied. Then $\mathcal{P}$ has a fixed point.*

Proof. Following the proof of Theorem 2.2, we can construct the sequence $\{u_n\}$ in $\mathcal{F}$ by $u_{n+1}=\mathcal{P}u_n$, $\forall\, n\ge 1$ which satisfies the following:

$$u_1\preceq u_2\preceq u_3\cdots\preceq u_n\preceq u_{n+1}\preceq\cdots$$

and

$$\beta(u_n,u_{n+1})\ge 1 \text{ and } \beta(u_{n+1},u_{n+2})\ge 1\ \forall\, n\in\mathbb{N}.\tag{31}$$

Therefore using (29) for $u=u_n$ and $v=u_{n+1}$, we have

$$\begin{aligned}\psi_1(\sigma_b(u_{n+1},u_{n+2}))&\le\beta(u_n,\mathcal{P}u_{n+1})\beta(u_{n+1},\mathcal{P}u_{n+2})\psi_1(k\sigma_b(\mathcal{P}u_n,\mathcal{P}u_{n+1}))\\ &\le\psi_1((\Delta_I)_k^{\mathcal{P}}(u_n,u_{n+1}))-\psi_2((\Delta_I)_k^{\mathcal{P}}(u_n,u_{n+1})),\end{aligned}\tag{32}$$

where

$$
\begin{aligned}
&(\Delta_I)_k^{\mathcal{P}}(u_n,u_{n+1})\\
&=\max\left\{\begin{array}{l}
\sigma_b(u_n,u_{n+1}),\sigma_b(u_n,\mathcal{P}u_n),\sigma_b(u_{n+1},\mathcal{P}u_{n+1}),\dfrac{\sigma_b(u_n,\mathcal{P}u_{n+1})+\sigma_b(u_{n+1},\mathcal{P}u_n)}{4k},\\
\dfrac{\sigma_b(u_n,\mathcal{P}u_n)\sigma_b(u_n,\mathcal{P}u_{n+1})+\sigma_b(u_{n+1},\mathcal{P}u_{n+1})\sigma_b(u_{n+1},\mathcal{P}u_n)}{1+k[\sigma_b(u_n,\mathcal{P}u_n)+\sigma_b(u_{n+1},\mathcal{P}u_{n+1})]},\\
\dfrac{\sigma_b(u_n,\mathcal{P}u_n)\sigma_b(u_n,\mathcal{P}u_{n+1})+\sigma_b(u_{n+1},\mathcal{P}u_{n+1})\sigma_b(u_{n+1},\mathcal{P}u_n)}{1+\sigma_b(u_n,\mathcal{P}u_{n+1})+\sigma_b(u_{n+1},\mathcal{P}u_n)}
\end{array}\right\}\\
&=\max\left\{\begin{array}{l}
\sigma_b(u_n,u_{n+1}),\sigma_b(u_{n+1},u_{n+2}),\dfrac{\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+1})}{4k},\\
\dfrac{\sigma_b(u_n,u_{n+1})\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+2})\sigma_b(u_{n+1},u_{n+1})}{1+k[\sigma_b(u_n,u_{n+1})+\sigma_b(u_{n+1},u_{n+2})]},\\
\dfrac{\sigma_b(u_n,u_{n+1})\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+2})\sigma_b(u_{n+1},u_{n+1})}{1+\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+1})}
\end{array}\right\}\\
&\leq\max\left\{\begin{array}{l}
\sigma_b(u_n,u_{n+1}),\sigma_b(u_{n+1},u_{n+2}),\\
\dfrac{k\sigma_b(u_n,u_{n+1})+k\sigma_b(u_{n+1},u_{n+2})+2k\sigma_b(u_{n+1},u_{n+2})}{4k}\\
\dfrac{\sigma_b(u_n,u_{n+1})\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+2})\sigma_b(u_{n+1},u_{n+1})}{1+k[\sigma_b(u_n,u_{n+1})+\sigma_b(u_{n+1},u_{n+2})]},\\
\dfrac{\sigma_b(u_n,u_{n+1})\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+2})\sigma_b(u_{n+1},u_{n+1})}{1+\sigma_b(u_n,u_{n+2})+\sigma_b(u_{n+1},u_{n+1})}
\end{array}\right\}\\
&<\max\{\sigma_b(u_n,u_{n+1}),\sigma_b(u_{n+1},u_{n+2})\}.
\end{aligned}
\tag{33}
$$

Repeating the arguments of the proof of Theorem 2.2, we obtain the equation (9).

Now, we have to show that $\{u_n\}$ is a σ_b-Cauchy sequence in $(\mathcal{F},\sigma_b)$. Suppose the contrary; then, using proof of Theorem 2.2, we obtain relations (10)–(19) between the terms of the sequence $\{u_n\}$. Since $\mathcal{P}$ is L_β-admissible and using (31), we obtain $\beta(u_{l(r)},u_{l(r)+1})\geq 1$ and $\beta(u_{h(r)},u_{h(r)+1})\geq 1$.

On using (29) we get

$$
\begin{aligned}
\psi_1(k\sigma_b(u_{l(r)+1},u_{h(r)}))&\leq\beta(u_{l(r)},u_{l(r)+1})\beta(u_{h(r)},u_{h(r)+1})\psi_1(k\sigma_b(\mathcal{P}u_{l(r)},\mathcal{P}u_{h(r)-1}))\\
&\leq\psi_1((\Delta_I)_k^{\mathcal{P}}(u_{l(r)},u_{h(r)-1}))-\psi_2((\Delta_I)_k^{\mathcal{P}}(u_{l(r)},u_{h(r)-1})),
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
&(\Delta_I)_k^{\mathcal{P}}(u_{l(r)},u_{h(r)-1})\\
&=\max\left\{\begin{array}{l}
\sigma_b(u_{l(r)},u_{h(r)-1}),\sigma_b(u_{l(r)},\mathcal{P}u_{l(r)}),\sigma_b(u_{h(r)-1},\mathcal{P}u_{h(r)-1}),\\
\dfrac{\sigma_b(u_{l(r)},\mathcal{P}u_{h(r)-1})+\sigma_b(u_{h(r)-1},\mathcal{P}u_{l(r)})}{4k},\\
\dfrac{\sigma_b(u_{l(r)},\mathcal{P}u_{l(r)})\sigma_b(u_{l(r)},\mathcal{P}u_{h(r)-1})+\sigma_b(u_{h(r)-1},\mathcal{P}u_{h(r)-1})\sigma_b(u_{h(r)-1},\mathcal{P}u_{l(r)})}{1+k[\sigma_b(u_{l(r)},\mathcal{P}u_{l(r)})+\sigma_b(u_{h(r)-1},\mathcal{P}u_{h(r)-1})]},\\
\dfrac{\sigma_b(u_{l(r)},\mathcal{P}u_{l(r)})\sigma_b(u_{l(r)},\mathcal{P}u_{h(r)-1})+\sigma_b(u_{h(r)-1},\mathcal{P}u_{h(r)-1})\sigma_b(u_{h(r)-1},\mathcal{P}u_{l(r)})}{1+\sigma_b(u_{l(r)},\mathcal{P}u_{h(r)-1})+\sigma_b(u_{h(r)-1},\mathcal{P}u_{l(r)})}
\end{array}\right\}\\
&=\max\left\{\begin{array}{l}
\sigma_b(u_{l(r)},u_{h(r)-1}),\sigma_b(u_{l(r)},u_{l(r)+1}),\sigma_b(u_{h(r)-1},u_{h(r)}),\\
\dfrac{\sigma_b(u_{l(r)},u_{h(r)})+\sigma_b(u_{h(r)-1},u_{l(r)+1})}{4k},\\
\dfrac{\sigma_b(u_{l(r)},u_{l(r)+1})\sigma_b(u_{l(r)},u_{h(r)})+\sigma_b(u_{h(r)-1},u_{h(r)})\sigma_b(u_{h(r)-1},u_{l(r)+1})}{1+k[\sigma_b(u_{l(r)},u_{l(r)+1})+\sigma_b(u_{h(r)-1},u_{h(r)})]},\\
\dfrac{\sigma_b(u_{l(r)},u_{l(r)+1})\sigma_b(u_{l(r)},u_{h(r)})+\sigma_b(u_{h(r)-1},u_{h(r)})\sigma_b(u_{h(r)-1},u_{l(r)+1})}{1+\sigma_b(u_{l(r)},u_{h(r)})+\sigma_b(u_{h(r)-1},u_{l(r)+1})}
\end{array}\right\}.
\end{aligned}
\tag{35}
$$

Applying the upper limit as $r \to \infty$ in (35) and using (9), (16), (17), (19) we get

$$\begin{aligned}
&\limsup_{r\to\infty} (\Delta_I)_k^{\mathcal{P}}(u_{l(r)}, u_{h(r)-1}) \\
&= \max\left\{\begin{array}{l}
\limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}), \limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{l(r)+1}), \limsup_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{h(r)}),\\
\frac{\limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{h(r)})+\limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{l(r)+1})}{4k},\\
\frac{\limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{l(r)+1}) \limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{h(r)})+\limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{h(r)}) \limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{l(r)+1})}{1+k[\limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{l(r)+1})+\limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{h(r)})]},\\
\frac{\limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{l(r)+1}) \limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{h(r)})+\limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{h(r)}) \limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{l(r)+1})}{1+\limsup_{r\to\infty} \sigma_b(u_{l(r)},u_{h(r)})+\limsup_{r\to\infty} \sigma_b(u_{h(r)-1},u_{l(r)+1})}
\end{array}\right\}. \\
&= \max\left\{\begin{array}{l}
\limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)-1}), 0, 0,\\
\frac{\limsup_{r\to\infty} \sigma_b(u_{l(r)}, u_{h(r)}) + \limsup_{r\to\infty} \sigma_b(u_{h(r)-1}, u_{l(r)+1})}{4k}, 0, 0
\end{array}\right\} \\
&\le \max\left\{\epsilon, \frac{\epsilon}{2}\right\} \\
&= \epsilon.
\end{aligned}$$

Repeating the remaining arguments of the proof of Theorem 2.2, we obtain a contradiction. Thus, $\{u_n\}$ is a σ_b-Cauchy sequence in b-metric-like space $(\mathcal{F}, \sigma_b)$. Since $(\mathcal{F}, \sigma_b)$ is σ_b-complete, it follows that there exists $z \in \mathcal{F}$ such that

$$\lim_{n\to\infty} \sigma_b(u_n, z) = 0.$$

Therefore, by using (9), the condition $\sigma_b(u_n, u_n) \le \sigma_b(z, u_n)$ and $\lim_{n\to\infty} \sigma_b(u_n, u_n) = 0$ we get

$$\lim_{n\to\infty} \sigma_b(u_n, z) = \lim_{n\to\infty} \sigma_b(u_n, u_n) = \sigma_b(z, z) = 0.$$

By $(\sigma_b 3)$, we obtain

$$\sigma_b(z, \mathcal{P}z) \le k\sigma_b(z, \mathcal{P}u_n) + k\sigma_b(\mathcal{P}u_n, \mathcal{P}z). \tag{36}$$

Therefore by applying the limit as $n \to \infty$ in (36) and using the continuity of $\mathcal{P}$ we get

$$\sigma_b(z, \mathcal{P}z) \le k \lim_{n\to\infty} \sigma_b(z, u_{n+1}) + k \lim_{n\to\infty} \sigma_b(\mathcal{P}u_n, \mathcal{P}z) = k\sigma_b(\mathcal{P}z, \mathcal{P}z). \tag{37}$$

Since $\beta(z, z) \ge 1$ and using (29) we get

$$\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) \le \beta(z, \mathcal{P}z)\beta(z, \mathcal{P}z)\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) \le \psi_1(\Delta^{\mathcal{P}}_{k\ I}(z, z)) - \psi_2(\Delta^{\mathcal{P}}_{k\ I}(z, z)),$$

where

$$\begin{aligned}
\Delta^{\mathcal{P}}_{k\ I}(z, z) &= \max\left\{\begin{array}{c}
\sigma_b(z, z), \sigma_b(z, \mathcal{P}z), \sigma_b(z, \mathcal{P}z), \frac{\sigma_b(z, \mathcal{P}z) + \sigma_b(z, \mathcal{P}z)}{4k},\\
\frac{\sigma_b(z, \mathcal{P}z)\sigma_b(z, \mathcal{P}z) + \sigma_b(z, \mathcal{P}z)\sigma_b(z, \mathcal{P}z)}{1 + k[\sigma_b(z, \mathcal{P}z) + \sigma_b(z, \mathcal{P}z)]},\\
\frac{\sigma_b(z, \mathcal{P}z)\sigma_b(z, \mathcal{P}z) + \sigma_b(z, \mathcal{P}z)\sigma_b(z, \mathcal{P}z)}{1 + \sigma_b(z, \mathcal{P}z) + \sigma_b(z, \mathcal{P}z)}
\end{array}\right\} \\
&< \sigma_b(z, \mathcal{P}z).
\end{aligned}$$

Therefore

$$\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) \le \beta(z, \mathcal{P}z)\beta(z, \mathcal{P}z)\psi_1(k\sigma_b(\mathcal{P}z, \mathcal{P}z)) \le \psi_1(\sigma_b(z, \mathcal{P}z)) - \psi_2(\sigma_b(z, \mathcal{P}z))). \tag{38}$$

Since ψ_1 is nondecreasing, $k\sigma_b(\mathcal{P}z, \mathcal{P}z) \le \sigma_b(z, \mathcal{P}z)$ and $k\sigma_b(\mathcal{P}z, \mathcal{P}z) = \sigma_b(z, \mathcal{P}z)$, which is possible only when $(\sigma_b(z, \mathcal{P}z)) = 0$ and $\mathcal{P}z = z$. Thus, z is a fixed point of $\mathcal{P}$. □

Similarly to Theorem 2.3, we can be designed a result for (β-ψ_1-ψ_2)-contractive mapping of type-II as follows:

Theorem 2.6. *Let all the conditions of Theorem 2.3 are satisfied, apart from the condition (3) which is replaced by (29). Then $\mathcal{P}$ has a fixed point.*

Now we have following consequences from our main results:

Corollary 2.7. *Let $(\mathcal{F}, \preceq, \sigma_b)$ be a σ_b-complete ordered b-metric-like space with the coefficient $k \geq 1$ and let $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ be an increasing map w.r.t. $\preceq$ such that an element $u_1 \in \mathcal{F}$ exists with $u_1 \preceq \mathcal{P}^m(u_1)$. Let us assume that there exist altering distance functions ψ_1, ψ_2 and $\beta\colon \mathcal{F} \times \mathcal{F} \to [0, \infty)$ such that*

$$\beta(u, \mathcal{P}^m u)\beta(v, \mathcal{P}^m v)\psi_1(k\sigma_b(\mathcal{P}^m u, \mathcal{P}^m v)) \leq \psi_1(\Delta_k^{\mathcal{P}}(u, v)) - \psi_2(\Delta_k^{\mathcal{P}}(u, v)), \tag{39}$$

for all comparable $u, v \in \mathcal{F}$, where

$$\Delta_k^{\mathcal{P}}(u, v) = \max \left\{ \begin{array}{l} \sigma_b(u, v), \sigma_b(u, \mathcal{P}^m u), \sigma_b(v, \mathcal{P}^m v), \dfrac{\sigma_b(u, \mathcal{P}^m v) + \sigma_b(v, \mathcal{P}^m u)}{4k}, \\ \dfrac{\sigma_b(u, \mathcal{P}^m u)\sigma_b(v, \mathcal{P}^m v)}{1 + \sigma_b(u, v)}, \dfrac{\sigma_b(u, \mathcal{P}^m u)\sigma_b(v, \mathcal{P}^m v)}{1 + \sigma_b(\mathcal{P}^m u, \mathcal{P}^m v),} \end{array} \right\}.$$

for some positive integer m. Assume that the following assertions hold:

(1) $\mathcal{P}$ is β-admissible and L_β-admissible (or R_β-admissible);
(2) $u_1 \in \mathcal{F}$ exists such that $u_1 \preceq \mathcal{P}u_1$ and $\beta(u_1, \mathcal{P}u_1) \geq 1$;
(3) $\mathcal{P}$ is continuous and if $\mathcal{P}^n u_1 \to z$ then $\beta(z, z) \geq 1$.

Then $\mathcal{P}$ has a fixed point.

Corollary 2.8. *Let $(\mathcal{F}, \preceq, \sigma_b)$ be a σ_b-complete ordered b-metric-like space with the coefficient $k \geq 1$ and let $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ be a continuous, nondecreasing mapping. Let us assume that there exist altering distance functions ψ_1, ψ_2 such that*

$$\psi_1(k\sigma_b(\mathcal{P}u, \mathcal{P}v)) \leq \psi_1(\Delta_k^{\mathcal{P}}(u, v)) - \psi_2(\Delta_k^{\mathcal{P}}(u, v)), \tag{40}$$

for all comparable $u, v \in \mathcal{F}$, where

$$\Delta_k^{\mathcal{P}}(u, v) = \max \left\{ \begin{array}{l} \sigma_b(u, v), \sigma_b(u, \mathcal{P}u), \sigma_b(v, \mathcal{P}v), \dfrac{\sigma_b(u, \mathcal{P}v) + \sigma_b(v, \mathcal{P}u)}{4k}, \\ \dfrac{\sigma_b(u, \mathcal{P}u)\sigma_b(v, \mathcal{P}v)}{1 + \sigma_b(u, v)}, \dfrac{\sigma_b(u, \mathcal{P}u)\sigma_b(v, \mathcal{P}v)}{1 + \sigma_b(\mathcal{P}u, \mathcal{P}v)} \end{array} \right\}.$$

If there exists $u_1 \in \mathcal{F}$ such that $u_1 \preceq \mathcal{P}u_1$, then $\mathcal{P}$ has a fixed point.

Corollary 2.9. *Let all the conditions of Corollary 2.8 be satisfied, apart from the condition (40) which is replaced by*

$$\psi_1(k\sigma_b(\mathcal{P}u, \mathcal{P}v)) \leq \psi_1((\Delta_I)_k^{\mathcal{P}}(u, v)) - \psi_2((\Delta_I)_k^{\mathcal{P}}(u, v)),$$

for all comparable $u, v \in \mathcal{F}$, where

$$(\Delta_I)_k^{\mathcal{P}}(u, v) = \max \left\{ \begin{array}{l} \sigma_b(u, v), \sigma_b(u, \mathcal{P}u), \sigma_b(v, \mathcal{P}v), \dfrac{\sigma_b(u, \mathcal{P}v) + \sigma_b(v, \mathcal{P}u)}{4k}, \\ \dfrac{\sigma_b(u, \mathcal{P}u)\sigma_b(u, \mathcal{P}v) + \sigma_b(v, \mathcal{P}v)\sigma_b(v, \mathcal{P}u)}{1 + k[\sigma_b(u, \mathcal{P}u) + \sigma_b(v, \mathcal{P}v)]}, \\ \dfrac{\sigma_b(u, \mathcal{P}u)\sigma_b(u, \mathcal{P}v) + \sigma_b(v, \mathcal{P}v)\sigma_b(v, \mathcal{P}u)}{1 + \sigma_b(u, \mathcal{P}v) + \sigma_b(v, \mathcal{P}u)} \end{array} \right\}.$$

Then $\mathcal{P}$ has a fixed point.

3. Illustrations

The first two examples demonstrate possible usage of Theorem 2.2. In the first one, the involvement of rational terms in contractive conditions of type-I is shown while in the second one we show that contraction condition considered in the paper [12] is not satisfied in our example. Similar practices are adopted for Theorem 2.5 in the remaining two examples. In addition, one nontrivial example is given for the usage of Theorem 2.2.

Example 3.1. *Consider $\mathcal{F} = \{a, b, c\}$. Let $\sigma_b : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ be defined by*

$$\sigma_b(a,a) = 0, \qquad \sigma_b(b,b) = \frac{10}{3}, \qquad \sigma_b(c,c) = \frac{5}{2}, \quad \sigma_b(a,b) = \sigma_b(b,a) = \frac{46}{5},$$

$$\sigma_b(a,c) = \sigma_b(c,a) = \frac{14}{5}, \quad \sigma_b(b,c) = \sigma_b(c,b) = \frac{13}{3}.$$

It is clear that $(\mathcal{F}, \sigma_b)$ is a b-complete b-metric like space with constant $k = \frac{5}{4}$. Note that $\sigma_b(b,b) \neq 0$, so σ_b is not a metric space and $\frac{46}{5} = \sigma_b(a,b) \nleq \sigma_b(a,c) + \sigma_b(c,b) - \sigma_b(c,c) = \frac{139}{30}$, so it is not a partial metric space or partial b-metric space ($\frac{46}{5} = \sigma_b(a,b) \nleq k[\sigma_b(a,c) + \sigma_b(c,b)] - \sigma_b(c,c) = \frac{77}{12}$). Define mappings $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ and $\beta : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by

$$\mathcal{P}a = a, \qquad \mathcal{P}b = c, \qquad \mathcal{P}c = a,$$

and

$$\beta(x,y) = \begin{cases} 1, & x, y \in \{a,b,c\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}$ is continuous and increasing. Take altering distance functions $\psi_1(t) = 2t$ and $\psi_2(t) = \frac{t}{1+t}$. We will prove the following:

i) $\mathcal{P}\colon \mathcal{F} \to \mathcal{F}$ is an (β-ψ_1-ψ_2)-contractive mapping of type-I;
ii) $\mathcal{P}$ is β-admissible;
iii) there exists $x_1 \in \mathcal{F}$ such that $x_1 \preceq \mathcal{P}x_1$ and $\beta(x_1, \mathcal{P}x_1) \geq 1$;
iv) If a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ is such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, as $n \to \infty$, then $\beta(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.

Proof. i) Clearly, $\sigma_b(\mathcal{P}a, \mathcal{P}c) = \sigma_b(\mathcal{P}a, \mathcal{P}a) = \sigma_b(\mathcal{P}c, \mathcal{P}c) = 0$. In order to prove that $\mathcal{P}$ is a (β-ψ_1-ψ_2)-contractive mapping the following three cases have to be considered:

Case I: $x = b$ and $y = b$ (this case shows the reason for involvement of rational terms).

We have $\psi_1\left(\frac{5}{4}\sigma_b(\mathcal{P}b, \mathcal{P}b)\right) = \psi_1\left(\frac{5}{4}\sigma_b(c,c)\right) = \frac{25}{4}$ and

$$\begin{aligned} \Delta_k^{\mathcal{P}}(b,b) &= \max\left\{ \begin{array}{c} \sigma_b(b,b), \sigma_b(b,c), \sigma_b(b,c), \dfrac{\sigma_b(b,c) + \sigma_b(b,c)}{5}, \\ \dfrac{\sigma_b(b,c)\sigma_b(b,c)}{1 + \sigma_b(b,b)}, \dfrac{\sigma_b(b,c)\sigma_b(b,c)}{1 + \sigma_b(c,c)} \end{array} \right\} \\ &= \max\left\{\frac{10}{3}, \frac{13}{3}, \frac{13}{3}, \frac{26}{15}, \frac{13}{3}, \frac{338}{63}\right\} = \frac{338}{63}. \end{aligned}$$

Then

$$\begin{aligned} \beta(b, \mathcal{P}b)\beta(b, \mathcal{P}b)\psi_1\left(\frac{5}{4}\sigma_b(c,c)\right) = \psi_1\left(\frac{5}{4}\left(\frac{5}{2}\right)\right) = 2\left(\frac{25}{8}\right) = \frac{25}{4} &\leq \frac{676}{63} - \frac{338}{401} \\ &= \psi_1\left(\frac{338}{63}\right) - \psi_2\left(\frac{338}{63}\right). \end{aligned}$$

Case II: $x = b$ and $y = c$.

It is $\psi_1\left(\frac{5}{4}\sigma_b(\mathcal{P}b,\mathcal{P}c)\right)=\psi_1\left(\frac{5}{4}\sigma_b(c,a)\right)=7$ and

$$\begin{aligned}\Delta_k^{\mathcal{P}}(b,c)&=\max\left\{\begin{array}{c}\sigma_b(b,c),\sigma_b(b,c),\sigma_b(c,a),\dfrac{\sigma_b(b,a)+\sigma_b(c,c)}{5},\\ \dfrac{\sigma_b(b,c)\sigma_b(c,a)}{1+\sigma_b(b,c)},\dfrac{\sigma_b(b,c)\sigma_b(c,a)}{1+\sigma_b(c,a)}\end{array}\right\}\\ &=\max\left\{\frac{13}{3},\frac{13}{3},\frac{14}{5},\frac{117}{50},\frac{91}{40},\frac{182}{57}\right\}=\frac{13}{3}.\end{aligned}$$

Then

$$\begin{aligned}\beta(b,\mathcal{P}b)\beta(c,\mathcal{P}c)\psi_1\left(\frac{5}{4}\sigma_b(c,a)\right)=\psi_1\left(\frac{5}{4}\left(\frac{14}{5}\right)\right)=7&\le\frac{26}{3}-\frac{26}{29}\\ &=\psi_1\left(\frac{13}{3}\right)-\psi_2\left(\frac{13}{3}\right).\end{aligned}$$

Case III: $x=b$ and $y=a$.
We have $\psi_1\left(\frac{5}{4}\sigma_b(\mathcal{P}b,\mathcal{P}a)\right)=\psi_1\left(\frac{5}{4}\sigma_b(c,a)\right)=\psi_1\left(\frac{5}{4}(\frac{14}{5})\right)=7$ and

$$\begin{aligned}\Delta_k^{\mathcal{P}}(b,a)&=\max\left\{\begin{array}{c}\sigma_b(b,a),\sigma_b(b,c),\sigma_b(a,a),\dfrac{\sigma_b(b,a)+\sigma_b(a,c)}{5},\\ \dfrac{\sigma_b(b,c)\sigma_b(a,a)}{1+\sigma_b(b,a)},\dfrac{\sigma_b(b,c)\sigma_b(a,a)}{1+\sigma_b(c,a)}\end{array}\right\}\\ &=\max\left\{\frac{46}{5},\frac{13}{3},0,\frac{12}{5},0,0\right\}=\frac{46}{5}.\end{aligned}$$

Then

$$\begin{aligned}\beta(b,\mathcal{P}b)\beta(b,\mathcal{P}b)\psi_1\left(\frac{5}{4}\sigma_b(c,c)\right)=\psi_1\left(\frac{5}{4}\left(\frac{14}{5}\right)\right)=7&\le\frac{92}{5}-\frac{46}{51}\\ &=\psi_1\left(\frac{46}{5}\right)-\psi_2\left(\frac{46}{5}\right).\end{aligned}$$

Therefore, $\mathcal{P}$ is a (β-ψ_1-ψ_2)-contractive mapping of type-I.

ii) Let $(x,y)\in\mathcal{F}\times\mathcal{F}$ be such that $\beta(x,y)\ge1$. From the definition of $\mathcal{P}$ and β we have $\mathcal{P}x,\mathcal{P}y\in\{a,b,c\}$, so we have $\beta(\mathcal{P}x,\mathcal{P}y)=1\ge1$. Hence, $\mathcal{P}$ is β-admissible.

iii) Taking $x_1=a\in\mathcal{F}$, we have

$$\beta(x_1,\mathcal{P}x_1)=\beta(a,\mathcal{P}a)=\beta(a,a)=1\ge1.$$

iv) Let $\{x_n\}$ be a sequence in $\mathcal{F}$ such that $\beta(x_n,x_{n+1})\ge1$, $\forall\, n\in\mathbb{N}$ and $x_n\to x\in\mathcal{F}$ as $n\to\infty$. By using the definiiton of β we have $x_n\in\{a,b,c\}$, $\forall\, n\in\mathbb{N}$ and $x\in\{a,b,c\}$. Then $\beta(x_n,x)=1\ge1$.

Now, all the hypothesis of Theorem 2.3 are satisfied. Therefore, $\mathcal{P}$ has a fixed point (the point a). □

Example 3.2. *Let $\mathcal{F}=[0,\infty)$ be equipped with the partial order $\preceq$ defined by*

$$x\preceq y\Leftrightarrow x\le y$$

and with the functional $\sigma_b\colon\mathcal{F}\times\mathcal{F}\to\mathbb{R}_+$ defined by $\sigma_b(x,y)=(\max\{x,y\})^2$, for all $x,y\in\mathcal{F}$. Clearly, $(\mathcal{F},\sigma_b)$ is an ordered b-metric-like space with $k=2$. Define a mapping $\mathcal{P}\colon\mathcal{F}\to\mathcal{F}$ by

$$\mathcal{P}(x)=\begin{cases}\dfrac{x}{\sqrt{3+2x}} & \text{if } x\in[0,2],\\ \dfrac{3x}{2} & \text{otherwise},\end{cases}$$

and $\beta\colon \mathcal{F}\times\mathcal{F}\to[0,\infty)$ *by*

$$\beta(x,y)=\begin{cases}1 & \text{if } x,y\in[0,2],\\ 0 & \text{otherwise }.\end{cases}$$

Consider control functions $\psi_1(t)=t$ *and*

$$\psi_2(t)=\begin{cases}\dfrac{t(2\sqrt{t}+1)}{2\sqrt{t}+3} & \text{if } t\in[0,2],\\ \dfrac{3t}{4} & \text{if } t>2.\end{cases}$$

Then $\mathcal{P}$ *is continuous and increasing,* $0\preceq\mathcal{P}0$.

We will prove the following:

i) $\mathcal{P}\colon\mathcal{F}\to\mathcal{F}$ *is a* (β-ψ_1-ψ_2)*-contractive mapping of type-I;*
ii) $\mathcal{P}$ *is* β*-admissible;*
iii) $x_1\in\mathcal{F}$ *exists such that* $x_1\preceq\mathcal{P}x_1$ *and* $\beta(x_1,\mathcal{P}x_1)\geq 1$;
iv) If a sequence $\{x_n\}_{n=1}^{\infty}$ *in* $\mathcal{F}$ *such that* $\beta(x_n,x_{n+1})\geq 1$ *and* $x_n\to x$, *as* $n\to\infty$, *then* $\beta(x_n,x)\geq 1$, *for all* $n\in\mathbb{N}$.

Proof. i) For all comparable $x,y\in\mathcal{F}$ we have

$$\begin{aligned}\beta(x,\mathcal{P}x)\beta(y,\mathcal{P}y)\psi_1(k\sigma_b(\mathcal{P}x,\mathcal{P}y))&=\psi_1\left(2\sigma_b\Big(\frac{x}{\sqrt{3+2x}},\frac{y}{\sqrt{3+2y}}\Big)\right)\\&=\psi_1\left(2\Big(\max\Big\{\frac{x}{\sqrt{3+2x}},\frac{y}{\sqrt{3+2y}}\Big\}\Big)^2\right).\end{aligned}$$

Without loss of generality we can consider $0\leq y\leq x\leq 2$. Then

$$\begin{aligned}\beta(x,\mathcal{P}x)\beta(y,\mathcal{P}y)\psi_1(k\sigma_b(\mathcal{P}x,\mathcal{P}y))&=\psi_1\left(2\sigma_b\Big(\frac{x}{\sqrt{3+2x}},\frac{y}{\sqrt{3+2y}}\Big)\right)\\&=\psi_1\left(2\Big(\frac{x}{\sqrt{3+2x}}\Big)^2\right)\\&=\frac{2x^2}{3+2x}\end{aligned}$$

and

$$\Delta_s^{\mathcal{P}}(x,y)=\max\left\{x^2,x^2,y^2,\frac{x^2+\Big(\max\Big\{y,\frac{x}{\sqrt{3+2x}}\Big\}\Big)^2}{4},\frac{(x^2)(y^2)}{1+x^2},\frac{(x^2)(y^2)}{1+\Big(\frac{x}{\sqrt{3+2x}}\Big)}\right\}=x^2.$$

Then

$$\begin{aligned}\beta(x,\mathcal{P}x)\beta(y,\mathcal{P}y)\psi_1(k\sigma_b(\mathcal{P}x,\mathcal{P}y))&=\frac{2x^2}{3+2x}\leq x^2-\frac{2x^3+x^2}{3+2x}\\&\leq\psi_1(x^2)-\psi_2(x^2)\\&=\psi_1(\Delta_s^{\mathcal{P}}(x,y))-\psi_2(\Delta_s^{\mathcal{P}}(x,y)).\end{aligned}$$

ii) Let $(x,y)\in\mathcal{F}\times\mathcal{F}$ be such that $\beta(x,y)\geq 1$. From the definition of $\mathcal{P}$ and β we have both $\mathcal{P}x=\dfrac{2x}{\sqrt{3+2x}}$, and $\mathcal{P}y=\dfrac{2y}{\sqrt{3+2y}}$ are in $[0,2]$, so we have $\beta(\mathcal{P}x,\mathcal{P}y)=1\geq 1$. Then $\mathcal{P}$ is β-admissible.

iii) Taking $x_1 = 0 \in \mathcal{F}$, we have

$$\beta(x_1, \mathcal{P}x_1) = \beta(0, \mathcal{P}0) = \beta(0,0) = 1 \geq 1.$$

iv) Let $\{x_n\}$ be a sequence in $\mathcal{F}$ such that $\beta(x_n, x_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$ and $x_n \to x \in \mathcal{F}$ as $n \to \infty$. By using the definition of β, we have $x_n \in [0,2]$, $\forall\, n \in \mathbb{N}$ and $x \in [0,2]$. Then $\beta(x_n, x) = 1 \geq 1$.

Thus all the hypotheses of Theorem 2.3 are fulfilled. Therefore, $\mathcal{P}$ has a fixed point in $\mathcal{F}$ (which is 0).

It can be remarked that the contraction condition (3) is not true without taking β term into account. For example at $x = 0$ and $y = 4$, we get

$$\begin{aligned}\psi_1(2\sigma_b(\mathcal{P}0, \mathcal{P}4)) = \psi_1(2\sigma_b(0,6)) = \psi_1(2(36)) = 72 \nleq 9 &= 36 - 27\\ &= \psi_1(36) - \psi_2(36)\\ &= \psi_1(\Delta_s^{\mathcal{P}}(0,4)) - \psi_2(\Delta_s^{\mathcal{P}}(0,4)).\end{aligned}$$

Now, in this problem, we check the contraction condition due to Mukheimer [12, Definition 2.1] for comparable $x, y \in \mathcal{F}$ and $p_b(x,y) = (\max\{x,y\})^2$. If we take $x = \frac{1}{3}$ and $y = \frac{1}{2}$, then

$$\beta\left(\frac{1}{3}, \frac{1}{2}\right)\psi_1\left(2p_b\left(\mathcal{P}\left(\frac{1}{3}\right), \mathcal{P}\left(\frac{1}{2}\right)\right)\right) = \psi_1\left(2p_b\left(\frac{1}{9}, \frac{1}{4}\right)\right) = \psi_1\left(2\left(\frac{1}{16}\right)\right) = \frac{1}{8}$$

and

$$M(x,y) = M\left(\frac{1}{3}, \frac{1}{2}\right) = \max\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{9}, \frac{13}{36}\right\} = \frac{1}{4}$$

$$\psi_1\left(M\left(\frac{1}{3}, \frac{1}{2}\right)\right) - \psi_2\left(M\left(\frac{1}{3}, \frac{1}{2}\right)\right) = \psi_1\left(\frac{1}{4}\right) - \psi_2\left(\frac{1}{4}\right) = 0 < \frac{1}{8}.$$

Therefore contraction condition due to Mukheimer [12] is not satisfied in this example. □

Example 3.3. *Let $\mathcal{F} = [0,\infty)$ be equipped with the partial order $\preceq$ defined by*

$$x \preceq y \Leftrightarrow x \leq y$$

and with the functional $\sigma_b \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+$ defined by $\sigma_b(x,y) = x^2 + y^2 + |x-y|^2$, for all $x, y \in \mathcal{F}$. Clearly, $(\mathcal{F}, \sigma_b)$ is an ordered b-metric-like space with $k = 2$.

Define a mapping $\mathcal{P} \colon \mathcal{F} \to \mathcal{F}$ by $\mathcal{P}(x) = \ln\left(1 + \frac{|x|}{16}\right)$ and $\beta \colon \mathcal{F} \times \mathcal{F} \to [0,\infty)$ by

$$\beta(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,2], \\ 0 & \text{otherwise}. \end{cases}$$

Consider control functions $\psi_1(t) = \frac{3t}{2}$ and $\psi_2(t) = \frac{t}{2}$. Then $\mathcal{P}$ is continuous and increasing, $0 \preceq \mathcal{P}0$.

We will prove the following:

i) $\mathcal{P} \colon \mathcal{F} \to \mathcal{F}$ is a (β-ψ_1-ψ_2)-contractive mapping of type-I, with $\psi_1(t) = \frac{3t}{2}$ for all $t \geq 0$;
ii) $\mathcal{P}$ is β-admissible;
iii) $x_1 \in \mathcal{F}$ exists such that $x_1 \preceq \mathcal{P}x_1$ and $\beta(x_1, \mathcal{P}x_1) \geq 1$;
iv) If a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, as $n \to \infty$, then $\beta(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.

Proof. i) Clearly, $\sigma_b(\mathcal{P}0, \mathcal{P}0) = \sigma_b(0,0) = 0$. Further, the following five cases can be distinguished:

Case I: $x = 2$ and $y = 1$ (this case shows the reason for involvement of rational terms). We have $\psi_1(2\sigma_b(\mathcal{P}2, \mathcal{P}1)) = \psi_1\left(2\sigma_b\left(\frac{59}{500}, \frac{3}{50}\right)\right) = \frac{101}{5000} = 0.0202$ and

$$\begin{aligned}\Delta_k^{\mathcal{P}}(2,1) &= \max\left\{\begin{array}{c}\sigma_b(2,1), \sigma_b\left(2, \frac{59}{500}\right), \sigma_b\left(1, \frac{3}{50}\right), \dfrac{\sigma_b\left(2, \frac{3}{50}\right) + \sigma_b\left(1, \frac{59}{500}\right)}{8}, \\ \dfrac{\sigma_b\left(2, \frac{59}{500}\right)\sigma_b\left(1, \frac{3}{50}\right)}{1 + \sigma_b(2,1)}, \dfrac{\sigma_b\left(2, \frac{59}{500}\right)\sigma_b\left(1, \frac{3}{50}\right)}{1 + \sigma_b\left(\frac{59}{500}, \frac{3}{50}\right)}\end{array}\right\}\\ &= \max\{6, 7.6817, 1.8872, 1.1951, 2.0709, 14.2019\} = 14.2019 = \frac{1776}{125}.\end{aligned}$$

Then

$$\begin{aligned}\beta(2,\mathcal{P}2)\beta(1,\mathcal{P}1)\psi_1\left(\sigma_b\left(2\sigma_b\left(\frac{59}{500},\frac{3}{50}\right)\right)\right) &= \psi_1\left(2\left(\frac{59}{500},\frac{3}{50}\right)\right)\\ &= \frac{101}{5000} = 0.0202\\ &< \frac{2131}{100} - \frac{888}{125}\\ &= \psi_1\left(\frac{1776}{125}\right) - \psi_2\left(\frac{1776}{125}\right).\end{aligned}$$

Case II: For $x = 1$ and $y = 0$, it is
$\psi_1\left(2\sigma_b(\mathcal{P}1,\mathcal{P}0)\right) = \psi_1\left(2\sigma_b\left(\frac{3}{50},0\right)\right) = \frac{27}{1250} = 0.0216$ and

$$\begin{aligned}\Delta_k^{\mathcal{P}}(1,0) &= \max\left\{\begin{array}{c}\sigma_b(1,0),\sigma_b(0,0),\sigma_b\left(1,\frac{3}{50}\right),\dfrac{\sigma_b(1,0)+\sigma_b\left(0,\frac{3}{50}\right)}{8},\\ \dfrac{\sigma_b(0,0)\sigma_b\left(1,\frac{3}{50}\right)}{1+\sigma_b(1,0)},\dfrac{\sigma_b(0,0)\sigma_b\left(1,\frac{3}{50}\right)}{1+\sigma_b\left(\frac{3}{50},0\right)}\end{array}\right\}\\ &= \max\{2,0,1.8872,0.2509,0,0\} = 2.\end{aligned}$$

Then

$$\begin{aligned}\beta(1,\mathcal{P}1)\beta(0,\mathcal{P}0)\psi_1\left(\sigma_b\left(2\sigma_b\left(\frac{3}{50},0\right)\right)\right) &= \psi_1\left(2\left(\frac{3}{50},0\right)\right)\\ &= \frac{27}{1250} = 0.0216\\ &< 2 = \psi_1(2) - \psi_2(2).\end{aligned}$$

Case III: $x = 1$ and $y = 1$ (this case also shows the reason for involvement of rational terms). We have $\psi_1\left(2\sigma_b(\mathcal{P}1,\mathcal{P}1)\right) = \psi_1\left(2\sigma_b\left(\frac{3}{50},\frac{3}{50}\right)\right) = \frac{27}{1250} = 0.0216$ and

$$\begin{aligned}\Delta_k^{\mathcal{P}}(1,1) &= \max\left\{\begin{array}{c}\sigma_b(1,1),\sigma_b\left(1,\frac{3}{50}\right),\sigma_b\left(1,\frac{3}{50}\right),\dfrac{\sigma_b\left(1,\frac{3}{50}\right)+\sigma_b\left(1,\frac{3}{50}\right)}{8},\\ \dfrac{\sigma_b\left(1,\frac{3}{50}\right)\sigma_b\left(1,\frac{3}{50}\right)}{1+\sigma_b(1,1)},\dfrac{\sigma_b\left(1,\frac{3}{50}\right)\sigma_b\left(1,\frac{3}{50}\right)}{1+\sigma_b\left(\frac{3}{50},\frac{3}{50}\right)}\end{array}\right\}\\ &= \max\{2,1.8872,1.8872,0.4718,1.1872,3.5361\} = 3.5361 = \frac{442}{125}.\end{aligned}$$

Then

$$\begin{aligned}\beta(1,\mathcal{P}1)\beta(1,\mathcal{P}1)\psi_1\left(\sigma_b\left(2\sigma_b\left(\frac{3}{50},\frac{3}{50}\right)\right)\right) &= \psi_1\left(2\left(\frac{3}{50},\frac{3}{50}\right)\right)\\ &= \frac{27}{1250} = 0.0216\\ &< \frac{663}{125} - \frac{221}{125}\\ &= \psi_1\left(\frac{27}{1250}\right) - \psi_2\left(\frac{27}{1250}\right).\end{aligned}$$

Case IV: For $x = 2$ and $y = 0$, it is
$\psi_1\left(2\sigma_b(\mathcal{P}2,\mathcal{P}0)\right) = \psi_1\left(2\sigma_b\left(\frac{59}{500},0\right)\right) = \frac{51}{625} = 0.0816$ and

$$\Delta_k^{\mathcal{P}}(2,0) = \max\left\{\begin{array}{c} \sigma_b(2,0), \sigma_b\left(2,\frac{59}{500}\right), \sigma_b(0,0), \dfrac{\sigma_b(2,0)+\sigma_b\left(0,\frac{59}{500}\right)}{8}, \\ \dfrac{\sigma_b\left(2,\frac{59}{500}\right)\sigma_b(0,0)}{1+\sigma_b(2,0)}, \dfrac{\sigma_b\left(2,\frac{59}{500}\right)\sigma_b(0,0)}{1+\sigma_b\left(\frac{59}{500},0\right)} \end{array}\right\}$$
$$= \max\{8, 7.6817, 0, 1.0034, 0, 0\} = 8.$$

Then

$$\begin{aligned}\beta(2,\mathcal{P}2)\beta(0,\mathcal{P}0)\psi_1\left(\sigma_b\left(2\left(\frac{59}{500},0\right)\right)\right) &= \psi_1\left(2\left(\frac{59}{500},0\right)\right)\\ &= \frac{51}{625} = 0.0816\\ &\leq \psi_1(8) - \psi_2(8).\end{aligned}$$

Case V: For $x = 2$ and $y = 2$, we have
$\psi_1\left(2\sigma_b(\mathcal{P}2,\mathcal{P}2)\right) = \psi_1\left(2\sigma_b\left(\frac{59}{500},\frac{59}{500}\right)\right) = \frac{411}{5000} = 0.0822$ and

$$\Delta_k^{\mathcal{P}}(2,2) = \max\left\{\begin{array}{c} \sigma_b(2,2), \sigma_b\left(2,\frac{59}{500}\right), \sigma_b(0,0), \dfrac{\sigma_b(2,\frac{59}{500})+\sigma_b\left(2,\frac{59}{500}\right)}{8}, \\ \dfrac{\sigma_b\left(2,\frac{59}{500}\right)\sigma_b\left(2,\frac{59}{500}\right)}{1+\sigma_b(2,2)}, \dfrac{\sigma_b\left(2,\frac{59}{500}\right)\sigma_b\left(2,\frac{59}{500}\right)}{1+\sigma_b\left(\frac{59}{500},\frac{59}{500}\right)} \end{array}\right\}$$
$$= \max\{8, 7.6817, 7.6817, 1.9204, 6.5565, 57.4348\} = 57.4348 = \frac{11487}{200}.$$

Then

$$\begin{aligned}\beta(2,\mathcal{P}2)\beta(2,\mathcal{P}2)\psi_1\left(\sigma_b\left(2\left(\frac{59}{500},\frac{59}{500}\right)\right)\right) &= \psi_1\left(2\left(\frac{59}{500},\frac{59}{500}\right)\right)\\ &= \frac{411}{5000} = 0.0822\\ &\leq \psi_1\left(\frac{11487}{200}\right) - \psi_2\left(\frac{11487}{200}\right).\end{aligned}$$

Similarly, we can consider other cases. Therefore, $\mathcal{P}$ is a (β-ψ_1-ψ_2)-contractive mapping of type-I.

ii) Let $(x, y) \in \mathcal{F}\times\mathcal{F}$ be such that $\beta(x, y) \geq 1$. From the definition of $\mathcal{P}$ and β we have both $\mathcal{P}x = \ln\left(1 + \frac{|x|}{16}\right)$, and $\mathcal{P}y = \ln\left(1 + \frac{|y|}{16}\right)$) are in $[0, 2]$, so we have $\beta(\mathcal{P}x, \mathcal{P}y) = 1 \geq 1$. Then $\mathcal{P}$ is β-admissible.

iii) Taking $x_1 = 0 \in \mathcal{F}$, we have

$$\beta(x_1, \mathcal{P}x_1) = \beta(0, \mathcal{P}0) = \beta(0, \ln(2)) = 1 \geq 1.$$

iv) Let $\{x_n\}$ be a sequence in $\mathcal{F}$ be such that $\beta(x_n, x_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$ and $x_n \to x \in \mathcal{F}$ as $n \to \infty$. Since $\beta(x_n, x_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$ and by using the β definition, we have $x_n \in [0, 2]$, $\forall\, n \in \mathbb{N}$ and $x \in [0, 2]$. Then $\beta(x_n, x) = 1 \geq 1$.

Now, all the hypotheses of Theorem 2.3 are satisfied. 0 is the fixed point of $\mathcal{P}$. □

Example 3.4. *Consider $\mathcal{F} = \{0, 1, 2\}$ and let $\sigma_b : \mathcal{F}\times\mathcal{F} \to [0, \infty)$ be defined by*

$$\sigma_b(0,0) = 0, \qquad \sigma_b(1,1) = 3, \qquad \sigma_b(2,2) = \frac{3}{2}, \quad \sigma_b(0,1) = \sigma_b(1,0) = 13,$$

$$\sigma_b(0,2) = \sigma_b(2,0) = 1, \quad \sigma_b(1,2) = \sigma_b(2,1) = 8.$$

It is clear that $(\mathcal{F}, \sigma_b)$ is a b-complete b-metric like space with constant $k = \frac{9}{8}$. Note that $\sigma_b(1,1) \neq 0$, so σ_b is not a metric and $13 = \sigma_b(0,1) \nleq \sigma_b(0,2) + \sigma_b(2,1) - \sigma_b(2,2) = \frac{15}{2}$, so it is not a partial metric, nor a partial b-metric ($13 = \sigma_b(0,1) \nleq \frac{5}{4}[\sigma_b(0,2) + \sigma_b(2,1)] - \sigma_b(2,2) = \frac{69}{8}$). Define mappings $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ and $\beta : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by

$$\mathcal{P}0 = 0, \qquad \mathcal{P}1 = 2, \qquad \mathcal{P}2 = 0,$$

and

$$\beta(x,y) = \begin{cases} 1, & x, y \in [0,2] \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}$ is continuous and increasing. Consider the altering distance functions $\psi_1(t) = t$ and $\psi_2(t) = \frac{t}{4}$. We will prove the following:

i) $\mathcal{P}: \mathcal{F} \to \mathcal{F}$ is a $(\beta$-ψ_1-$\psi_2)$-contractive mapping of type-II;
ii) $\mathcal{P}$ is β-admissible;
iii) $x_1 \in \mathcal{F}$ exists such that $x_1 \preceq \mathcal{P}x_1$ and $\beta(x_1, \mathcal{P}x_1) \geq 1$;
iv) If a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, as $n \to \infty$, then $\beta(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.

Proof. i) Observe that $\sigma_b(\mathcal{P}0, \mathcal{P}2) = \sigma_b(\mathcal{P}0, \mathcal{P}0) = \sigma_b(\mathcal{P}2, \mathcal{P}2) = 0$, Therefore, in order to prove that $\mathcal{P}$ is a $(\beta$-ψ_1-$\psi_2)$-contractive mapping, the following three cases have to be considered:

Case I: $x = 1$ and $y = 2$ (this case shows the reason for involvement of rational terms).

We have $\psi_1\left(\frac{9}{8}\sigma_b(\mathcal{P}1, \mathcal{P}2)\right) = \psi_1\left(\frac{9}{8}\sigma_b(2,0)\right) = \frac{9}{8}$ and

$$(\Delta_I)_k^{\mathcal{P}}(1,2) = \max \left\{ \begin{array}{c} \sigma_b(1,2), \sigma_b(2,0), \sigma_b(1,2), \dfrac{\sigma_b(2,2) + \sigma_b(1,0)}{9/2}, \\ \dfrac{\sigma_b(1,2)\sigma_b(1,0) + \sigma_b(2,0)\sigma_b(2,2)}{1 + \frac{9}{8}[\sigma_b(1,2) + \sigma_b(2,0)]}, \\ \dfrac{\sigma_b(1,2)\sigma_b(1,0) + \sigma_b(2,0)\sigma_b(2,2)}{1 + \sigma_b(1,0) + \sigma_b(2,2)} \end{array} \right\}$$
$$= \max\left\{8, 1, 8, \frac{29}{9}, \frac{844}{89}, \frac{211}{31}\right\} = \frac{844}{89}.$$

Then

$$\beta(1, \mathcal{P}1)\beta(2, \mathcal{P}2)\psi_1\left(\frac{9}{8}\sigma_b(2,0)\right) = \frac{9}{8} \leq \frac{844}{89} - \frac{844}{356} = \psi_1\left(\frac{844}{89}\right) - \psi_2\left(\frac{844}{89}\right).$$

Case II: For $x = 0$ and $y = 1$, it is $\psi_1\left(\frac{9}{8}\sigma_b(\mathcal{P}0, \mathcal{P}1)\right) = \psi_1\left(\frac{9}{8}\sigma_b(0,2)\right) = \frac{9}{8}$ and

$$(\Delta_I)_k^{\mathcal{P}}(0,1) = \max \left\{ \begin{array}{c} \sigma_b(0,1), \sigma_b(1,2), \sigma_b(0,0), \dfrac{\sigma_b(0,2) + \sigma_b(1,0)}{9/2}, \\ \dfrac{\sigma_b(0,0)\sigma_b(0,2) + \sigma_b(1,2)\sigma_b(1,0)}{1 + \frac{9}{8}[\sigma_b(0,0) + \sigma_b(1,2)]}, \\ \dfrac{\sigma_b(0,0)\sigma_b(0,2) + \sigma_b(1,2)\sigma_b(1,0)}{1 + \sigma_b(0,2) + \sigma_b(1,0)} \end{array} \right\}$$
$$= \max\left\{13, 8, 0, \frac{28}{9}, \frac{104}{10}, \frac{104}{15}\right\} = 13.$$

Then

$$\beta(0, \mathcal{P}0)\beta(1, \mathcal{P}1)\psi_1\left(\frac{9}{8}\sigma_b(\mathcal{P}0, \mathcal{P}1)\right) = \frac{9}{8} \leq 13 - \frac{13}{4}$$
$$= \psi_1(13) - \psi_2(13).$$

Case III: For $x = 1$ and $y = 1$, we have $\psi_1(\frac{9}{8}\sigma_b(\mathcal{P}1, \mathcal{P}1)) = \psi_1\left(\frac{9}{8}\sigma_b(2,2)\right) = \psi_1\left(\frac{9}{8}\left(\frac{3}{2}\right)\right) = \frac{27}{16}$ and

$$
\begin{aligned}
(\Delta_I)_k^{\mathcal{P}}(1,1) &= \max\left\{\begin{array}{c} \sigma_b(1,1), \sigma_b(1,2), \sigma_b(1,2), \dfrac{\sigma_b(1,2)+\sigma_b(1,2)}{9/2}, \\ \dfrac{\sigma_b(1,2)\sigma_b(1,2)+\sigma_b(1,2)\sigma_b(1,2)}{1+\frac{9}{8}[\sigma_b(1,2)+\sigma_b(1,2)]}, \\ \dfrac{\sigma_b(1,2)\sigma_b(1,2)+\sigma_b(1,2)\sigma_b(1,2)}{1+\sigma_b(1,2)+\sigma_b(1,2)} \end{array}\right\} \\
&= \max\left\{3, 8, 8, \frac{32}{9}, \frac{128}{19}, \frac{128}{17}\right\} \\
&= 8.
\end{aligned}
$$

Then

$$
\begin{aligned}
\beta(1, \mathcal{P}1)\beta(1, \mathcal{P}1)\psi_1\left(\frac{9}{8}\sigma_b(2,2)\right) = \frac{27}{16} &\leq 8 - 2 \\
&= \psi_1(8) - \psi_2(8).
\end{aligned}
$$

Thus in all cases $\mathcal{P}$ satisfies (β-ψ_1-ψ_2)-contractive condition of type-II.

ii) Let $(x, y) \in \mathcal{F} \times \mathcal{F}$ be such that $\beta(x, y) \geq 1$. From the definition of $\mathcal{P}$ and β we have $\mathcal{P}x$, $\mathcal{P}y$ are in $[0, 2]$, so we have $\beta(\mathcal{P}x, \mathcal{P}y) = 1 \geq 1$. Hence, $\mathcal{P}$ is β-admissible.

iii) Taking $x_1 = 0 \in \mathcal{F}$, we have

$$\beta(x_1, \mathcal{P}x_1) = \beta(0, \mathcal{P}0) = \beta(0, 0) = 1 \geq 1.$$

iv) Let $\{x_n\}$ be a sequence in $\mathcal{F}$ such that $\beta(x_n, x_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$ and $x_n \to x \in \mathcal{F}$ as $n \to \infty$. By using the definition of β, we have $x_n \in [0, 2]$, $\forall\, n \in \mathbb{N}$ and $x \in [0, 2]$. Then $\beta(x_n, x) = 1 \geq 1$.

Thus, all the conditions of Theorem 2.6 are satisfied. Therefore, $\mathcal{P}$ has a fixed point (which is 0). □

Example 3.5. *Let $\mathcal{F} = [0, \infty)$ be equipped with the partial order $\preceq$ defined by*

$$x \preceq y \Leftrightarrow x \leq y$$

and with the functional $\sigma_b \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+$ defined by $\sigma_b(x, y) = (\max\{x, y\})^2$ for all $x, y \in \mathcal{F}$. Clearly, $(\mathcal{F}, \sigma_b)$ is an ordered b-metric-like space with $k = 2$. Define the mapping $\mathcal{P} \colon \mathcal{F} \to \mathcal{F}$ by

$$
\mathcal{P}(x) = \begin{cases} \dfrac{x}{\sqrt{3+x^2}} & \text{if } x \in [0, 3], \\ 2x + 3 & \text{if } x > 3, \end{cases}
$$

and $\beta \colon \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by

$$
\beta(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 3], \\ 0 & \text{otherwise.} \end{cases}
$$

Then $\mathcal{P}$ is continuous and increasing, $0 \preceq \mathcal{P}0$. Consider control functions $\psi_1(t) = 2t$ and

$$
\psi_2(t) = \begin{cases} \dfrac{2t(t+1)}{t+3} & \text{if } t \in [0, 3], \\ \dfrac{3(t+1)}{2} & \text{if } t > 3. \end{cases}
$$

We will prove the following:

i) $\mathcal{P} \colon \mathcal{F} \to \mathcal{F}$ is a (β-ψ_1-ψ_2)-contractive mapping of type-II, *ii)* $\mathcal{P}$ *is* β*-admissible;*

iii) There exists $x_1 = 0 \in \mathcal{F}$ *such that* $x_1 \preceq \mathcal{P}x_1$ *and* $\beta(x_1, \mathcal{P}x_1) \geq 1$*;*

iv) If a sequence $\{x_n\}_{n=1}^{\infty}$ *in* $\mathcal{F}$ *is such that* $\beta(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x$*, as* $n \to \infty$*, then* $\beta(x_n, x) \geq 1$*, for all* $n \in \mathbb{N}$*.*

Proof. i) We have

$$\begin{aligned}\beta(x,\mathcal{P}x)\beta(y,\mathcal{P}y)\psi_1(k\sigma_b(\mathcal{P}x,\mathcal{P}y)) &= \psi_1\left(2\sigma_b\Big(\frac{x}{\sqrt{3+x^2}}, \frac{y}{\sqrt{3+y^2}}\Big)\right)\\ &= \psi_1\left(2\Big(\max\Big\{\frac{x}{\sqrt{3+x^2}}, \frac{y}{\sqrt{3+y^2}}\Big\}\Big)^2\right).\end{aligned}$$

Without loss of generality, we can take $0 \leq y \leq x \leq 3$. Then

$$\begin{aligned}\beta(x,\mathcal{P}x)\beta(y,\mathcal{P}y)\psi_1(k\sigma_b(\mathcal{P}x,\mathcal{P}y)) &= \psi_1\left(2\sigma_b\Big(\frac{x}{\sqrt{3+x^2}}, \frac{x}{\sqrt{3+y^2}}\Big)\right)\\ &= \psi_1\Big(2\frac{x^2}{3+x^2}\Big)\\ &= \Big(\frac{4x^2}{3+x^2}\Big)\end{aligned}$$

and

$$\Delta_k^{\mathcal{P}}(x,y) = \max\left\{\begin{array}{l} x^2, y^2, x^2, \dfrac{x^2 + \Big(\max\Big\{y, \dfrac{x}{\sqrt{3+x^2}}\Big\}\Big)^2}{2},\\ \dfrac{(x^2)(x^2) + (y^2)\Big(\max\Big\{y, \dfrac{x}{\sqrt{3+x^2}}\Big\}\Big)^2}{1 + 1(x^2+y^2)},\\ \dfrac{(x^4) + (y^2)\Big(\max\Big\{y, \dfrac{x}{\sqrt{3+x^2}}\Big\}\Big)^2}{1 + x^2 + y^2} \end{array}\right\} = x^2.$$

Then

$$\begin{aligned}\beta(x,\mathcal{P}x)\beta(y,\mathcal{P}y)\psi_1(k\sigma_b(\mathcal{P}x,\mathcal{P}y)) &= \frac{4x^2}{x^2+3} \leq 2x^2 - \frac{2x^4+2x^2}{x^2+3}\\ &\leq \psi_1(x^2) - \psi_2(x^2)\\ &= \psi_1(\Delta_k^{\mathcal{P}}(x,y)) - \psi_2(\Delta_k^{\mathcal{P}}(x,y)).\end{aligned}$$

ii) Let $(x,y) \in \mathcal{F} \times \mathcal{F}$ be such that $\beta(x,y) \geq 1$. From the definition of $\mathcal{P}$ and β we have $\mathcal{P}x = \dfrac{x}{\sqrt{3+x^2}}$, $\mathcal{P}y = \dfrac{y}{\sqrt{3+y^2}}$ are in $[0,3]$, so we have $\beta(\mathcal{P}x, \mathcal{P}y) = 1 \geq 1$. Hence, $\mathcal{P}$ is β-admissible.

iii) Taking $x_1 = 0 \in \mathcal{F}$, we have

$$\beta(x_1, \mathcal{P}x_1) = \beta(0, \mathcal{P}0) = \beta(0,0) = 1 \geq 1.$$

iv) Let $\{x_n\}$ be a sequence in $\mathcal{F}$ such that $\beta(x_n, x_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$ and $x_n \to x \in \mathcal{F}$ as $n \to \infty$. Then we have $x_n \in [0,3]$, $\forall\, n \in \mathbb{N}$ and $x \in [0,3]$. Hence, $\beta(x_n, x) = 1 \geq 1$.

Now, all the hypotheses of Theorem 2.6 are satisfied. Therefore, $\mathcal{P}$ has a fixed point (which is 0).

For checking the contraction condition (30) (without β term) for comparable $x, y \in \mathcal{F}$, we take $x = 1$ and $y = 5$, and then

$$\begin{aligned}\psi_1(\sigma_b(\mathcal{P}1, \mathcal{P}5)) = \psi_1\left(\sigma_b\left(\frac{1}{2}, 13\right)\right) = \psi_1(169) = 338 &\nleq 83 = 338 - 255 \\ &= \psi_1(169) - \psi_2(169) \\ &= \psi_1(\Delta_k^{\mathcal{P}}(1,5)) - \psi_2(\Delta_k^{\mathcal{P}}(1,5)).\end{aligned}$$

Now consider the same problem and check the contraction condition due to Mukheimer [12, Definition 2.1] for comparable $x, y \in \mathcal{F}$ and $p_b(x, y) = (\max\{x, y\})^2$. Take $x = 1$ and $y = 2$, and then

$$\beta(1,2)\psi_1(2p_b(\mathcal{P}1, \mathcal{P}2)) = \psi_1\left(2p_b\left(\frac{1}{2}, \frac{2}{\sqrt{7}}\right)\right) = \psi_1(\frac{8}{7}) = \frac{16}{7}$$

and

$$M(x, y) = M(1, 2) = \max\{4, 1, 4, \frac{5}{4}\} = 4,$$

$$\psi_1(M(1, 2)) - \psi_2(M(1, 2)) = \psi_1(4) - \psi_2(4) = 8 - 6 = 2 < \frac{16}{7}.$$

Therefore the contraction condition due to Mukheimer [12, Definition 2.1] is not satisfied in this example, too. $\square$

4. An Application to Fourth-order Two-point Boundary Value Problem

Consider the fourth-order two-point boundary value problem

$$\begin{cases} u''''(t) = h(t, u(t)), \ 0 < t < 1; \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{41}$$

with $I = [0, 1]$ and $h \in C(I \times \mathbb{R}, \mathbb{R})$.

This problem is equivalent to the integral equation

$$u(t) = \int_0^1 K(t, r)h(r, u(r))dr, \ \forall\, t \in I = [0, 1], \tag{42}$$

where $h : I \times \mathbb{R} \to \mathbb{R}$ and $K : [0, 1] \times [0, 1] \to [0, \infty)$ is the Green function given by

$$K(t, r) = \frac{1}{6}\begin{cases} r^2(3t - r), & 0 \le r \le t \le 1 \\ t^2(3r - t), & 0 \le t \le r \le 1. \end{cases}$$

Consider the space $\mathcal{F} = C(I, \mathbb{R}) := \{x : I \to \mathbb{R} \mid x \text{ is continuous on } I\}$ and define a b-metric-like $\sigma_b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ by

$$\sigma_b(x, y) = \sup_{t \in [0,1]} \max\{|x(t)|, |y(t)|\}^2 \ \forall\, x, y \in \mathcal{F}.$$

From Proposition 1.9, $(\mathcal{F}, \sigma_b, 2)$ is a σ_b-complete b-metric like space. Define an order relation $\preceq$ on $\mathcal{F}$ by

$$u \preceq v \ \text{ iff } \ u(t) \le v(t), \ \forall\, t \in I.$$

Then $(\mathcal{F}, \preceq)$ is a partially ordered set. Therefore, $(\mathcal{F}, \sigma_b, 2, \preceq)$ is a σ_b-complete ordered b-metric like space.

Theorem 4.1. *Consider the mapping $\mathcal{P} : \mathcal{F} \times \mathcal{F}$ defined by*

$$\mathcal{P}u(t) = \int_0^1 K(t,r)h(r,u(r))dr \;\forall\; t \in [0,1].$$

Assume that the following assertions hold:

(i) $h : I \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

(ii) For all $k \in [0,1]$, *there exists a nondecreasing function* $h(k,\cdot)$ *i.e.,*

$$u, v \in \mathbb{R},\; u \le v \;\Rightarrow\; h(r,u) \le h(r,v);$$

such that

$$\max\{h(r,u)|, |h(r,v)\} \le \Theta_k(u,v), \tag{43}$$

where

$$\Theta_k(u,v) = \max\left\{\begin{array}{l} (\max\{u,v\})^2, (\max\{u,\mathcal{P}u\})^2, (\max\{v,\mathcal{P}v\})^2, \\ \dfrac{(\max\{u,\mathcal{P}v\})^2 + (\max\{v,\mathcal{P}v\})^2}{4k}, \\ \dfrac{(\max\{u,\mathcal{P}u\})^2(\max\{v,\mathcal{P}v\})^2}{1+(\max\{u,v\})^2}, \dfrac{(\max\{u,\mathcal{P}u\})^2(\max\{v,\mathcal{P}v\})^2}{1+(\max\{\mathcal{P}u,\mathcal{P}v\})^2} \end{array}\right\}^{\frac{1}{2}},$$

for all $u, v \in \mathcal{F}$ *with* $u \le v$ *and for all* $r \in [0,1]$;

(iii) there exists $u_1 \in C(I,\mathbb{R})$ *such that*

$$u_1(t) \le \int_0^1 K(t,r)h(t,u_1(r))\,dr, \quad t \in I;$$

(iv)
$$\sup_{t\in[0,1]} \int_0^1 K(t,r)dr \le \frac{1}{2}.$$

Then there exists a solution of the integral equation (42), and hence, there exists a solution of the problem (41).

Proof. From (i)–(ii), it follows immediately that $\mathcal{P}$ is a continuous and non-decreasing mapping w.r.t. $\preceq$. Also, from (iii), there exists $u_1 \in \mathcal{F}$ such that $u_1 \preceq \mathcal{P}u_1$.

For all $t \in [0,1]$, from the conditions (ii) and (iv), we get

$$\begin{aligned}
\sigma_b(\mathcal{P}u,\mathcal{P}v) &= \left(\max_{t\in[0,1]}\left\{\left|\int_0^1 K(t,r)h(r,u(r))dr\right|,\left|\int_0^1 K(t,r)h(r,v(r))dr\right|\right\}\right)^2\\
&\leq \left(\max_{t\in[0,1]}\left\{\int_0^1 K(t,r)\left|h(r,u(r))\right|dr,\int_0^1 K(t,r)\left|h(r,v(r))\right|dr\right\}\right)^2\\
&= \left(\max_{t\in[0,1]}\int_0^1 K(t,r)\max_{t\in[0,1]}(|h(r,u(r))|,|h(r,v(r))|)dr\right)^2\\
&\leq \left(\max_{t\in[0,1]}\int_0^1 K(t,r)(\Theta_k(u,v)(r))dr\right)^2\\
&\leq \left(\max_{t\in[0,1]}\int_0^1 K(t,r)\max\left\{\begin{array}{c}(\max\{u(r),v(r)\})^2,(\max\{u(r),\mathcal{P}u(r)\})^2,\\ (\max\{v(r),\mathcal{P}v(r)\})^2,\\ \frac{(\max\{u(r),\mathcal{P}v(r)\})^2+(\max\{v(r),\mathcal{P}v(r)\})^2}{4k},\\ \frac{(\max\{u(r),\mathcal{P}u(r)\})^2(\max\{v(r),\mathcal{P}v(r)\})^2}{1+(\max\{u(r),v(r)\})^2},\\ \frac{(\max\{u(r),\mathcal{P}u(r)\})^2(\max\{v(r),\mathcal{P}v(r)\})^2}{1+(\max\{\mathcal{P}u(r),\mathcal{P}v(r)\})^2}\end{array}\right\}^{\frac{1}{2}}dr\right)^2\\
&\leq \frac{1}{4}\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,\mathcal{P}u),\sigma_b(v,\mathcal{P}v),\frac{\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}v)}{4k},\\ \frac{\sigma_b(u,\mathcal{P}u)\sigma_b(v,\mathcal{P}v)}{1+\sigma_b(u,v)},\frac{\sigma_b(u,\mathcal{P}u)\sigma_b(v,\mathcal{P}v)}{1+\sigma_b(\mathcal{P}u,\mathcal{P}v)}\end{array}\right\}.
\end{aligned}$$

Now, by considering the control functions $\psi_1,\psi_2:[0,+\infty)\to[0,+\infty)$ given by:

$$\psi_1(t)=t,\quad \text{and}\quad \psi_2(t)=\frac{1}{2},\ \text{for } t\geq 0,$$

we get

$$\begin{aligned}
\psi_1(k\sigma_b(\mathcal{P}u,\mathcal{P}v)) \leq\ & \psi_1\left(\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,\mathcal{P}u),\sigma_b(v,\mathcal{P}v),\frac{\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}v)}{4k},\\ \frac{\sigma_b(u,\mathcal{P}u)\sigma_b(v,\mathcal{P}v)}{1+\sigma_b(u,v)},\frac{\sigma_b(u,\mathcal{P}u)\sigma_b(v,\mathcal{P}v)}{1+\sigma_b(\mathcal{P}u,\mathcal{P}v)}\end{array}\right\}\right)\\
&-\psi_2\left(\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,\mathcal{P}u),\sigma_b(v,\mathcal{P}v),\frac{\sigma_b(u,\mathcal{P}v)+\sigma_b(v,\mathcal{P}v)}{4k},\\ \frac{\sigma_b(u,\mathcal{P}u)\sigma_b(v,\mathcal{P}v)}{1+\sigma_b(u,v)},\frac{\sigma_b(u,\mathcal{P}u)\sigma_b(v,\mathcal{P}v)}{1+\sigma_b(\mathcal{P}u,\mathcal{P}v)}\end{array}\right\}\right).
\end{aligned}$$

Thus all the hypotheses of Corollary 2.8 are fulfilled for $k=2$. Therefore, there exists a fixed point of $\mathcal{P}$, which is equivalent to the existence of a solution of the equation (41). □

Finally we present the following numerical example.

Example 4.2. *Let us consider the problem*

$$\begin{cases}u''''(t)=1-\frac{1}{1+(u(t))^2},\ 0<t<1;\\ u(0)=u'(0)=u''(1)=u'''(1)=0.\end{cases}\tag{44}$$

In this case $h(t,u(t))=1-\frac{1}{1+(u(t))^2}$. *Consider* $\mathcal{F}=C(I,[0,1])$. *Then it is easily seen that* $h(t,u(t))$ *satisfies all conditions of Theorem 4.1. Indeed* $h(t,u(t))$ *is continuous (as differentiable), nondecreasing,* $u_1=0$ *and the inequality (43) is satisfied on* $\mathcal{F}$. *Thus from Theorem 4.1, the problem (44) has a solution in* $\mathcal{F}$.

5. Conclusion

Our results extend and generalize the results of Mukheimer [12] in partially ordered b-metric-like spaces. Two new concepts of (β-ψ_1-ψ_2)-contractive mapping are introduced using two altering distance functions in ordered b-metric-like spaces and the respective fixed point results are studied. Numerical examples are presented to illustrate our work in an appropriate manner. An immediate application of our investigation towards the solution of a boundary-value problem for a fourth-order ODE is discussed and rationalized by a numerical example.

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