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A Note on Positivity of Two-Dimensional Differential Operators

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Abstract. We consider the two-dimensional differential operator $A^{(t,x)}u(t,x) = -a_{11}(t,x)\,u_{tt} - a_{22}(t,x)u_{xx} + \sigma u$ defined on functions on the half-plane $\mathbb{R}^+ \times \mathbb{R}$ with the boundary condition $u(0,x) = 0$, $x \in \mathbb{R}$ where $a_{ii}(t,x)$, $i = 1,2$ are continuously differentiable and satisfy the uniform ellipticity condition $a_{11}^2(t,x) + a_{22}^2(t,x) \geq \delta > 0$, $\sigma > 0$. The structure of fractional spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}^+ \times \mathbb{R}\right), A^{(t,x)}\right)$ generated by the operator $A^{(t,x)}$ is investigated. The positivity of $A^{(t,x)}$ in $L_1\left(W_1^{2\alpha}(\mathbb{R}^+ \times \mathbb{R})\right)$ spaces is established. In applications, theorems on well-posedness in $L_1\left(W_1^{2\alpha}(\mathbb{R}^+ \times \mathbb{R})\right)$ spaces of elliptic problems are obtained.

1. Introduction

The theory of positivity of differential and difference operators in Hilbert and Banach spaces is important in the study of various properties of boundary value problems for partial differential equations, of difference schemes for partial differential equations, and of summation Fourier series converging in $C-norm$ (see, for examples, [1]-[3]).

Let us give the definition of positive operators and introduce the fractional spaces and preliminary facts that will be needed in the sequel.

An operator A densely defined in a Banach space E with domain $D(A)$ is called positive in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0 < \varphi < \pi$, symmetric with respect to the real axis, and moreover on the edges of this sector $S_1(\varphi) = \{\rho e^{i\varphi} : 0 \leq \rho \leq \infty\}$ and $S_2(\varphi) = \{\rho e^{-i\varphi} : 0 \leq \rho \leq \infty\}$, and outside of the sector the resolvent $(A - \lambda I)^{-1}$ is subject to the bound (see, [1])

$$\left\|(A - \lambda I)^{-1}\right\|_{E \to E} \leq \frac{M}{1 + |\lambda|}.$$

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A,E)$. The operator A is said to be strongly positive in a Banach space E if $\varphi(A,E) < \frac{\pi}{2}$.

Throughout the present paper, we will indicate with M positive constants which can be different from time to time and we are not interested in precise. We will write $M(\alpha, \beta, \cdots)$ to stress the fact that the constant depends only on $\alpha, \beta, \cdots$.

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The theory of differential and difference operators and their related applications have been investigated by many researchers (see, for example,[4]-[15], [24]-[31]).

For a positive operator A in the Banach space $E_{\alpha,1}$, let us introduce the fractional spaces $E_{\alpha,1} = E_{\alpha,1}(E, A)$ $(0 < \alpha < 1)$ consisting of those $v \in E$ for which norms

$$\|v\|_{E_{\alpha,1}} = \int\limits_0^\infty \lambda^\alpha \|A(A+\lambda I)^{-1}v\|_E \frac{d\lambda}{\lambda} + \|v\|_E$$

are finite. Clearly, the positive operator commutes A and its resolvent $(A - \lambda I)^{-1}$. By the definition of the norm in the fractional space $E_{\alpha,1} = E_{\alpha,1}(E, A)$ $(0 < \alpha < 1)$, we get

$$\|(A-\lambda I)^{-1}\|_{E_{\alpha,1}\to E_{\alpha,1}} \le \|(A-\lambda I)^{-1}\|_{E\to E}. \tag{1}$$

Thus, from the positivity of operator A in the Banach space E it follows the positivity of this operator in fractional spaces $E_{\alpha,1} = E_{\alpha,1}(E, A)$ $(0 < \alpha < 1)$.

Theorem 1.1. ([30]) *Let p and q be mutually conjugate exponents, that is, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be any two non-negative Lebesque measurable functions such that $0 < \int\limits_0^\infty f^p(x)\,dx < \infty$ and $0 < \int\limits_0^\infty g^q(y)\,dy < \infty$. Then, the following Hilbert's inequality holds:*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)\,g(y)}{x+y} < \pi \csc\left(\frac{\pi}{p}\right)\left[\int\limits_0^\infty f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int\limits_0^\infty g^q(y)\,dy\right]^{\frac{1}{q}}. \tag{2}$$

S.I. Danelich in [13] considered the positivity of a difference analog A_h^x of the $2m$-th order multi-dimensional elliptic operator A^x with dependent coefficients on half-spaces $\mathbb{R}^+ \times \mathbb{R}^{n-1}$.

The positivity of differential and difference operators $2m - th$ order in Hölder spaces and structure of fractional spaces generated by these operators were established in [16, 17].

The structure of fractional spaces generated by positive multi-dimensional differential and difference operators on space $\mathbb{R}^n$ in Banach spaces has been well investigated (see [21]-[23] and the references given therein).

In papers [18]-[20], the structure of fractional spaces generated by positive one-dimensional differential and difference operators in Banach spaces was studied. Note that the structure of fractional spaces generated by positive multi-dimensional differential and difference operators with local and nonlocal conditions on $\Omega \subset \mathbb{R}^n$ in Banach spaces $L_1(\Omega)$ has not been well studied.

In the present paper, we will study the structure of fractional spaces generated by the two-dimensional differential operator

$$A^{(t,x)}u(t,x) = -a_{11}(t,x)u_{tt}(t,x) - a_{22}(t,x)u_{xx}(t,x) + \delta u(t,x), \tag{3}$$

defined over the region $\mathbb{R}_+^2 = \mathbb{R}^+ \times \mathbb{R}$ with the boundary condition $u(0,x) = 0$, $x \in \mathbb{R}$. Here, the coefficients $a_{ii}(t,x)$, $i = 1,2$ are continuously differentiable and satisfy the uniform ellipticity

$$a_{11}^2(t,x) + a_{22}^2(t,x) \ge \delta > 0, \tag{4}$$

and $\sigma > 0$.

In the space $L_1 = L_1(\mathbb{R}_+^2)$ of all absolutely integrable functions $\varphi(t,x)$ defined on $\mathbb{R}_+^2$ with the norm

$$\|\varphi\|_{L_1(\mathbb{R}_+^2)} = \int\limits_0^\infty \int\limits_{-\infty}^\infty |\varphi(t,x)|dtdx$$

we will consider the problem of finding the resolvent of the operator $-A^{(t,x)}$

$$A^{(t,x)}u(t,x)+\lambda u(t,x)=f(t,x),\ t,x\in\mathbb{R}_+^2. \tag{5}$$

Following the paper [13], passing limit when $h\to 0$ in the special case $m=1$ and $n=2$, we get that there exists the inverse operator $(A^{(t,x)}+\lambda I)^{-1}$ for all $\lambda\geq 0$ and the following formula

$$(A^{(t,x)}+\lambda I)^{-1}f(t,x)=\int_0^\infty\int_{-\infty}^\infty G(t,x,p,s,\lambda)f(p,s)dsdp \tag{6}$$

is valid. Here $G(t,x,p,s,\lambda)$ is the Green's function of differential operator (5).

Lemma 1.2. *The following identities hold:*

$$\int_0^\infty\int_{-\infty}^\infty G(0,x,p,s,\lambda)dsdp=0,$$

$$\int_0^\infty\int_{-\infty}^\infty G(t,x,p,s,\lambda)dsdp=\frac{1}{\lambda+\delta}(1-v(t,x)),\ t>0,\ x\in\mathbb{R}, \tag{7}$$

where $v(t,x)$ is the solution of the following problem

$$-a_{11}(t,x)v_{tt}(t,x)-a_{22}(t,x)v_{xx}(t,x)+(\lambda+\delta)v(t,x)=0$$

$$v(0,x)=0,\ x\in\mathbb{R}. \tag{8}$$

We have the following estimates of the Green's function $G(t,x,p,s;\lambda)$ and its derivatives

$$\left|G(t,x,p,s;\lambda)\right| \tag{9}$$

$$\leq C\exp\left\{-a(\lambda+1)^{1/2}\left(\left|t-p\right|+|x-s|\right)\right\}\left(1+\ln\left\{1+\left((\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)\right)^{-1}\right\}\right)$$

and

$$\left|G_t(t,x,p,s,\lambda)\right|,\left|G_x(t,x,p,s,\lambda)\right| \tag{10}$$

$$\leq C\exp\left\{-a(\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)\right\}\left(\left|t-p\right|+|x-s|\right)^{-1}$$

hold. Here $a=a(\delta)$.

Note that under the assumption (4) there exists a unique solution $v(t,x)$ of problem $v(t,x)$ (8) and the following estimate holds:

$$|v(t,x)|\leq C\exp\left\{-a(\lambda+\delta)^{1/2}(|t|+|x|)\right\}. \tag{11}$$

Here, the structure of fractional spaces generated by the operator $A^{(t,x)}$ is investigated. The positivity of $A^{(t,x)}$ in Hölder spaces is studied. The organization of the present paper is as follows. In section 2, the positivity of $A^{(t,x)}$ in Hölder spaces is established. In section 3, the main theorem on the structure of fractional spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}_+^2\right),A^{(t,x)}\right)$ generated by $A^{(t,x)}$ is investigated. In section 4, applications on theorems on well-posedness in $L_1\left(W_1^{2\alpha}(\mathbb{R}_+^2)\right)$ space of elliptic problems are presented. Finally, the conclusion is given.

2. Positivity of $A^{(t,x)}$ in $L_1\left(\mathbb{R}_+^2\right)$ Spaces

Theorem 2.1. *$A^{(t,x)}$ is the positive operator in the Banach space $L_1\left(\mathbb{R}_+^2\right)$.*

Proof. Applying formula (6), the triangle inequality, the definition of L_1-norm, we get

$$\left|(A^{(t,x)}+\lambda I)^{-1}f(t,x)\right| \leq \int_0^\infty\int_{-\infty}^\infty \left|G(t,x,p,s,\lambda)\right|\left|f(p,s\right|dsdp.$$

Using estimate (9), we obtain

$$\int_0^\infty\int_{-\infty}^\infty \left|(A^{(t,x)}+\lambda I)^{-1}f(t,x)\right|dxdt$$

$$\leq \int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^\infty e^{-a(\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)}\left(1+\ln\left(1+\frac{1}{(\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)}\right)\right)\left|f(p,s\right|dsdpdxdt.$$

Before, by Hilbert's inequality, we have

$$\int_0^\infty\int_{-\infty}^\infty e^{-a(\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)}\left(1+\ln\left(1+\frac{1}{(\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)}\right)\right)dxdt$$

$$\leq \frac{M}{\lambda+\delta}\int_0^\infty\int_0^\infty e^{-a\left(\mu+z\right)}\left(1+\ln\left(1+\frac{1}{\mu+z}\right)\right)d\mu dz$$

$$\leq \frac{M}{\lambda+\delta}\left[\int_0^\infty\int_0^\infty e^{-a\left(\mu+z\right)}d\mu dz+\int_0^\infty\int_0^\infty \frac{e^{-a\left(\mu+z\right)}}{\mu+z}\left(1+\ln\left(1+\frac{1}{\mu+z}\right)\right)d\mu dz\right]$$

$$\leq \frac{M}{\lambda+\delta}\left\|f\right\|_{L_1}. \tag{12}$$

From (12) inequality that it follows

$$\int_0^\infty\int_{-\infty}^\infty \left|(A^{(t,x)}+\lambda I)^{-1}f(t,x)\right|dxdt \leq \frac{M}{\lambda+\delta}\int_0^\infty\int_{-\infty}^\infty \left|f(p,s\right|dsdp \leq \frac{M}{\lambda+\delta}\left\|f\right\|_{L_1}.$$

Then,

$$\left\|(A^{(t,x)}+\lambda I)^{-1}\right\|_{L_1\to L_1} \leq \frac{M}{\lambda+\delta}.$$

This finishes the proof of Theorem 2.1. □

We will introduce the Banach space $W_1^{2\alpha}(\mathbb{R}_+^2)$ $(0<\mu<1)$ of all functions φ defined on $\mathbb{R}_+^2$ and satisfying a Hölder condition for which the following norm is finite:

$$\left\|\varphi\right\|_{W_1^{2\alpha}(\mathbb{R}_+^2)} = \int_0^\infty\int_{-\infty}^\infty \left|\varphi(t,x)\right|dxdt+\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^\infty \frac{\left|\varphi(t,x)-\varphi(p,s)\right|}{\left(\left|t-p\right|^2+|x-s|^2\right)^{1+\alpha}}dxdtdsdp$$

$$+\int_0^\infty\int_{-\infty}^\infty \frac{\left|\varphi(t,x)-\varphi(0,x)\right|}{|t|^{2\alpha}}dxdt.$$

3. On Structure of Space $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_+\right), A^{(t,x)}\right)$

In this section, we prove the following theorem on structure of space $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_+\right), A^{(t,x)}\right)$.

Theorem 3.1. *$E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_+\right), A^{(t,x)}\right) = W_1^{2\alpha}(\mathbb{R}^2_+)$ for all $0 < 2\alpha < 1$.*

Proof. Assume that $f \in W_1^{2\alpha}(\mathbb{R}^2_+)$. Let $(t,x) \in \mathbb{R}^2_+$ and $\lambda > 0$ be fixed. From formulas (6) and (7) it follows that

$$A^{(t,x)}\left(A^{(t,x)} + \lambda I\right)^{-1} f(t,x)$$

$$= \frac{1}{\lambda+\delta} f(t,x) + \lambda \int_0^\infty \int_{-\infty}^\infty G(t,x,p,s;\lambda)\left(f(t,x) - f(p,s)\right) dsdp + \frac{\lambda}{\lambda+\delta} v(t,x) f(t,x). \tag{13}$$

Using equation (13) and the triangle inequality, we obtain

$$\int_0^\infty \int_{-\infty}^\infty \lambda^\alpha \left| A^{(t,x)}\left(A^{(t,x)} + \lambda\right)^{-1} f(t,x)\right| dxdt \le \int_0^\infty \int_{-\infty}^\infty \frac{\lambda^\alpha}{\lambda+\delta} \left|f(t,x)\right| dxdt$$

$$+ \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \lambda^{\alpha+1} \left|G(t,x,p,s;\lambda)\right| \left|f(t,x) - f(p,s)\right| dsdpdxdt + \int_0^\infty \int_{-\infty}^\infty \frac{\lambda^{\alpha+1}}{\lambda+\delta} \left|v(t,x)\right| \left|f(t,x)\right| dxdt.$$

Consequently,

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \lambda^\alpha \left| A^{(t,x)}\left(A^{(t,x)} + \lambda\right)^{-1} f(t,x)\right| dxdt \frac{d\lambda}{\lambda} \le \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\lambda^\alpha}{\lambda(\lambda+\delta)} \left|f(t,x)\right| dxdtd\lambda$$

$$+ \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \lambda^\alpha \left|G(t,x,p,s;\lambda)\right| \left|f(t,x) - f(p,s)\right| dsdpdxdtd\lambda$$

$$+ \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\lambda^\alpha}{\lambda+\delta} \left|v(t,x)\right| \left|f(t,x)\right| dxdtd\lambda$$

$$= I_1 + I_2 + I_3. \tag{14}$$

We will estimate $I_i,\ i = 1,2,3$, separately. First, let us estimate I_1. Since

$$\int_0^\infty \frac{\lambda^\alpha}{\lambda(\lambda+1)} d\lambda = \int_0^1 \frac{1}{\lambda^{1-\alpha}} d\lambda + \int_1^\infty \frac{1}{\lambda^{2-\alpha}} d\lambda = \frac{1}{\alpha(1-\alpha)}$$

we get

$$I_1 \le \frac{1}{\alpha(1-\alpha)} \int_0^\infty \int_{-\infty}^\infty \left|f(t,x)\right| dxdt \le \frac{1}{\alpha(1-\alpha)} \left\|f\right\|_{W_1^{2\alpha}}. \tag{15}$$

Second, let us estimate I_2. From estimate (9) it follows that

$$I_2 \leq M_1 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \lambda^\alpha e^{-a\sqrt{\lambda+\delta}\left(\left|t-p\right|+|x-s|\right)} \left(1+\ln\left(1+\frac{1}{\sqrt{\lambda+\delta}\left(\left|t-p\right|+|x-s|\right)}\right)\right)$$

$$\times \left|f(t,x)-f(p,s)\right| dsdpdxdtd\lambda$$

$$= M_1 \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty I(t,x,p,s) \left|f(t,x)-f(p,s)\right| dsdpdxdtd\lambda$$

where

$$I(t,x,p,s) = \int_0^\infty \lambda^\alpha e^{-a\sqrt{\lambda+\delta}\left(\left|t-p\right|+|x-s|\right)} \left(1+\ln\left(1+\frac{1}{\sqrt{\lambda+\delta}\left(\left|t-p\right|+|x-s|\right)}\right)\right) d\lambda.$$

Since $\ln(1+x) \leq x$, for all $x > -1$ we have that

$$I(t,x,p,s) \leq \int_0^\infty \lambda^\alpha e^{-a\sqrt{\lambda}\left(\left|t-p\right|+|x-s|\right)} d\lambda + \int_0^\infty \lambda^\alpha e^{-a\sqrt{\lambda}\left(\left|t-p\right|+|x-s|\right)} \frac{1}{\sqrt{\lambda}\left(\left|t-p\right|+|x-s|\right)} d\lambda.$$

Applying the change of the variable $a\sqrt{\lambda}\left(\left|t-p\right|+|x-s|\right) = y$, we can write

$$I(t,x,p,s) \leq \frac{2}{a^{2\alpha+2}\left(\left|t-p\right|+|x-s|\right)^{2\alpha+2}} \int_0^\infty e^{-y} y^{2\alpha+1} dy + \frac{2}{a^{2\alpha+1}\left(\left|t-p\right|+|x-s|\right)^{2\alpha+2}} \int_0^\infty e^{-y} y^{2\alpha} dy$$

$$= \frac{2\Gamma(2\alpha+2)}{a^{2\alpha+2}\left(\left|t-p\right|+|x-s|\right)^{2\alpha+2}} + \frac{2\Gamma(2\alpha+1)}{a^{2\alpha+1}\left(\left|t-p\right|+|x-s|\right)^{2\alpha+2}}$$

$$\leq M \frac{1}{\left(\left|t-p\right|^2+|x-s|^2\right)^{\alpha+1}}.$$

Therefore,

$$I_2 \leq M_2 \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\left|f(t,x)-f(p,s)\right|}{\left(\left|t-p\right|^2+|x-s|^2\right)^{\alpha+1}} dsdpdxdtd\lambda \leq M_3 \left\|f\right\|_{W_1^{2\alpha}}. \tag{16}$$

Third, let us estimate I_3. By estimate (11), we have

$$I_3 \leq M_1 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\lambda^\alpha}{\lambda+1} e^{-a\sqrt{\lambda}(|t|+|x|)} \left|f(t,x)\right| dxdtd\lambda$$

$$= M_1 \int_0^\infty \int_{-\infty}^\infty J(t,x,p,s) \left|f(t,x)\right| dxdtd\lambda$$

where

$$J(t,x,p,s)=\int_{0}^{\infty}\frac{\lambda^{\alpha}}{\lambda+1}e^{-a\sqrt{\lambda+\delta}(|t|+|x|)}d\lambda.$$

Using the change of the variable $a\sqrt{\lambda}\,(|t|+|x|)=y$, we obtain

$$J\leq\frac{2}{a^{2\alpha}\left(|t|+|x|\right)^{2\alpha}}\int_{0}^{\infty}y^{2\alpha-1}e^{-y}dy=\frac{2\Gamma\left(2\alpha\right)}{a^{2\alpha}\left(|t|+|x|\right)^{2\alpha}}\leq\frac{M}{\left(|t|+|x|\right)^{2\alpha}}.$$

Consequently,

$$I_{3}\leq M_{2}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{\left|f\left(t,x\right)\right|}{\left(|t|+|x|\right)^{2\alpha}}dxdtd\lambda\leq M_{3}\left\|f\right\|_{W_{1}^{2\alpha}}. \tag{17}$$

Combining estimates (15), (16), and (17), we get

$$\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\lambda^{\alpha}\left|A^{(t,x)}\left(A^{(t,x)}+\lambda\right)^{-1}f\left(t,x\right)\right|dxdt\frac{d\lambda}{\lambda}\leq\frac{M}{\alpha\left(1-\alpha\right)}\left\|f\right\|_{W_{1}^{2\alpha}}. \tag{18}$$

From estimate (18) yield

$$W_{1}^{2\alpha}(\mathbb{R}_{+}^{2})\subset E_{\alpha,1}\left(L_{1}\left(\mathbb{R}_{+}^{2}\right),A^{(t,x)}\right).$$

Now we will proof $E_{\alpha,1}\left(L_{1}\left(\mathbb{R}_{+}^{2}\right),A^{(t,x)}\right)\subset W_{1}^{2\alpha}(\mathbb{R}_{+}^{2})$. Using the definition of $E_{\alpha,1}\left(L_{1}\left(\mathbb{R}_{+}^{2}\right),A^{(t,x)}\right)$, we obtain

$$\int_{0}^{\infty}\int_{-\infty}^{\infty}|f(t,x)|dtdx\leq\left\|f^{h}\right\|_{E_{\alpha,1}\left(L_{1}\left(\mathbb{R}_{+}^{2}\right),A^{(t,x)}\right)}.$$

By Theorem 2, $A^{(t,x)}$ is a positive operator in the Banach space $E_{\alpha,1}\left(L_{1}\left(\mathbb{R}_{+}^{2}\right),A^{(t,x)}\right)$. Hence, for positive operator $A^{(t,x)}$ we have

$$f=\int_{0}^{\infty}A^{(t,x)}\left(A^{(t,x)}+\lambda I\right)^{-2}fd\lambda. \tag{19}$$

Then, using formula (6) and equation (19), we get

$$f\left(t,x\right)=\int_{0}^{\infty}\left(A^{(t,x)}+\lambda I\right)^{-1}A^{(t,x)}\left(A^{(t,x)}+\lambda I\right)^{-1}f\left(t,x\right)d\lambda$$

$$=\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}G\left(t,x,p,s;\lambda\right)A^{(t,x)}\left(A^{(t,x)}+\lambda I\right)^{-1}f\left(p,s\right)dsdpd\lambda. \tag{20}$$

Without loss of generality, we can put $\tau,h>0$. Using formula (20) and the triangle inequality yield

$$\frac{\left|f\left(t+\tau,x+h\right)-f\left(t,x\right)\right|}{\left(\tau^{2}+h^{2}\right)^{1+\alpha}}$$

$$\leq \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{|G(t+\tau,x+h,\lambda)-G(t,x,\lambda)|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right| dsdpd\lambda. \tag{21}$$

Thus, from inequality (21), we obtain

$$\int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\left| f(t+\tau,x+h)-f(t,x) \right|}{(\tau^2+h^2)^{1+\alpha}} dhd\tau dxdt$$

$$\leq \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\left| G(t+\tau,x+h,p,s;\lambda)-G(t,x,p,s;\lambda) \right|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right| dsdpd\lambda dhd\tau dxdt.$$

We will estimate two cases $\tau^2+h^2 \geq 1$ and $\tau^2+h^2 \leq 1$ separately. First, let us estimate $\tau^2+h^2 \geq 1$. Then

$$\int_0^\infty \int_{-\infty}^\infty \iint_{\tau^2+h^2\geq 1} \frac{\left| f(t+\tau,x+h)-f(t,x) \right|}{(\tau^2+h^2)^{1+\alpha}} dhd\tau dxdt$$

$$\leq \int_0^\infty \int_{-\infty}^\infty \iint_{\tau^2+h^2\geq 1} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\left| G(t+\tau,x+h,p,s;\lambda) \right| + G(t,x,p,s;\lambda)}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right| dsdpd\lambda dhd\tau dxdt$$

$$\leq M_1 \iint_{\tau^2+h^2\geq 1} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty e^{-a(\lambda+\delta)^{1/2}\left(\left|t+\tau-p\right|+|x+h-s|\right)}$$

$$\times \left(1+\ln\left(1+\frac{1}{\sqrt{\lambda+\delta}\left(\left|t+\tau-p\right|+|x+h-s|\right)} \right) \right) \frac{\lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} dxdtdsdpd\lambda dhd\tau$$

$$+M_1 \iint_{\tau^2+h^2\geq 1} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty e^{-a(\lambda+\delta)^{1/2}\left(\left|t-p\right|+|x-s|\right)}$$

$$\times \left(1+\ln\left(1+\frac{1}{\sqrt{\lambda+\delta}\left(\left|t-p\right|+|x-s|\right)} \right) \right) \frac{\lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} dxdtdsdpd\lambda dhd\tau$$

$$\leq M[I_1+I_2].$$

Now, we will estimate $I_i,\ i=1,2$, separately. Let us estimate I_1. By estimate (12), we obtain

$$I_1 \leq M_2 \iint_{\tau^2+h^2\geq 1} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{\lambda^\alpha (\lambda+\delta)(\tau^2+h^2)^{1+\alpha}} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right| dxdtdsdpd\lambda dhd\tau$$

$$\leq M_2 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \left(\iint_{\tau^2+h^2\geq 1} \frac{1}{(\tau^2+h^2)^{1+\alpha}} dhd\tau \right) \frac{1}{\lambda^\alpha (\lambda+\delta)} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)}+\lambda I \right)^{-1} f(p,s) \right| dsdpd\lambda.$$

Using the change of variables $\tau = r\sin\theta,\ h = r\cos\theta,\ r^2 \geq 1,\ 0 < \theta < 2\pi$, we get

$$I_1 \leq M_3 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \left(\int_0^{2\pi} \int_1^\infty \frac{r}{r^{2(1+\alpha)}} dr d\theta \right) \frac{1}{\lambda^\alpha(\lambda+\delta)} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)} + \lambda I \right)^{-1} f(p,s) \right| dsdpd\lambda$$

$$\leq M_4 \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{2\alpha} \frac{\lambda}{\lambda^\alpha(\lambda+\delta)} \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)} + \lambda I \right)^{-1} f(p,s) \right| dsdp \frac{d\lambda}{\lambda}$$

$$\leq \sup_{\lambda>0} \frac{\lambda^{1-\alpha}}{\lambda+\delta} \frac{M_5}{2\alpha} \int_0^\infty \lambda^\alpha \left\| A^{(t,x)} \left(A^{(t,x)} + \lambda I \right)^{-1} f \right\|_{E_{\alpha,1}} \frac{d\lambda}{\lambda}$$

$$\leq \frac{M}{2\alpha} \|f\|_{E_{\alpha,1}}. \tag{22}$$

Similarly,

$$I_2 \leq \frac{M}{2\alpha} \|f\|_{E_{\alpha,1}}. \tag{23}$$

From estimates (22) and (23), we obtain

$$\int_0^\infty \int_{-\infty}^\infty \iint_{\tau^2+h^2\geq 1} \frac{\left| f(t+\tau, x+h) - f(t,x) \right|}{(\tau^2+h^2)^{1+\alpha}} dhd\tau dxdt \leq \frac{M}{2\alpha} \|f\|_{E_{\alpha,1}}. \tag{24}$$

Second, let estimate $\tau^2 + h^2 \leq 1$. By the triangle inequality, we have

$$\int_0^\infty \int_{-\infty}^\infty \iint_{\tau^2+h^2\leq 1} \frac{\left| f(t+\tau, x+h) - f(t,x) \right|}{(\tau^2+h^2)^{1+\alpha}} dhd\tau dxdt$$

$$\leq \iint_{\tau^2+h^2\leq 1} \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{\left| G(t+\tau, x+h, p, s; \lambda) - G(t, x, p, s; \lambda) \right|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}}$$

$$\times \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)} + \lambda \right)^{-1} f(p,s) \right| dsdpd\lambda dhd\tau dxdt$$

$$\leq \iint_{\tau^2+h^2\leq 1} \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \left[\int_0^\tau \frac{\left| G_t(t+q, x+h, p, s; \lambda) \right|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} dq + \int_0^h \frac{\left| G_x(t, x+\mu, p, s; \lambda) \right|}{\lambda^\alpha (\tau^2+h^2)^{1+\alpha}} d\mu \right]$$

$$\times \lambda^\alpha \left| A^{(t,x)} \left(A^{(t,x)} + \lambda \right)^{-1} f(p,s) \right| dsdpd\lambda dxdtdhd\tau.$$

Using the triangle inequality, estimates (10), the following estimate

$$u^\theta e^{-u} \leq 1,\ u \geq 0,\ \theta \in [0,1], \tag{25}$$

the Lagrange theorem, Hilbert's inequality, we obtain

$$\int_0^\infty \int_{-\infty}^\infty \left| G_t(t+q, x+h, p, s; \lambda) \right| dxdt \leq \int_{-\infty}^\infty \int_{-\infty}^\infty \left| G_t(t+q, x+h, p, s; \lambda) \right| dxdt$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-a(\lambda+\delta)^{1/2}\left(\left|t-p\right|+\left|x-s\right|\right)}}{\sqrt{\lambda+\delta}\left(\left|t+q-p\right|+\left|x+h-s\right|\right)}dxdt \leq \frac{M}{\lambda+\delta}.$$

Similarly, we get

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} \left|G_x\left(t,x+\mu,p,s;\lambda\right)\right| dxdt \leq \frac{M}{\lambda+\delta}.$$

Then

$$\int_{0}^{\infty}\int_{-\infty}^{\infty}\iint_{\tau^2+h^2\leq 1} \frac{\left|f\left(t+\tau,x+h\right)-f\left(t,x\right)\right|}{\left(\tau^2+h^2\right)^{1+\alpha}}dhd\tau dxdt$$

$$\leq \iint_{\tau^2+h^2\leq 1}\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\left[\int_{0}^{\tau}\frac{dq}{\left(\tau^2+h^2\right)^{1+\alpha}}+\int_{0}^{h}\frac{d\mu}{\left(\tau^2+h^2\right)^{1+\alpha}}\right]\left|A^{(t,x)}\left(A^{(t,x)}+\lambda\right)^{-1}f\left(p,s\right)\right| dsdpd\lambda dhd\tau$$

$$\leq \iint_{\tau^2+h^2\leq 1}\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\tau}{\left(\tau^2+h^2\right)^{1+\alpha}}+\frac{h}{\left(\tau^2+h^2\right)^{1+\alpha}}\right]\lambda\left|A^{(t,x)}\left(A^{(t,x)}+\lambda\right)^{-1}f\left(p,s\right)\right| dsdpd\lambda dhd\tau.$$

Applying the change of variables $\tau = r\sin\theta,\ h = r\cos\theta,\ r^2 \geq 1,\ 0 < \theta < 2\pi$, we have

$$\int_{0}^{\infty}\int_{-\infty}^{\infty}\iint_{\tau^2+h^2\leq 1} \frac{\left|f\left(t+\tau,x+h\right)-f\left(t,x\right)\right|}{\left(\tau^2+h^2\right)^{1+\alpha}}dhd\tau dxdt$$

$$\leq M\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\left(\int_{0}^{2\pi}\int_{0}^{1}\frac{r^2\left(\sin\theta+\cos\theta\right)}{r^{2(1+\alpha)}}drd\theta\right)\frac{1}{\left(\lambda+\delta\right)}\left|A^{(t,x)}\left(A^{(t,x)}+\lambda\right)^{-1}f\left(p,s\right)\right| dsdpd\lambda$$

$$\leq M_1\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\left(1-2\alpha\right)}\frac{\lambda}{\lambda^{\alpha}\left(\lambda+\delta\right)}\lambda^{\alpha}\left|A^{(t,x)}\left(A^{(t,x)}+\lambda\right)^{-1}f\left(p,s\right)\right| dsdp\frac{d\lambda}{\lambda}$$

$$\leq \sup_{\lambda>0}\frac{\lambda^{1-\alpha}}{\lambda+\delta}\frac{M_2}{\left(1-2\alpha\right)}\int_{0}^{\infty}\lambda^{\alpha}\left\|A^{(t,x)}\left(A^{(t,x)}+\lambda\right)^{-1}f\right\|_{E_{\alpha,1}}\frac{d\lambda}{\lambda}$$

$$\leq \frac{M_3}{\left(1-2\alpha\right)}\left\|f\right\|_{E_{\alpha,1}}. \tag{26}$$

Combining estimates (24) and (26), we get

$$\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{\left|f\left(t+\tau,x+h\right)-f\left(t,x\right)\right|}{\left(\tau^2+h^2\right)^{1+\alpha}}dhd\tau dxdt \leq \frac{M}{2\alpha\left(1-2\alpha\right)}\left\|f\right\|_{E_{\alpha,1}}. \tag{27}$$

From estimate (27), we obtain

$$E_{\alpha,1}\left(L_1\left(\mathbb{R}_+^2\right),A^{(t,x)}\right)\subset W_1^{2\alpha}(\mathbb{R}_+^2).$$

This finishes the proof of Theorem 3.1. □

From the positivity of an elliptic operator $A^{(t,x)}$ in the Banach space $L_1(\mathbb{R}_+^2)$ and estimate (1) it follows the positivity of this operator in Banach spaces $W_1^{2\alpha}(\mathbb{R}_+^2)$.

4. Applications

In this section, we consider some applications of Theorem 3.1. First, we will consider the boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(y,x_1,x_2)}{\partial y^2} - a_{11}(x_1,x_2)\frac{\partial^2 u(y,x_1,x_2)}{\partial x_1^2} - a_{22}(x_1,x_2)\frac{\partial^2 u(y,x_1,x_2)}{\partial x_2^2} + \sigma u(y,x_1,x_2) \\ = f(y,x_1,x_2),\ 0 < y < T,\ x_1 \in \mathbb{R}^+,\ x_2 \in \mathbb{R}, \\ u(0,x_1,x_2) = \varphi(x_1,x_2),\ u(T,x_1,x_2) = \psi(x_1,x_2),\ x_1 \in \mathbb{R}^+,\ x_2 \in \mathbb{R}, \\ u(y,0,x_2) = 0,\ 0 \le y \le T,\ x_2 \in \mathbb{R}. \end{cases} \tag{28}$$

Here, $a_{11}(x_1,x_2)$, $a_{22}(x_1,x_2)$, $\varphi(x_1,x_2)$, $\psi(x_1,x_2)$, and $f(y,x_1,x_2)$ are sufficiently smooth functions and they satisfy every compatibility conditions which guarantee problem (28) has a smooth solution $u(y,x_1,x_2)$. Assume that the assumption of the uniform ellipticity holds.

Theorem 4.1. *Let* $0 < 2\alpha < 1$. *Then, for the solution of boundary value problem (28), we have the following coercive stability inequality*

$$\begin{aligned} &\|u_{yy}\|_{L_1(W_1^{2\alpha}(\mathbb{R}_+^2))} + \|u_{x_1x_1}\|_{L_1(W_1^{2\alpha}(\mathbb{R}_+^2))} + \|u_{x_2x_2}\|_{L_1(W_1^{2\alpha}(\mathbb{R}_+^2))} \\ \le\ & M(\alpha)\Big[\|\varphi_{x_1x_1}\|_{W_1^{2\alpha}(\mathbb{R}_+^2)} + \|\varphi_{x_2x_2}\|_{W_1^{2\alpha}(\mathbb{R}_+^2)} + \|\psi_{x_1x_1}\|_{W_1^{2\alpha}(\mathbb{R}_+^2)} \\ &+\|\psi_{x_2x_2}\|_{W_1^{2\alpha}(\mathbb{R}_+^2)} + \|f\|_{L_1(W_1^{2\alpha}(\mathbb{R}_+^2))}\Big], \end{aligned}$$

where $M(\alpha)$ *is independent of* φ, ψ *and* f.

The proof of Theorem 4.1 is based on Theorem 3.1 on the structure of the fractional spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}_+^2\right), A^{(t,x)}\right)$, Theorem 2.1 on the positivity of the operator $A^{(t,x)}$, on the following theorems on coercive stability of boundary value for the abstract elliptic equation and on the structure of the fractional space $E'_{\alpha,1} = E_{\alpha,1}(E, A^{1/2})$ which is the Banach space consists of those $v \in E$ for which the norm

$$\|v\|_{E'_{\alpha,1}} = \int_0^\infty \lambda^\alpha \left\|A^{1/2}\left(\lambda + A^{1/2}\right)^{-1} v\right\|_E \frac{d\lambda}{\lambda}$$

is finite.

Theorem 4.2. ([21]) *The spaces* $E_{\alpha,1}(E,A)$ *and* $E'_{2\alpha,1}(A^{1/2},E)$ *coincide for any* $0 < \alpha < \frac{1}{2}$, *and their norms are equivalent.*

Theorem 4.3. ([3]) *Let A be positive operator in a Banach space E and* $f \in L_1([0,T], E'_{\alpha,1})$ $(0 < \alpha < 1)$. *Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases} -u''(t) + Au(t) = f(t),\ 0 < t < T, \\ u(0) = \varphi,\ u(T) = \psi \end{cases} \tag{29}$$

in a Banach space E with positive operator A the coercive inequality

$$\|u''\|_{L_1([0,T],E'_{\alpha,1})} + \|Au\|_{L_1([0,T],E'_{\alpha,1})}$$

$$\leq M\left[\|A\varphi\|_{E'_{\alpha,1}} + \|A\psi\|_{E'_{\alpha,1}} + \frac{M}{\alpha(1-\alpha)}\|f\|_{L_1([0,T],E'_{\alpha,1})}\right]$$

holds.

Second, we will consider the nonlocal-boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(y,x_1,x_2)}{\partial y^2} - a_{11}(x_1,x_2)\frac{\partial^2 u(y,x_1,x_2)}{\partial x_1^2} - a_{22}(x_1,x_2)\frac{\partial^2 u(y,x_1,x_2)}{\partial x_2^2} + \sigma u(y,x_1,x_2) \\ = f(y,x_1,x_2),\ 0 < y < T,\ x_1 \in \mathbb{R}^+,\ x_2 \in \mathbb{R}, \\ u(0,x_1,x_2) = u(T,x_1,x_2),\ u_y(0,x_1,x_2) = u_y(T,x_1,x_2),\ x_1 \in \mathbb{R}^+,\ x_2 \in \mathbb{R}, \\ u(y,0,x_2) = 0,\ 0 \leq y \leq T,\ x_2 \in \mathbb{R}. \end{cases} \tag{30}$$

Here, $a_{11}(x_1,x_2)$, $a_{22}(x_1,x_2)$, and $f(y,x_1,x_2)$ are sufficiently smooth functions and they satisfy every compatibility conditions which guarantee problem (30) has a smooth solution $u(y,x_1,x_2)$. Assume that the assumption of the uniform ellipticity holds.

Theorem 4.4. *Let $0 < 2\alpha < 1$. Then, for the solution of boundary value problem (30), we have the following coercive stability inequality*

$$\|u_{yy}\|_{L_1(W_1^{2\alpha}(\mathbb{R}^2_+))} + \|u_{x_1x_1}\|_{L_1(W_1^{2\alpha}(\mathbb{R}^2_+))} + \|u_{x_2x_2}\|_{L_1(W_1^{2\alpha}(\mathbb{R}^2_+))}$$

$$\leq \quad M(\alpha)\|f\|_{L_1(W_1^{2\alpha}(\mathbb{R}^2_+))}$$

where $M(\alpha)$ is independent of f.

The proof of Theorem 4.4 is based on Theorem 3.1 on the structure of the fractional spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_+\right), A^{(t,x)}\right)$, Theorem 2.1 on the positivity of the operator $A^{(t,x)}$, Theorem 5 on the structure of the fractional space $E'_{\alpha,1} = E_{\alpha,1}(E, A^{1/2})$ and on the following theorem on coercive stability of nonlocal boundary value for the abstract elliptic equation.

Theorem 4.5. ([27]) *Let A be positive operator in a Banach space E and $f \in C([0,T], E'_{\alpha,1})$ $(0 < \alpha < 1)$. Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases} -u''(t) + Au(t) = f(t),\ 0 < t < T, \\ u(0) = u(T),\ u'(0) = u'(T) \end{cases} \tag{31}$$

in a Banach space E with positive operator A the coercive inequality

$$\|u''\|_{L_1([0,T],E'_{\alpha,1})} + \|Au\|_{L_1([0,T],E'_{\alpha,1})} \leq \frac{M}{\alpha(1-\alpha)}\|f\|_{L_1([0,T],E'_{\alpha,1})}$$

holds.

5. Conclusion

In the present article, the structure of the fractional spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_+\right), A^{(t,x)}\right)$generated by the two-dimensional elliptic differential operator $A^{(t,x)}$ is investigated. The positivity of this operator $A^{(t,x)}$ in Banach spaces is established. Of course, the difference operator A_h^x approximates of the operator $A^{(t,x)}$ can be presented. The positivity of this operator A_h^x in Banach spaces can be established.

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