On the $g$-Hypergroupoids Associated with $g$-Hypergraphs

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Abstract. In this paper, we associate a partial $g$-hypergroupoid with a given $g$-hypergraph and analyze the properties of this hyperstructure. We prove that a $g$-hypergroupoid may be a commutative hypergroup without being a join space. Next, we define diagonal direct product of $g$-hypergroupoids. Further, we construct a sequence of $g$-hypergroupoids and investigate some relationships between it’s terms. Also, we study the quotient of a $g$-hypergroupoid by defining a regular relation. Finally, we describe fundamental relation of an $H_v$-semigroup as a $g$-hypergroupoid.

1. Introduction and Preliminaries

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, let $P(X)$ be the set of all subsets of a given set $X$. A partial hypergroupoid is a pair $(X,\ast)$, where $X$ is a non-empty set and $\ast$ is a partial hyperoperation, i.e., $\ast: X \times X \rightarrow P(X), \ (x, y) \mapsto x \ast y$.

Every map from $X \times X$ to $P'(X)$ is called a hyperoperation, where $P'(X) = P(X) - \{\emptyset\}$. If $A, B \in P'(X)$, then we define $A \ast B = \bigcup \{a \ast b \mid a \in A, b \in B\}, x \ast B = \{x\} \ast B$ and $A \ast y = A \ast \{y\}$. If $A = \emptyset$ or $B = \emptyset$ we define $A \ast B = \emptyset$.

A partial hypergroupoid $(X,\ast)$ is called a hypergroupoid if $\ast$ is a hyperoperation. A hypergroupoid $(X,\ast)$ is called a semihypergroup if the associative axiom is valid, i.e., $x \ast (y \ast z) = (x \ast y) \ast z$, for all $x, y, z \in X$ and it is called reproductive if $x \ast X = X \ast x = X$, for all $x \in X$. A hypergroup is a reproductive semihypergroup.

A commutative hypergroupoid $(X,\ast)$ (i.e., $x \ast y = y \ast x$ for all $x, y \in X$) is called a join space if the following implication holds for all elements $a, b, c, d$ of $X$:

$$a/b \cap c/d \neq \emptyset \Rightarrow a \ast d \cap b \ast c \neq \emptyset,$$

where $a/b = \{x \mid a \in x \ast b\}$.

$H_v$-structures which satisfy the corresponding structure-like axioms are the largest class of algebraic hyperstructures. The notion of $H_v$-structures has been introduced by Vougiouklis [13] as a generalization...
of well-known algebraic hyperstructures (semihypergroups, hypergroups, hyperrings and so on) which satisfy the weak axioms where the non-empty intersection replaces the equality. A comprehensive review of the theory of $H_v$-structures appears in [1, 5, 6]. A hypergroupoid $(X, \ast)$ is called an $H_v$-semigroup if the weak associative axiom is valid, i.e.,

$$x \ast (y \ast z) \cap (x \ast y) \ast z \neq \emptyset, \quad \text{for all } x, y, z \in X$$

and it is called an $H_v$-group if it is a reproductive $H_v$-semigroup.

Let $X$ be a non-empty set. By an $h$-relation $R$ on $X$ we mean a subset of $X \times P^*(X)$. The domain of $R$ is the set $\text{Dom}(R) = \{x \in X \mid (x, A) \in R \text{ for some } A \in P(X)\}$ and codomain of $R$ is the set $\text{Cod}(R) = \{A \in P(X) \mid (x, A) \in R \text{ for some } x \in X\}$. Also, for any $x \in X$, we define $x_R = \{A \mid (x, A) \in R\}$.

The notion of hypergraph has been introduced around 1960 as a generalization of graph and one of the initial concerns was to extend some classical results of graph theory. In [2], there is a very good presentation of graph and hypergraph theory. Connections between hypergraphs and hyperstructures are studied by many authors, for example, see [4, 8, 10, 11]. A hypergraph is a pair $\Gamma = (X, A)$, where $X$ is a finite set of vertices and $A = \{A_1, \ldots, A_m\}$ is a set of hyperedges which are non-empty subsets of $X$. Figure 1 is an example of a hypergraph with 2 hyperedges $A_1 = \{1, 2, 3\}$ and $A_2 = \{2, 3, 4\}$.

![Figure 1](image1.png)

**Figure 1:** An example of hypergraph with 2 hyperedges.

2. Partial $g$-Hypergroupoids

In this section we generalize the notion of hypergraphs to generalized hypergraphs and then we associate a partial hypergroupoid to each generalized hypergraph.

**Definition 2.1.** [12] A generalized hypergraph or, in short, a $g$-hypergraph is an ordered pair $\mathcal{G} = (X, R)$, where $X$ is a non-empty set and $R$ is an $h$-relation on $X$. The elements of $X$ are called the vertices and the sets in $\mathcal{E} = \text{Cod}(R)$ are called the hyperedges of the $g$-hypergraph.

It is worth mentioning that in this paper we deal only with $g$-hypergraphs $\mathcal{G} = (X, R)$ in which $X$ is a finite set. A $g$-hypergraph $\mathcal{G} = (X, R)$ is called $v$-linked if $x_R \neq \emptyset$, for all $x \in X$ and it is called plenary if $\bigcup_{A \in \text{Cod}(R)} A = X$.

![Figure 2](image2.png)

**Figure 2:** An example of a $g$-hypergraph.
Let $G = (X, R)$ be a g-hypergraph. The partial hypergroupoid $X_G = (X, \circ)$ where the partial hyperoperation $\circ$ is defined by $x \circ y = N(x) \cup N(y)$, for all $(x, y) \in X^2$, is called the partial g-hypergroupoid associated with $G$, where $N(x) = \bigcup_{(x, A) \in R} A$. In the case that $\circ$ is a hyperoperation, $X_G$ is called a g-hypergroupoid.

Lemma 2.2. $X_G$ is a g-hypergroupoid if and only if $G$ is v-linked.

Proof. It is obvious. $\square$

Remark 2.3. In [4], Corsini associated to a given hypergraph $\Gamma = (H, \{A_i\})$ an h.g. hypergroupoid $H_\Gamma = (H, \circ)$ where the hyperoperation $\circ$ has defined as follows:

\[ x \circ y = E(x) \cup E(y), \text{ for all } x, y \in H^2, \]

where $E(x) = \bigcup_{x \in A_i} A_i$. Let $\Gamma = (H, \{A_i\})$ be a hypergraph. If we define the h-relation $R = \{(x, A_i) | x \in A_i\}$ on $H$, then $\Gamma$ becomes a v-linked and plenary g-hypergraph. Thus, every hypergraph can be considered as a g-hypergraph and there is no difference between h.g. hypergroupoids and g-hypergroupoids when we deal with hypergraphs. In other words, each h.g. hypergroupoid can be considered as a g-hypergroupoid. As we will see, h.g. hypergroupoids does not coincide with g-hypergroupoids. For example, by Theorem 3 of [4], each h.g. hypergroupoid is a join space whereas there are g-hypergroupoids which are not join spaces (see Example 2.4).

Example 2.4. Consider the following g-hypergraph and the table of it’s associated g-hypergroupoid:

\[
\begin{array}{c|ccc}
\circ & 1 & 2 & 3 \\
\hline
1 & \{2\} & \{1, 2\} & \{1, 2, 3\} \\
2 & \{1, 2\} & \{1, 2\} & \{1, 2, 3\} \\
3 & \{1, 2, 3\} & \{1, 2, 3\} & \{1, 3\} \\
\end{array}
\]

It is not difficult to see that $(X = \{1, 2, 3\}, \circ)$ is a hypergroup. We have $1/3 = \{1, 2, 3\}$ and $3/1 = \{3\}$. It implies that $1/3 \cap 3/1 \neq \emptyset$, but $1 \circ 1 \cap 3 \circ 3 = \emptyset$. Hence $(X, \circ)$ is not a join space.

Here, we give an example of a g-hypergraph such that it’s associated g-hypergroup is a join space.

Example 2.5. In the following, we have drawing a g-hypergraph $G$ and the table of the g-hypergroupoid associated with $G$: 

[Diagram of g-hypergraph]
Moreover, let the hypergroupoid associated with the v-linked g-hypergraph \( G = (X, R) \) be considered as a g-hypergroupoid. This implies that \((X, \circ)\) is a join space.

**Definition 2.6.** A partial hypergroupoid \((X, \circ)\) is called separable if the following property holds:

\[
x \circ y = x \circ x \cup y \circ y, \text{ for all } x, y \in X.
\]

**Remark 2.7.** Let \((X, \circ)\) be a separable hypergroupoid. Define \( R = \{(x, x \circ x) \mid x \in X\} \). Then, \((X, \circ)\) is the g-hypergroupoid associated with the v-linked g-hypergraph \( G = (X, R) \). Therefore, every separable hypergroupoid can be considered as a g-hypergroupoid.

The next lemma can be proved easily by using the previous notions.

**Lemma 2.8.** Let \((X, \circ)\) be a partial g-hypergroupoid. Then, for all \( x, y \in X \) and \( A \subseteq X \) we have

1. \( x \circ y = x \circ x \cup y \circ y \),
2. \( (x \circ x) \circ (x \circ x) = \bigcup_{t \in x \circ x} t \circ t \),
3. \( (A \circ A) \circ (A \circ A) = \bigcup_{t \in A \circ A} t \circ t \).

**Lemma 2.9.** Let \((X, \circ)\) be a separable hypergroupoid. Then

1. for each \( x, y, z \in X \) we have
   \[
   (x \circ y) \circ z = (x \circ x) \circ (x \circ y) \circ z \cup (y \circ y) \circ (y \circ y) \circ z.
   \]
   \[
   x \circ (y \circ z) = (y \circ y) \circ (y \circ y) \cup x \circ x \cup (z \circ z) \circ (z \circ z).
   \]
2. \((X, \circ)\) is an \( H_e \)-semigroup.

**Proof.** (1) For each \( x, y, z \in X \) we have

\[
(x \circ y) \circ z = (x \circ x) \cup y \circ y \circ z = (x \circ x) \circ z \cup (y \circ y) \circ z,
\]

and

\[
x \circ (y \circ z) = (y \circ y) \circ (y \circ y) \cup x \circ x \cup (z \circ z) \circ x.
\]

Moreover,

\[
(x \circ x) \circ z = \bigcup_{t \in x \circ x} t \circ z = \bigcup_{t \in x \circ x} t \circ t \cup z \circ z = [(x \circ x) \circ (x \circ x)] \cup z \circ z.
\]

Therefore, we have

\[
(x \circ y) \circ z = [(x \circ x) \circ (x \circ x)] \cup z \circ z \cup [(y \circ y) \circ (y \circ y)]
\]

and

\[
x \circ (y \circ z) = [(y \circ y) \circ (y \circ y)] \cup x \circ x \cup [(z \circ z) \circ (z \circ z)].
\]

(2) We have \( \emptyset \neq [(y \circ y) \circ (y \circ y)] \subset (x \circ y) \circ z \cap x \circ (y \circ z) \). This completes the proof. \( \Box \)
Notice that every partial g-hypergroupoid is separable and so we have the following corollary.

**Corollary 2.10.** Every g-hypergroupoid is an $H_v$-semigroup.

**Corollary 2.11.** A partial g-hypergroupoid $X_G$ is an $H_v$-semigroup if and only if $G$ is $v$-linked.

**Theorem 2.12.** Let $G = (X, R)$ be a $v$-linked g-hypergraph. Then, the g-hypergroupoid $X_G = (X, o)$ is an $H_v$-group if and only if $G$ is plenary.

**Proof.** Suppose that $X_G = (X, o)$ is an $H_v$-group. It suffices to show that $X \subseteq \bigcup_{A \in \text{Cod}(R)} A$. Let $x \in X$ be an arbitrary element. By assumption, we have $x \circ X = X$ and so there is $y \in X$ such that $x \circ y = N(x) \cup N(y)$. Thus, there is $A \in \text{Cod}(R)$ such that $x \in A \subseteq \bigcup_{A \in \text{Cod}(R)} A$.

Conversely, let $G$ be plenary and $x \in X$ be an arbitrary element. By Corollary 2.10, it is sufficient to show that $x \circ X = X \circ x = X$. It is obvious that $x \circ X \subseteq X$. We show that $X \subseteq x \circ X$. Since $G$ is plenary, if $z \in X$ is an arbitrary element, then there is $A \in \text{Cod}(R)$ such that $z \in A$. Since $A \in \text{Cod}(R)$, it follows that there is $y \in X$ such that $(y, A) \in R$ and so we have $z \in x \circ y \subseteq x \circ X$. This implies that $X \subseteq x \circ X$ and so $x \circ X = X$. Clearly $X \circ x = X$ since $\circ$ is commutative. Therefore, $X_G$ is an $H_v$-group. □

**Corollary 2.13.** $X_G$ is a reproductive g-hypergroupoid if and only if $G$ is $v$-linked and plenary.

**Theorem 2.14.** Let $(X, o)$ be a separable hypergroupoid. Then, $\circ$ is associative if and only if the following conditions hold:

1. $x \circ x \subseteq (x \circ x) \circ (x \circ x)$, for all $x \in X$,
2. $[(x \circ x) \circ (x \circ x)] - x \circ x \subseteq (y \circ y) \circ (y \circ y)$, for all $x, y \in X$.

**Proof.** Suppose that $\circ$ is associative and $x, y$ are arbitrary elements of $X$. First, we show that $x \circ x \subseteq (x \circ x) \circ (x \circ x)$. Suppose that $x \circ x = \{x_1, \ldots, x_n\}$ and $x_i \in x \circ x$ is an arbitrary element. Since $x \circ x$ is an arbitrary element, then $x_i \circ x_i \subseteq (x \circ x) \circ (x \circ x)$, it follows that $x_i \in x \circ x$, and so $x_i \in x \circ x$. Associativity of $\circ$ implies that

$$x_i \in (x \circ x) \circ x_i = x_1 \circ x_1 \cup \ldots \cup x_n \circ x_n = (x \circ x) \circ (x \circ x).$$

Thus (1) holds. Now, to prove the condition (2) we have

$$(y \circ y) \circ x = \bigcup_{t \in x} t \circ x = \bigcup_{t \in y} t \circ y \cup x \circ x = (y \circ y) \circ (y \circ y) \cup x \circ x,$$

and

$$x \circ (y \circ z) = \bigcup_{t \in z} x \circ t \circ z = \bigcup_{t \in x} (y \circ y) \circ (y \circ y) \cup x \circ n \circ (z \circ z).$$

Consequently, (2) holds.

Conversely, suppose that $x, y, z$ are arbitrary elements of $X$ and the conditions (1) and (2) hold. From point (1) of Lemma 2.9, we have

$$(x \circ y) \circ z = [(x \circ x) \circ (x \circ x)] \cup z \circ z \cup [(y \circ y) \circ (y \circ y)].$$

and

$$x \circ (y \circ z) = [(y \circ y) \circ (y \circ y)] \cup x \circ x \cup [(z \circ z) \circ (z \circ z)].$$

By setting $A = [(x \circ x) \circ (x \circ x)] \cup z \circ z$ and $B = [(z \circ z) \circ (z \circ z)] \cup x \circ x$ we have $(x \circ y) \circ z = [(y \circ y) \circ (y \circ y)] \cup A$ and $x \circ (y \circ z) = [(y \circ y) \circ (y \circ y)] \cup B$. By using the conditions (1) and (2) we have

$$A = [(x \circ x) \circ (x \circ x)] - x \circ x \cup x \circ x \cup z \circ z$$

and

$$B = [(z \circ z) \circ (z \circ z)] \cup z \circ z \cup x \circ x = B.$$
Theorem 2.15. Let \((X, \circ)\) be a separable hypergroupoid. Then, \(\circ\) is associative if and only if the following conditions hold:

1. \(A \circ A \subseteq (A \circ A) \circ (A \circ A), \) for all \(A \subseteq X,\)
2. \([(A \circ A) \circ (A \circ A)] - A \circ A \subseteq (B \circ B) \circ (B \circ B), \) for all \(A, B \subseteq X.\)

Proof. Suppose that \(\circ\) is associative and \(A, B\) are arbitrary subsets of \(X.\) Then, by using Theorem 2.14 we have

\[A \circ A = \bigcup_{a \in A} a \circ a \subseteq \bigcup_{a \in A} (a \circ a) \circ (a \circ a) = \bigcup_{a \in A} \left( \bigcup_{b \in A} b \circ a \right) = \bigcup_{b \in A} b \circ a = (A \circ A) \circ (A \circ A).\]

Hence, (1) is true. For every \(b \in B\) we have

\[[(A \circ A) \circ (A \circ A)] - A \circ A \subseteq \bigcup_{a \in A} [(a \circ a) \circ (a \circ a)] - a \circ a \subseteq (b \circ b) \circ (b \circ b).\]

On the other hand, we have \((b \circ b) \circ (b \circ b) \subseteq (B \circ B) \circ (B \circ B).\) Hence, the assertion (2) holds too.

Conversely, suppose that the assertions (1) and (2) hold for all subsets \(A\) and \(B\) of \(X.\) Let \(x, y\) be arbitrary elements of \(X.\) By setting \(A = \{x\}\) and \(B = \{y\},\) the assertions (1) and (2) of Theorem 2.14 hold and therefore \(\circ\) is associative.

\[\square\]

Corollary 2.16. If a reproductive g-hypergroupoid \(X_G = (X, \circ)\) satisfies anyone of the following conditions:

\[(x \circ x) \circ (x \circ x) = x \circ x, \text{ for all } x \in X,\]

\[(x \circ x) \circ (x \circ x) = X, \text{ for all } x \in X,\]

then it is a hypergroup.

Example 2.17. The g-hypergroupoid associated with the g-hypergraph of Figure 2 has the following table:

<table>
<thead>
<tr>
<th>(\circ)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 2)</td>
<td>(X)</td>
<td>(1, 2, 3)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(X)</td>
<td>(X)</td>
<td>(X)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(1, 2)</td>
<td>(X)</td>
<td>(1, 2, 3)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(1, 2, 3)</td>
<td>(X)</td>
<td>(1, 2, 3)</td>
<td>(1, 2, 3)</td>
</tr>
</tbody>
</table>

where \(X = \{1, 2, 3, 4\}.\) It is easy to verify that \((x \circ x) \circ (x \circ x) = X, \) for all \(x \in X.\) On the other hand, for every \(x \in X\) we have \(x \circ X = X \circ x = X.\) So, by Corollary 2.16, \((X, \circ)\) is a hypergroup.

Example 2.18. Consider the g-hypergroupoid of Example 2.4. By Theorem 2.14, \((X, \circ)\) is a hypergroup. Also, we have \((1 \circ 1) \circ (1 \circ 1) = \{1, 2\}.\) This shows that the converse of Corollary 2.16 is not true.

3. Higher-order Hypergroupoids

Let \((X, \circ)\) be a separable hypergroupoid. We construct a sequence of hypergroupoids \(X_0 = (X, \circ_0), X_1 = (X, \circ_1), X_2 = (X, \circ_2), \ldots\) recursively as follows: for all \(x, y \in X\) we define \(x \circ_0 y = x \circ y, x \circ_{k+1} x = (x \circ_k x) \circ (x \circ_k x)\) and \(x \circ_{k+1} y = x \circ_{k+1} x \cup y \circ_{k+1} y, \) where \(k \geq 0.\) Set \(N_k(x) = x \circ_k x.\) We define \(N_k(A) = \bigcup_{a \in A} N_k(a),\) where \(A\) is a subset of \(X.\) The following properties are immediate:

1. \(N_k(A) = A \circ_k A, \text{ for all } A \subseteq X,\)
2. \(N_{k+1}(x) = N_k(N_k(x)), \text{ for all } x \in X \text{ and } k \geq 0,\)
Corollary 3.3. Theorem 3.2. By Theorem 2.14, it follows that there is a separable semihypergroup if and only if the following conditions hold:

(a) \( N_k(x) \subseteq N_{k+1}(x) \), for all \( x \in X \),

(b) \( N_{k+1}(x) - N_k(x) \subseteq N_{k+1}(y) \), for all \( x, y \in X \).

**Lemma 3.1.** The above hyperoperation \( o_k \) has the following properties:

(1) \( A o_k A = (A o_k A) o_k (A o_k A) \), for all \( A \subseteq X \),

(2) \( x o_{k+2} x = ((x o_{k+1} x) o_k (x o_{k+1} x)) o_k ((x o_{k+1} x) o_k (x o_{k+1} x)) \), for all \( x \in X \).

**Proof.** (1) Let \( A \) be a subset of \( X \). Then,

\[
A o_k A = N_{k+1}(A) = N_k(N(A)) = N_k(A) o_k N_k(A) = (A o_k A) o_k (A o_k A).
\]

(2) The result follows from part (1) and the definition of \( o_{k+2} \). □

**Theorem 3.2.** Let \((X, o)\) be a separable hypergroupoid.

(1) If \( X_k = (X, o_k) \) satisfies condition (a) for some \( k \geq 0 \), then \( N_r(x) \subseteq N_{r+1}(x) \), for all \( x \in X \) and \( r \geq k \).

(2) If \( X_k = (X, o_k) \) satisfies condition (b) for some \( k \geq 0 \), then \( N_{r+1}(x) \subseteq N_r(x) \), for all \( x \in X \) and \( r \geq k \).

**Proof.** (1) Let \( x \in X \) be an arbitrary element. We prove the result by induction on \( r \). If \( r = k \), then there is nothing to prove. Assume that \( N_{r-1}(x) \subseteq N_r(x) \) for \( r > k \), the induction hypothesis. Thus we have

\[
N_r(x) = N_{r-1}(N_{r-1}(x)) \subseteq N_{r-1}(N_r(x)) = N_r(N_{r-1}(x)) \subseteq N_r(N_r(x)) = N_{r+1}(x).
\]

(2) Let \( x \in X \) be an arbitrary element. First, we show that \( N_k(N_{k+1}(x)) \subseteq N_{k+1}(x) \). Assume to the contrary that \( t \in N_k(N_{k+1}(x)) \) but \( t \notin N_{k+1}(x) \). Then, \( t \notin N_k(x) \) and there is \( a \in N_{k+1}(x) \) such that \( t \in N_k(a) \). Since \( a \in N_{k+1}(x) \), it follows that there is \( b \in N_k(x) \) such that \( a \in N_k(b) \) and so \( t \in N_k(N_k(b)) = N_{k+1}(b) \). On the other hand, \( t \notin N_{k+1}(x) \) implies that \( t \notin N_k(b) \) and so \( t \in N_{k+1}(b) - N_k(b) \). By hypothesis we have \( N_{k+1}(b) - N_k(b) \subseteq N_{k+1}(x) \) which implies that \( t \in N_{k+1}(x) \) contradicting to \( t \notin N_{k+1}(x) \). Now, we prove the result by induction on \( r \). We have \( N_{r+2}(x) = N_k(N_k(N_{k+1}(x))) \subseteq N_k(N_{k+1}(x)) \subseteq N_{k+1}(x) \). So, we are done with the initial step. Assume that \( N_{r+1}(x) \subseteq N_r(x) \) for \( r > k \), the induction hypothesis. We obtain

\[
N_{r+2}(x) = N_{r+1}(N_{r+1}(x)) \subseteq N_{r+1}(N_r(x)) = N_r(N_{r+1}(x)) \subseteq N_r(N_r(x)) = N_{r+1}(x).
\]

□

**Corollary 3.3.** If \((X, o_k)\) is a separable semihypergroup, then \( N_r(x) = N_{r+1}(x) \), for all \( x \in X \) and \( r > k \).

**Corollary 3.4.** If \((X, o_k)\) is a separable semihypergroup, then \( N_r(A) = N_{r+1}(A) \), for all \( A \subseteq X \) and \( r > k \).

Next example shows that the converse of Corollary 3.3 is not true.
Example 3.5. Let \((X = \{1, 2, 3\}, \circ)\) be a hypergroupoid with the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>X</td>
<td>[2]</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>{1,3}</td>
<td>X</td>
</tr>
</tbody>
</table>

We can see that \(N_0(1) = \{2\}\) and \(N_1(1) = \{1, 3\}\). Since \(N_0(x) \not\subset N_1(x)\) by Theorem 2.14, \((X, \circ)\) is not a semihypergroup. We can check that \(N_k(x) = N_{k+1}(x)\) for all \(x \in X\) and \(k > 0\). This means that the converse of Corollary 3.3 is not true.

Proposition 3.6. If there exists a natural number \(k\) such that \(N_k(x) = N_{k+1}(x)\), for all \(x \in X\), then

1. \(X_k = (X, \circ_k)\) is a semihypergroup,
2. \(X_r = X_k\) for all \(r \geq k\).

At the beginning of this section, we construct a sequence of separable hypergroupoids \(X_0 = (X, \circ_0), X_1 = (X, \circ_1), X_2 = (X, \circ_2), \ldots\) by a given separable hypergroupoid \((X, \circ)\). Let \(G\) be a v-linked g-hypergraph and \(X_G\) be the g-hypergroupoid associated with \(G\). Set \(X_0 = X_G\). For \(k > 0\), we define an h-relation \(R_k\) on \(X\) as follows:

\[
R_k = \{(x, x \circ_k x) \mid x \in X\}
\]

and therefore we have a sequence \(G_0, G_1, G_2, \ldots\) of g-hypergraphs where \(G_0 = G\) and \(G_k = (X, R_k)\), for \(k > 0\). It is easy to verify that \(X_k\) is the g-hypergroupoid associated with \(G_k\). Now, by Corollary 3.3 and Proposition 3.6 we conclude that if \(X_k\) is an associative g-hypergroupoid, then \(G_r = G_{k+1}\) and \(X_r = X_k\), for all \(r > k\). For a given g-hypergraph \(G\) we define

\[
n(G) = \min\{k \mid N_k(x) = N_{k+1}(x) \text{ for all } x \in X\}
\]

and

\[
s(G) = \min\{k \mid X_k \text{ is a semihypergroup}\}.
\]

Obviously, \(s(G) \leq n(G)\). Consider the g-hypergraph \(G\) of Figure 2. In Example 2.17 we showed that \(X_G\) is a hypergroup and so we have \(s(G) = 0\) whereas \(n(G) = 1\). This means that the inequality \(s(G) \leq n(G)\) may be hold strictly.

4. Quotient g-Hypergroupoids

In this section, by considering a regular equivalence relation on a g-hypergroupoid, we define a quotient g-hypergroupoid. Next, we investigate some relationships between diagonal direct product of hypergroupoids and direct product of g-hypergraphs. In this regards we recall some definitions and results which we need for the development of the rest of paper.

Let \((X, *)\) be a hypergroupoid and \(\rho\) be an equivalence relation on \(X\). If \(A\) and \(B\) are non-empty subsets of \(X\), then \(A \rho B\) means that for all \(a \in A\), there exists \(b \in B\) such that \(apb\) and for all \(b' \in B\), there exists \(a', a' \in A\) such that \(a' \rho b'\). We say that \(\rho\) is regular if for all \(a \in X\) from \(x \rho y\), it follows that \((a \ast x) \rho (a \ast y)\) and \((x \ast a) \rho (y \ast a)\). For an equivalence relation \(\rho\) on \(X\), we may use \(\rho(x)\) to denote the equivalence class of \(x \in X\). Moreover, generally, if \(A\) is a non-empty subset of \(X\), then \(\rho(A) = \bigcup\{\rho(x) \mid x \in A\}\). Let \(X/\rho\) be the family \(\{\rho(x) \mid x \in X\}\) of classes of \(\rho\). By Theorem 2.5.2 of [6], if \((X, *)\) is a hypergroupoid and \(\rho\) is a regular equivalence relation on \(X\), then the following hyperoperation on \(X/\rho\) is well defined:

\[
\rho(x) \circ \rho(y) = \{\rho(z) \mid z \in x \ast y\}.
\]

Let \(G = (X, R)\) be a v-linked g-hypergraph and \((X, \circ)\) be the g-hypergroupoid associated with \(G\). We define the relation \(\rho_G\) on \(X\) as follows:

\[
x \rho_G y \text{ if and only if } x_R = y_R.
\]
Lemma 4.1. The relation $\rho_{\mathcal{G}}$ is a regular equivalence relation.

Proof. Obviously, $\rho_{\mathcal{G}}$ is an equivalence relation. Let $z \in X$ be an arbitrary element and $x \rho_{\mathcal{G}} y$. First, we show that $x \circ z = y \circ z$ which implies that $(x \circ z)\overline{\rho_{\mathcal{G}}} (y \circ z)$. Let $r \in x \circ z = N(x) \cup N(z)$ be an arbitrary element. In the case that $r \in N(z)$, there is nothing to prove. If $r \in N(x)$, then there is a hyperedge $A$ such that $(x, A) \in R$ and $r \in A$. By assumption, we have $x_r = y_r$ and therefore we have $(y, A) \in R$. This implies that $r \in N(y)$. Hence $x \circ z \subseteq y \circ z$. The reverse inclusion can be shown similarly. In a similar way we can show that $(z \circ x)\overline{\rho_{\mathcal{G}}} (z \circ y)$. □

Definition 4.2. Let $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$ be two g-hypergraphs. Then, the direct product of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the g-hypergraph $\mathcal{G}_1 \times \mathcal{G}_2 = (X_1 \times X_2, R_1 \times R_2)$ where $R_1 \times R_2 = \{(x, y), A \times B \mid (x, A) \in R_1, (y, B) \in R_2\}$.

Lemma 4.3. Let $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$ be two g-hypergraphs. Then, for every $(x, y), (u, v) \in X_1 \times X_2$,

$$(x, y)_{\rho_{\mathcal{G}_1 \times \mathcal{G}_2}} (u, v) \Leftrightarrow x_{\rho_{\mathcal{G}_1}, u} \text{ and } y_{\rho_{\mathcal{G}_2}, v}.$$ 

Proof. It is obvious. □

Definition 4.4. Let $(X, *)$ and $(Y, \circ)$ be two hypergroupoids. We define the hyperoperation $\times_d$ on the Cartesian product $X \times Y$ as follows:

$$(x_1, y_1) \times_d (x_2, y_2) = d((x_1, y_1)) \cup d((x_2, y_2)),$$

where $d(a, b) = \{(x, y) \mid a \in x * a \text{ and } y \in b \circ b\}$. The hypergroupoid $(X \times Y, \times_d)$ is called the diagonal direct product of $(X, *)$ and $(Y, \circ)$.

Theorem 4.5. Let $(X_1, *)$ and $(X_2, \circ)$ be the g-hypergroupoids associated with the $v$-linked g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively. Then, the diagonal direct product $(X_1, *)$ and $(X_2, \circ)$ is the g-hypergroupoid associated with $\mathcal{G}_1 \times \mathcal{G}_2$.

Proof. Let $(X_1 \times X_2, \times_d)$ be the diagonal direct product of $(X_1, *)$ and $(X_2, \circ)$. It suffices to show that

$$(x, y) \times_d (x, y) = \bigcup \{A \times B \mid (x, y), A \times B \in R_1 \times R_2\},$$

where $(x, y)$ is an arbitrary element of $X_1 \times X_2$. This can be seen by the following argument. Let $(r, s) \in (x, y) \times_d (x, y)$ be an arbitrary element. Then, $r \in x * x$ and $s \in y \circ y$. Since $(X_1, *)$ and $(X_2, \circ)$ are the g-hypergroupoids associated with the g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively, there are hyperedges $A$ and $B$ such that $(x, A) \in R_1, (y, B) \in R_2$ and $(r, s) \in A \times B$. By the definition of $R_1 \times R_2$ we have $(x, y), A \times B \in R_1 \times R_2$ and therefore $(r, s) \in \bigcup \{A \times B \mid (x, y), A \times B \in R_1 \times R_2\}$. Hence $(x, y) \times_d (x, y) \subseteq \bigcup \{A \times B \mid (x, y), A \times B \in R_1 \times R_2\}$. The reverse inclusion can be shown similarly. □

Definition 4.6. Let $(X_1, *)$ and $(X_2, \circ)$ be two hypergroupoids. A map $\varphi : X_1 \to X_2$ is called a homomorphism if for all $x, y \in X_1$ we have $\varphi(x * y) = \varphi(x) * \varphi(y)$. If $\varphi$ is one to one (onto) we say that $\varphi$ is a monomorphism (epimorphism). If there exists a one to one epimorphism from $X_1$ onto $X_2$ we say that $X_1$ is isomorphic to $X_2$ and we write $X_1 \cong X_2$.

Theorem 4.7. Let $(X_1, *)$ and $(X_2, \circ)$ be the g-hypergroupoids associated with the $v$-linked g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively. Then,

$$X_1/\rho_{\mathcal{G}_1} \times_d X_2/\rho_{\mathcal{G}_2} \cong (X_1, *_{\mathcal{G}_1}) \times (X_2, *_{\mathcal{G}_2}).$$
Proof. We equip \(X_1/\rho_{\mathcal{G}_1}, X_2/\rho_{\mathcal{G}_2}\) and \((X_1 \times_d X_2)/\rho_{\mathcal{G}_1 \times \mathcal{G}_2}\) with hyperoperations \(\odot, \sqcap\) and \(\odot\), respectively. Define \(\varphi : X_1/\rho_{\mathcal{G}_1} \times_d X_2/\rho_{\mathcal{G}_2} \rightarrow (X_1 \times_d X_2)/\rho_{\mathcal{G}_1 \times \mathcal{G}_2}\) by

\[
\varphi((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))) = \rho_{\mathcal{G}_1 \times \mathcal{G}_2}(x, y), \text{ for all } (x, y) \in X_1 \times X_2.
\]

First, we prove \(\varphi\) is well defined. Consider

\[
(\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)) = (\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y')).
\]

Hence, we have \(x_{R_1} = x'_{R_1}\) and \(y_{R_2} = y'_{R_2}\). Since

\[
A \times B \in (x, y)_{R_1 \times R_2} \iff A \in x_{R_1}, B \in y_{R_2} \iff A \times B \in (x', y')_{R_1 \times R_2},
\]

we obtain \(\rho_{\mathcal{G}_1 \times \mathcal{G}_2}(x, y) = \rho_{\mathcal{G}_1 \times \mathcal{G}_2}(x', y')\), i.e., \(\varphi\) is well defined. Now, we check that \(\varphi\) is one to one. Suppose that \(\rho_{\mathcal{G}_1 \times \mathcal{G}_2}(x, y) = \rho_{\mathcal{G}_1 \times \mathcal{G}_2}(x', y')\). We obtain

\[
A \in x_{R_1}, B \in y_{R_2} \iff A \times B \in (x, y)_{R_1 \times R_2} \iff A \times B \in (x', y')_{R_1 \times R_2} \iff A \in x'_{R_1}, B \in y'_{R_2}.
\]

This implies that \(\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y) = (\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y'))\). Clearly \(\varphi\) is onto. We need only to show that \(\varphi\) is a homomorphism. Before that we show that \(\varphi(\Delta((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)))) = \rho_{\mathcal{G}_1 \times \mathcal{G}_2}(\Delta(x, y))\), for all \((x, y) \in X_1 \times X_2\). We know that

\[
\Delta((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))) = \{(\rho_{\mathcal{G}_1}(r), \rho_{\mathcal{G}_2}(s)) \mid \rho_{\mathcal{G}_1}(r) \in \rho_{\mathcal{G}_1}(x) \odot \rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(s) \in \rho_{\mathcal{G}_2}(y) \sqcap \rho_{\mathcal{G}_2}(y)\},
\]

and so we have

\[
\varphi(\Delta((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)))) = \left\{\rho_{\mathcal{G}_1 \times \mathcal{G}_2}(r, s) \mid r \in \rho_{\mathcal{G}_1}(x) \odot \rho_{\mathcal{G}_1}(x), s \in \rho_{\mathcal{G}_2}(y) \sqcap \rho_{\mathcal{G}_2}(y)\right\}.
\]

But \(\rho_{\mathcal{G}_1}(r) \in \rho_{\mathcal{G}_1}(x) \odot \rho_{\mathcal{G}_1}(x)\) if and only if there is \(u \in x \times x\) such that \(\rho_{\mathcal{G}_1}(r) = \rho_{\mathcal{G}_1}(u)\) and \(\rho_{\mathcal{G}_2}(s) \in \rho_{\mathcal{G}_2}(y) \sqcap \rho_{\mathcal{G}_2}(y)\) if and only if there is \(v \in y \circ y\) such that \(\rho_{\mathcal{G}_2}(s) = \rho_{\mathcal{G}_2}(v)\). Now, by using Lemma 4.3 we have

\[
\varphi(\Delta((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)))) = \rho_{\mathcal{G}_1 \times \mathcal{G}_2}(\Delta(x, y)) = \rho_{\mathcal{G}_1 \times \mathcal{G}_2}(\Delta(x, y)).
\]
Hence, \( \varphi \) is an isomorphism.

**Theorem 4.8.** Let \( \mathcal{G} \) be a \( v \)-linked \( g \)-hypergraph and \((X_2, \circ)\) be the \( g \)-hypergroupoid associated with \( \mathcal{G} \). If \((X_1, \ast)\) is a separable hypergroupoid and \( \varphi : X_1 \to X_2 \) is an epimorphism, then there exists a regular equivalence relation \( \mu \) on \( X_1 \) such that
\[
X_1/\mu \cong X_2/\rho_g.
\]

**Proof.** Suppose that the relation \( \mu \) on \( X_1 \) is defined by \( x \mu y \Leftrightarrow \varphi(x) \rho_g \varphi(y) \), for all \( x, y \in X_1 \). Since \( \rho_g \) is an equivalence relation on \( X_2 \), then it is easy to check that \( \mu \) is an equivalence relation on \( X_1 \). Let \( x, y, z \in X_1 \) be arbitrary elements such that \( x \mu y \). We show that \((x \ast z) \bar{\mu} (y \ast z)\). From \( x \mu y \) it follows that \( \varphi(x) \circ \varphi(x) = \varphi(y) \circ \varphi(y) \) which implies that \( \varphi(x \ast x) = \varphi(y \ast y) \). Let \( r \in x \ast z \) be an arbitrary element. Then, we have \( \varphi(r) \in \varphi(x \ast z) = \varphi(x \ast x) \cup \varphi(z \ast z) = \varphi(y \ast y) \cup \varphi(z \ast z) = \varphi(y \ast z) \). Therefore, there is \( t \in y \ast z \) such that \( \varphi(t) = \varphi(r) \). This means that \( r \mu t \) and so \((x \ast z) \bar{\mu} (y \ast z)\). In a similar way we can show that \((z \ast x) \bar{\mu} (y \ast z)\). Thus \( \mu \) is regular. Now, let \( \psi : X_1/\mu \to X_2/\rho_g \) be defined by \( \psi(\mu(x)) = \rho_g(\varphi(x)) \). Suppose that \( x, y \in X_1 \). Then
\[
\mu(x) = \mu(y) \Leftrightarrow \varphi(x) \rho_g \varphi(y) \Leftrightarrow \rho_g(\varphi(x)) = \rho_g(\varphi(y)) \Leftrightarrow \psi(\mu(x)) = \psi(\mu(y)).
\]

Thus \( \psi \) is well defined and one to one. Since \( \varphi \) is onto, it follows that \( \psi \) is onto. We equip \( X_1/\mu \) and \( X_1/\rho_g \) with the hyperoperations \( \odot \) and \( \square \), respectively. Let \( x, y \in X_1 \). The following argument shows that \( \psi \) is a homomorphism.

\[
\psi(\mu(x)) \odot \psi(\mu(y)) = \rho_g(\varphi(x)) \odot \rho_g(\varphi(y)) = \rho_g(\varphi(x) \circ \varphi(y)) = \rho_g(\varphi(t) \text{ for some } t \in x \ast y) = \psi(\mu(t) | t \in x \ast y) = \psi(\mu(x) \odot \mu(y)).
\]

**Theorem 4.9.** Let \( \mathcal{G} \) be a \( v \)-linked \( g \)-hypergraph and \((X_1, \ast)\) be the \( g \)-hypergroupoid associated with \( \mathcal{G} \). If \((X_2, \circ)\) is a separable hypergroupoid and \( \varphi : X_1 \to X_2 \) is an epimorphism, then there exists a regular equivalence relation \( \mu' \) on \( \varphi(X_1) \) such that
\[
X_1/\rho_g \cong \varphi(X_1)/\mu'.
\]
Proof. Suppose that the relation $\mu'$ on $X_1$ is defined by $\varphi(x)\mu'(\varphi(y)) \Leftrightarrow x\rho\varphi, \forall x, y \in X_1$. It is easy to see that $\mu'$ is a regular equivalence relation. Define $\psi : X_1/\rho \to \varphi(X_1)/\mu'$ by $\psi(\rho(x)) = \mu'(\varphi(x))$. One can easily checks that $\psi$ is an isomorphism.

Lemma 4.10. Let $\rho$ be a regular equivalence relation on a hypergroupoid $(X, \circ)$. Then, $\pi : X \to X/\rho$ which is defined by $\pi(x) = \rho(x)$, for all $x \in X$, is an epimorphism which is called canonical epimorphism.

Proof. The proof is straightforward. ∎

Theorem 4.11. Let $(X_1, \ast)$ and $(X_2, \circ)$ be $g$-hypergroupoids associated with the $v$-linked $g$-hypergraphs $G_1 = (X_1, R_1)$ and $G_2 = (X_2, R_2)$, respectively. Let $\varphi : X_1 \to X_2$ be an epimorphism such that $\varphi(x)\rho_{G_2} \varphi(y)$ implies $x\rho_{G_1} y$. If $\mu = \{(x, y) \in X_1^2 \mid \varphi(x)\rho_{G_1} x\}$ and $\mu' = \{(\varphi(x), \varphi(y)) \in X_2^2 \mid x\rho_{G_1} y\}$, then there exists a unique homomorphism $\varphi' : X_1/\mu \to X_2/\mu'$ such that the following diagram is commutative: i.e., $\pi' \circ \varphi = \varphi' \circ \pi$, where $\pi$ and $\pi'$ denote the canonical epimorphisms.

Proof. The proof of the fact that $\mu$ and $\mu'$ are regular equivalence relations is analogous to the corresponding part of the proof of Theorem 4.8 and we omit the details. We equip $X_1/\mu$ and $X_2/\mu'$ with the hyperoperations $\odot$ and $\boxplus$, respectively. Let $\varphi' : X_1/\mu \to X_2/\mu'$ is defined by $\varphi'(\mu(x)) = \mu'(\varphi(x))$, for all $x \in X_1$. First, we show that $\varphi'$ is well defined. Let $x, y \in X_1$ and $\mu(x) = \mu(y)$. Then, $\varphi(x)\rho_{G_2} \varphi(y)$ and so $x\rho_{G_1} y$. Therefore, $\varphi'$ is well defined. Moreover, it is easy to prove that $\varphi'(\mu(x) \odot \mu(y)) = \varphi'(\mu(x)) \boxplus \varphi'(\mu(y))$ and $\pi' \circ \varphi = \varphi' \circ \pi$. Now, we show that $\varphi'$ is unique. Let $g : X_1/\mu \to X_2/\mu'$ be a homomorphism such that $\pi' \circ g = g \circ \pi$. Then, for all $x \in X_1, g(\mu(x)) = g(\pi(x)) = \pi' \circ \varphi(x) = \varphi' \circ \pi(x) = \varphi'(\mu(x))$. ∎

5. Fundamental Relation on a $g$-Hypergroupoid

One of the main tools to study hyperstructures is the fundamental relation $\beta^*$ in an $H_v$-semigroup $(X, \circ)$ as the smallest equivalence relation so that the quotient $X/\beta^*$ would be a semigroup. The relation $\beta^*$ was introduced on hypergroups by M. Koskas in 1970 [9] and was mainly studied intensively and in depth by Corsini [3], also see [7].

For a relation $\beta$ on a non-empty set $X$, we denote by $\overline{\beta}$ the transitive closure of $\beta$ and define it as follows:

\[ x\overline{\beta}y \quad \text{if and only if} \quad \text{there exists a natural number } k \text{ and elements } \]
\[ x = a_1, a_2, \ldots, a_{k-1}, a_k = y \text{ in } X \text{ such that } \]
\[ a_1\beta a_2, a_2\beta a_3, \ldots, a_{k-1}\beta a_k. \]

Obviously, $\beta = \overline{\beta}$ if $\beta$ is transitive.

The proof of following theorem is similar to the proof of Theorem 1.2.2 of [14].

Theorem 5.1. Let $(X, \circ)$ be an $H_v$-semigroup and denote $U$ the set of all finite products of elements of $X$. We define the relation $\beta$ on $X$ by setting $x\beta y$ if and only if $x = y$ or $\{x, y\} \subseteq u$ where $u \in U$. Then, $\beta^*$ is the transitive closure of $\beta$. 

Theorem 5.2. Let $G = (X, R)$ be a $v$-linked $g$-hypergraph and $X_G = (X, \circ)$ be the $g$-hypergroupoid associated with $G$. Then, we have

$$\beta^*(x) = \begin{cases} S & \text{if } x \in S \\ \{x\} & \text{if } x \notin S, \end{cases}$$

where $S = \bigcup_{A \in \text{Cod}(R)} A$ and $\beta$ is the relation defined in Theorem 5.1.

Proof. First, we show that $\bigcup \{ u \mid u \in U \} = S$. Let $x \in S$ be an arbitrary element. Then, there exist $A \in \text{Cod}(R)$ and $a \in X$ such that $(a, A) \in R$ and $x \in A$. Therefore, by setting $u = a \circ a$ we have $x \in u$. Hence $S \subseteq \bigcup \{ u \mid u \in U \}$. The reverse inclusion is obvious. We conclude that if $x, y \in S$ and $x \beta y$, then $x \in S$ if and only if $y \in S$. Now, let $x, y \in S$ be arbitrary elements. Then, there exist $A, B \in \text{Cod}(R)$ and $a, b \in X$ such that $(a, A) \in R$, $(b, B) \in R$, $x \in A$ and $y \in B$. Therefore, $(x, y) \subseteq a \circ b$ which implies that $x \beta y$. Consequently, for every $x, y \in S$ we have $x \beta y$. By the above argument, we obtain

$$\beta(x) = \begin{cases} S & \text{if } x \in S \\ \{x\} & \text{if } x \notin S. \end{cases}$$

It is easy to see that $\beta$ is transitive and so we have $\hat{\beta} = \beta$. By using Theorem 5.1, we have $\beta^* = \hat{\beta}$ which completes the proof.

Corollary 5.3. A $v$-linked $g$-hypergraph $G = (X, R)$ is plenary if and only if $\beta^* = X \times X$.

By using the above corollary and Remark 2.7, we obtain the following corollary.

Corollary 5.4. For every separable $H_v$-group, we have $\beta^* = \beta$.

References