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Angles and Quasiconformal Mappings Between Manifolds

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Abstract. In this paper we discuss the distortion of angles under quasiconformal deformation between manifolds. Moreover, we obtain some useful inequalities.

1. Introduction

First we introduce some basic concepts as follows.

1.1. Dilatations

Let D, D' be subdomains of $\mathbf{R}^n$ and $f : D \to D'$ be a differentiable homeomorphism and denote its Jacobian by $J(x, f)$, $x \in D$. If $x \in D$ and $J(x, f) \neq 0$, then the derivative of f at $x \in D$ is a bijective linear mapping $f'(x) : \mathbf{R}^n \to \mathbf{R}^n$ and we denote

$$H_I(f'(x)) = \frac{|J(x,f)|}{\lambda_f(x)^n}, \quad H_O(f'(x)) = \frac{\Lambda_f(x)^n}{|J(x,f)|}, \quad H(f'(x)) = \frac{\Lambda_f(x)}{\lambda_f(x)}, \tag{1}$$

where

$$\Lambda_f(x) := \max\{|f'(x)h| : |h| = 1\} \text{ and } \lambda_f(x) := \min\{|f'(x)h| : |h| = 1\}.$$

Sometimes instead of $\Lambda_f(x)$ we use notation $|f'(x)|$, to denote the norm of the matrix $A = f'(x)$. If $\lambda_1^2 \leq \cdots \leq \lambda_n^2$ $(\lambda_i > 0, i = 1, 2, \cdots, n)$ are eigenvalues of the symmetric matrix AA^t where A^t is the adjoint of A, then we have the following well-known formulas

$$|J(x,f)| = \prod_{k=1}^{n} \lambda_k, \quad \Lambda_f(x) = \lambda_n, \quad \lambda_f(x) = \lambda_1. \tag{2}$$

By (1) and (2), we arrive at the following simple inequalities [6, 14.3]

$$H(f'(x)) \leq \min\{H_I(f'(x)), H_O(f'(x))\} \leq H(f'(x))^{n/2}. \tag{3}$$

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$$H(f'(x))^{n/2} \le \max\{H_I(f'(x)), H_O(f'(x))\} \le H(f'(x))^{n-1}. \tag{4}$$

The quantities

$$K_I(f) = \sup_{x\in D} H_I(f'(x)), \quad K_O(f) = \sup_{x\in D} H_O(f'(x))$$

are called the inner and outer dilatation of f, respectively. The maximal dilatation of f is

$$K(f) = \max\{K_I(f), K_O(f)\}.$$

1.2. **Quasiconformal Mappings Between Open Sets**

In the literature, see e.g. [4], we can find various definitions of quasiconformality which are equivalent. The following analytic definition for quasiconformal mappings is from [6, Theorem 34.6]: a homeomorphism $f : D \to D'$ is C-quasiconformal if and only if the following conditions are satisfied: (i) f is ACL; (ii) f is differentiable a.e.; (iii) $\Lambda_f(x)^n/C \le |J(x, f)| \le C\lambda_f(x)^n$ for a.e. $x \in D$. By [6, Theorem 34.4], if f satisfies the conditions (i), (ii) and $J(x, f) \neq 0$ a.e., then

$$K_I(f) = \operatorname*{ess\,sup}_{x\in D} H_I(f'(x)), \quad K_O(f) = \operatorname*{ess\,sup}_{x\in D} H_O(f'(x)).$$

Hence (iii) can be written as $K(f) \le C$ which by (4) is equivalent to

$$H(f'(x)) \le K \text{ for a.e. } x \in D. \tag{5}$$

Here the constant $K \le C^{2/n}$. In this paper we say that a quasiconformal mapping $f : D \to D'$ is K-quasiconformal if K satisfies (5). For other definition of quasiconformal mappings we refer to [5],[7],[8].

It is important to notice that f is K-quasiconformal if and only if f^{-1} is K-quasiconformal and that the composition of K_1 and K_2 quasiconformal mappings is K_1K_2-quasiconformal. (It is well-known that this also holds for K - quasiconformality in Väisälä's sense, see [6, Corollary 13.3, Corollary 13.4]).

1.3. Quasiconformal Mappings Between Manifolds

Let M and N be connected separable, orientable n-dimensional ($n \ge 2$) differentiable manifolds of class C^1 The tangent bundle of M is denoted by TM. The derivative of a differentiable mapping $f : M \to N$ is a fibre mapping $Df : TM \to TN$. If we repeat the approach from the previous subsection to the linear mapping $A(p) = Df(p)$, we arrive to the notation of K - quasiconformality of f at $p \in M$.

1.4. Angles Between Two Vectors

Let $a, b \in \mathbf{R}^n$ be two vectors and $\langle a|b\rangle$ denotes the standard inner product of vectors. If θ is the angle of these two vectors, then we have

$$\cos(\theta) = \frac{\langle a|b\rangle}{|a| \cdot |b|}.$$

2. The Main Results

It is well-known that smooth conformal mappings preserves the angles between the curves. What is less-known is that to what extend the angles change under quasiconformal mappings. Two classical papers by Agard and Ghering [2] and by Agard [1], bring much light on this topic for two and three dimensional case. The main result of the paper is the following theorem.

Theorem 2.1. *Let f be a $K-$quasiconformal mapping between two orientable n-dimensional ($n \geq 2$) differentiable manifolds of class C^1 and let γ_1 and γ_2 be two smooth curves making the angle s in their intersection point $p \in M$, where the Jacobian of f does not vanish. Then the angle t between $\delta_1 = f(\gamma_1)$ and $\delta_2 = f(\gamma_2)$ in $q = f(p)$ satisfies the following inequality*

$$|\cos t| \leq \frac{H + \cos s}{1 + H \cos s}; \tag{6}$$

where $H = (K^2 - 1)/(K^2 + 1)$. Moreover if $B = Df(p)^* Df(p)$ and $t = t(s)$ is the infinum of all angles between curves γ_1 and γ_2 passing throughout p and making the angle s, then there are vectors h and k such that $|h| = |k|$ and $\langle Bh, h\rangle = \langle Bk, k\rangle = 1$ so that

$$\cos t = \langle Bh, k\rangle = \frac{K_{i,j} + \cos s}{1 + K_{i,j}^2 \cos s}$$

where

$$K_{i,j} = \frac{\lambda_i^2 - \lambda_j^2}{\lambda_i^2 + \lambda_j},$$

and λ_i^2, $i = 1, \dots, n$ are eigenvalues of B.

Remark 2.2. *Under the condition of the Theorem 2.1, for two-dimensional planar domains case Agard and Ghering in [2, Theorem 1], proved that*

$$t \geq \frac{s}{K}. \tag{7}$$

Let us show that (6) implies (7). It is enough to show that for $s \in [0, \pi/2]$,

$$\Phi(s) := \arccos \frac{H + \cos s}{1 + H \cos s} - \frac{s}{K} \geq 0,$$

where $H = (K^2 - 1)/(K^2 + 1)$. By differentiating Φ, we obtain

$$\Phi'(s) = \frac{2K}{1 + K^2 + (-1 + K^2)\cos s}.$$

Thus $\Phi'(s) \geq 0$, which implies that $\Phi(s) \geq \Phi(0) = 0$. Further in [1], Agard proved for three-dimensional case the inequality

$$\tan \frac{s}{2} \geq \frac{1}{K} \tan \frac{t}{2}. \tag{8}$$

It can be shown that (6) is equivalent with (8), but the proof given in [1] is applied only on the three-dimensional case, and the present proof is different and hold for an Euclidean space of arbitrary dimension and for manifolds as well.

Proof. Fix $p \in M$ and let $q = f(p) \in N$. Let $\gamma_i : [-1, 1] \to M$, $i = 1, 2$, and $\gamma_i(0) = p$ and assume that their angle is s, then the curves δ_1, δ_2 have the intersection point q and make the angle t at it. We should prove that

$$-\frac{H + \cos s}{1 + H \cos s} \leq \cos t \leq \frac{H + \cos s}{1 + H \cos s}.$$

Let $A = Df(p)$, $B = A^* A$, $h = \gamma_1'(0)$, $k = \gamma_2'(0)$. Since $TM_p \cong \mathbf{R}^n \cong TN_q$, we will identify both TM_p and TN_q by

$\mathbf{R}^n$. Let $\langle a|b\rangle$ denotes the standard inner product of vectors. Then

$$\begin{aligned}\cos t &= \frac{\left\langle \delta_1'(0)|\delta_2'(0)\right\rangle}{|\delta_1'(0)|\cdot|\delta_2'(0)|}\\ &= \frac{\left\langle Df(p)\gamma_1'(0)|Df(p)\gamma_2'(0)\right\rangle}{|Df(p)\gamma_1'(0)|\cdot|Df(p)\gamma_2'(0)|}\\ &= \frac{\left\langle A\gamma_1'(0)|A\gamma_2'(0)\right\rangle}{|A\gamma_1'(0)|\cdot|A\gamma_2'(0)|}\\ &= \frac{\langle Bh|k\rangle}{\sqrt{\langle Bh|h\rangle}\sqrt{\langle Bk|k\rangle}}.\end{aligned}$$

Here

$$h' = \frac{h}{\sqrt{\langle Bh|h\rangle}}$$

and

$$k' = \frac{k}{\sqrt{\langle Bk|k\rangle}}.$$

We see that

$$\langle Bh'|h'\rangle = \left\langle B\frac{h}{\sqrt{\langle Bh|h\rangle}}, \frac{h}{\sqrt{\langle Bk|k\rangle}}\right\rangle = 1$$

and

$$\langle Bk'|k'\rangle = \left\langle B\frac{k}{\sqrt{\langle Bk|k\rangle}}, \frac{k}{\sqrt{\langle Bk|k\rangle}}\right\rangle = 1.$$

Thus we solve the extremal problem

- $\langle Bh|k\rangle \to \mathrm{Ext}$

under the conditions

1. $\langle Bh|h\rangle = 1$,
2. $\langle Bk|k\rangle = 1$ and
3. $\langle h|k\rangle - \cos s|h|\cdot|k| = 0$.

We consider the set

$$\mathcal{K} = \{(h,k)\in\mathbf{R}^n\times\mathbf{R}^n : \langle Bh|h\rangle = 1,\ \ \langle Bk|k\rangle = 1, \langle h|k\rangle - \cos s|h|\cdot|k| = 0\},$$

which is compact, because $\det B \neq 0$. Then there exists $(h_0,k_0)\in\mathcal{K}$ such that

$$\langle Bh_0|k_0\rangle = \max_{(h,k)\in\mathcal{K}} \langle Bh|k\rangle.$$

Thus it is necessary and sufficient to find the maximum of the function $\langle Bh|k\rangle$ in $\mathcal{K}$. The Lagrangian is

$$\mathcal{L} = \langle Bh|k\rangle + \mu\langle Bh|h\rangle + \nu\langle Bk|k\rangle + \eta(\langle h|k\rangle - \cos s|h|\cdot|k|).$$

Then by differentiating $\mathcal{L}$ w.r.t. h and k, we obtain that the stationary points on the intersections of Descartes product of ellipsoids $\langle Bh|h\rangle = 1$, $\langle Bk|k\rangle = 1$ and the set $\langle h|k\rangle - \cos s|h|\cdot|k| = 0$ satisfy the equations

$$\mathcal{L}_h = Bk + 2\mu Bh + \eta(k - \cos(s)h\frac{|k|}{|h|}) = 0, \tag{9}$$

$$\mathcal{L}_k = Bh + 2\nu Bk + \eta(h - \cos(s)k\frac{|h|}{|k|}) = 0 \tag{10}$$

where μ, ν and η are some real constants. Then

$$\langle Bk|h\rangle + 2\mu\,\langle Bh|h\rangle = 0,\ \ \langle Bh|k\rangle + 2\nu\,\langle Bk|k\rangle = 0, \tag{11}$$

implying that $\mu = \nu$ and

$$\langle Bh|k\rangle = -2\mu \tag{12}$$

and

$$\eta(\langle k,k\rangle - \cos(s)\,\langle k,h\rangle\,\frac{|k|}{|h|}) = \eta(\langle h,h\rangle - \cos(s)\,\langle k,h\rangle\,\frac{|h|}{|k|}).$$

The last implies that

$$\eta\left(|k|^2 - |h|^2 - \cos(s)\,\langle k,h\rangle\,\frac{|k|^2 - |h|^2}{|k|\cdot|h|}\right) = 0.$$

Thus

$$\eta(|k|^2 - |h|^2)\sin^2(s) = 0.$$

This implies that $|h| = |k|$, or $s = 0$ or $\eta = 0$. Since the cases $s = 0$ and $\eta = 0$ are trivial we consider only the case $|h| = |k|$. Let

$$P = \begin{pmatrix} 2\mu & 1 \\ 1 & 2\mu \end{pmatrix}.$$

Then the system (9) and (10) can be written as

$$PB\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \eta\cos(s)\frac{|k|}{|h|} & -\eta \\ -\eta & \eta\cos(s)\frac{|h|}{|k|} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix},$$

or

$$B\begin{pmatrix} h \\ k \end{pmatrix} = Q\begin{pmatrix} h \\ k \end{pmatrix} \tag{13}$$

where

$$Q = P^{-1}\cdot\begin{pmatrix} \eta\cos(s)\frac{|k|}{|h|} & -\eta \\ -\eta & \eta\cos(s)\frac{|h|}{|k|} \end{pmatrix}.$$

So we need to consider the matrix Q and determine its eigenvectors and eigenvalues. First we have that

$$Q = \begin{pmatrix} \frac{2\mu}{4\mu^2-1} & \frac{1}{1-4\mu^2} \\ \frac{1}{1-4\mu^2} & \frac{2\mu}{4\mu^2-1} \end{pmatrix}\cdot\begin{pmatrix} \eta\cos(s)\frac{|k|}{|h|} & -\eta \\ -\eta & \eta\cos(s)\frac{|h|}{|k|} \end{pmatrix}$$

i.e.

$$Q = \begin{pmatrix} \frac{\eta(|h|+2|k|\mu\cos(s))}{|h|\left(-1+4\mu^2\right)} & \frac{\eta(2|h|\mu+|k|\cos(s))}{|h|-4|h|\mu^2} \\ \frac{\eta(2|h|\mu+|k|\cos(s))}{|h|-4|h|\mu^2} & \frac{\eta(|h|+2|k|\mu\cos(s))}{|h|\left(-1+4\mu^2\right)} \end{pmatrix}.$$

Then

$$\det(Q-\lambda I)=\frac{\eta^2\left(-|h|^2+|k|^2\cos^2 s\right)}{|h|^2\left(-1+4\mu^2\right)}+\frac{2\eta(|h|+2|k|\mu\cos s)\lambda}{|h|-4|h|\mu^2}+\lambda^2.$$

In view of the fact that $|h|=|k|$ we have

$$\det(Q-\lambda I)=0$$

if and only if

$$\lambda=\frac{\eta(-1+\cos(s))}{1+2\mu}\text{ or }\lambda=\frac{\eta(1+\cos(s))}{-1+2\mu}.$$

Thus in view of (13) we have

$$\frac{\eta(-1+\cos(s))}{1+2\mu}=\lambda_i^2,\text{ and }\frac{\eta(1+\cos(s))}{-1+2\mu}=\lambda_j^2,$$

where λ_i^2 and λ_j^2 are eigenvalues of the positive operator B.

Then

$$\mu=-\frac{\lambda_i^2-\lambda_j^2+\lambda_i^2\cos(s)+\lambda_j^2\cos(s)}{2(\lambda_i^2+\lambda_j^2+\lambda_i^2\cos(s)-\lambda_j^2\cos(s))}$$

and

$$\eta=-\frac{(2\lambda_i^2\lambda_j^2)}{\lambda_i^2+\lambda_j^2+\lambda_i^2\cos(s)-\lambda_j^2\cos(s)}.$$

Inserting μ in (12), we obtain that

$$\langle Bh|k\rangle=\frac{\lambda_i^2-\lambda_j^2+\lambda_i^2\cos(s)+\lambda_j^2\cos(s)}{(\lambda_i^2+\lambda_j^2+\lambda_i^2\cos(s)-\lambda_j^2\cos(s))}.\tag{14}$$

So

$$\langle Bh|k\rangle=\frac{K_{i,j}+\cos(s)}{1+K_{i,j}\cos(s)},\tag{15}$$

where

$$K_{i,j}=\frac{1-\frac{\lambda_j^2}{\lambda_i^2}}{1+\frac{\lambda_j^2}{\lambda_i^2}}.$$

This finishes the proof $\square$

Now we infer the following.

Theorem 2.3. *Let $0<\lambda_1\le\cdots\le\lambda_n$ and a_i, b_i, $i=1,\dots,n$ be real numbers such that*

$$\sum_{i=1}^n a_ib_i=\cos(s)\sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}.$$

Then

$$\sum_{i=1}^n\lambda_ia_ib_i\le\frac{H+\cos(s)}{1+H\cos(s)}\sqrt{\sum_{i=1}^n\lambda_ia_i^2}\sqrt{\sum_{i=1}^n\lambda_ib_i^2},$$

where $H=(\lambda_n-\lambda_1)/(\lambda_n+\lambda_1)$

Proof. Let $B = (b_{i,j})$ be a $n \times n$ diagonal matrix satisfies $b_{i,i} = \lambda_i$ for $i = 1, 2, \cdots, n$, where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are given by Theorem 2.3, and $b_{i,j} = 0$ for every $i \ne j$. Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two vectors in $\mathbf{R}^n$. If we set

$$h = \frac{a}{\sqrt{\langle Ba|a\rangle}}$$

and

$$k = \frac{b}{\sqrt{\langle Bb|b\rangle}},$$

then we have

$$\langle Bh|h\rangle = 1 = \langle Bk|k\rangle.$$

Since

$$\langle Ba|b\rangle = \sum_{i=1}^{n} \lambda_i a_i b_i,$$

we see that it is enough to find the maximum of $\langle Ba|b\rangle$. According to the proof of Theorem 2.1 we see that

$$\max \langle Bh|k\rangle = \frac{K_{i,j} + \cos(s)}{1 + K_{i,j}\cos(s)}$$

where $\cos(s) = \frac{\langle a|b\rangle}{|a|\cdot|b|}$,

$$K_{i,j} = \frac{1 - \frac{\lambda_j}{\lambda_i}}{1 + \frac{\lambda_j}{\lambda_i}},$$

and λ_i, λ_j are eigenvalues of the matrix B (c.f. (15), here we use $\lambda_i > 0$ instead of λ_i^2).

It is easy to see that the function $\varphi_1(t) := \frac{1-t}{1+t}$ is a decreasing function for $t > 0$. Therefore we have

$$K_{i,j} \le \frac{1 - \frac{\lambda_1}{\lambda_n}}{1 + \frac{\lambda_1}{\lambda_n}} := H.$$

The function $\varphi_2(t) := \frac{t+\cos(s)}{1+t\cos(s)}$ is an increasing function of $t > 0$. Hence we have

$$\langle Bh|k\rangle \le \frac{H + \cos(s)}{1 + H\cos(s)}.$$

By the assumption we know that

$$\langle Ba|b\rangle = \langle Bh|k\rangle \cdot \langle Ba|a\rangle \cdot \langle Bb|b\rangle.$$

This shows that

$$\langle Ba|b\rangle \le \frac{H + \cos(s)}{1 + H\cos(s)} \langle Ba|a\rangle \cdot \langle Bb|b\rangle$$

which implies that

$$\sum_{i=1}^{n} \lambda_i a_i b_i \le \frac{H + \cos(s)}{1 + H\cos(s)} \sqrt{\sum_{i=1}^{n} \lambda_i a_i^2} \sqrt{\sum_{i=1}^{n} \lambda_i b_i^2}.$$

The proof is completed. □

Corollary 2.4. *Let $0 < \lambda_1 \le \cdots \le \lambda_n$ and a_i and b_i be real numbers such that*

$$\sum_{i=1}^{n} a_i b_i = 0.$$

Then

$$\sum_{i=1}^{n} \lambda_i a_i b_i \leq \frac{K-1}{K+1} \sqrt{\sum_{i=1}^{n} \lambda_i a_i^2} \sqrt{\sum_{i=1}^{n} \lambda_i b_i^2},$$

where $K = \lambda_n / \lambda_1$.

Proof. This is a special case of Theorem 2.3 with $\cos(s) = 0$. □

Remark 2.5. *Let us explore the equality statement of Corollary 2.4 for the case* $n = 2$. *Assume that* $a = (a_1, a_2)$ *and* $b = (b_1, b_2)$. *Let* $a_2 = b_2 = \xi$, $a_1 = t$, $\lambda_1 < \lambda_2$. *Then the equality of the above Corollary 2.4 shows that*

$$\lambda_1 a_1 b_1 + \lambda_2 a_2 b_2 = \frac{\frac{\lambda_2}{\lambda_1} - 1}{\frac{\lambda_2}{\lambda_1} + 1} \sqrt{\lambda_1 a_1^2 + \lambda_2 a_2^2} \sqrt{\lambda_1 b_1^2 + \lambda_2 b_2^2}.$$

Using $\sum_{i=1}^{2} a_i b_i = 0$, *we have*

$$\xi^2(\lambda_2 + \lambda_1) = \sqrt{(\lambda_1 t^2 + \lambda_2 \xi^2)(\lambda_1 \frac{\xi^4}{t^2} + \lambda_2 \xi^2)}.$$

Take squared from the both side and Simply the equality we obtain

$$\lambda_1 \lambda_2 \xi^2 t^4 - 2\lambda_1 \lambda_2 \xi^4 t^2 + \lambda_1 \lambda_2 \xi^6 = 0,$$

which implies that $(t^2 - \xi^2) = 0$. *Therefore* $t = \pm\xi$.

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