Filomat 31:15 (2017), 4695–4702
https://doi.org/10.2298/FIL1715695W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An Einstein-like Metric on Almost Kenmotsu Manifolds

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Abstract. In this paper, we prove that if the metric of a three-dimensional $(k,\mu)'$-almost Kenmotsu manifold satisfies the Miao-Tam critical condition, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$. Moreover, we prove that if the metric of an almost Kenmotsu manifold with conformal Reeb foliation satisfies the Miao-Tam critical condition, then the manifold is either of constant scalar curvature or Einstein. Some corollaries of main results are also given.

1. Introduction

Kenmotsu manifolds known as not only a special case of almost contact metric manifolds (see Blair [2]) but also an analogous of Hermitian manifolds were investigated by many authors in the last four decades. Recently, G. Pitiş [18] published a book in which many interesting results on such manifolds were collected. After Kenmotsu manifolds was first introduced by K. Kenmotsu in [12], later such manifolds were generalized to almost Kenmotsu manifolds by D. Janssens and L. Vanhecke [11] (see also Kim and Pak [13]). Since then some authors started to study almost Kenmotsu manifolds under various conditions and many fundamental formula were obtained (see Dileo and Pastore [7, 8]). Among others, in the present paper we are concerned with the studies of the Ricci tensors on almost Kenmotsu manifolds satisfying certain nullity condition or conformal Reeb foliation.

From Binh, Tamassy, De and Tarafdar [1, Theorem 2], it is known that any Kenmotsu manifold of dimension three with parallel Ricci tensor, i.e.,

$$\nabla Q = 0, \tag{1}$$

is Einstein and hence it is of constant sectional curvature −1. De and Pathak in [4] proved that if the Ricci tensor of a Kenmotsu manifold of dimension three is cyclic-parallel, i.e.,

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0 \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 53D15; Secondary 53C25
Keywords. Almost Kenmotsu manifold, $(k,\mu)'$-nullity condition, Miao-Tam critical condition, Einstein-like metric
Received: 10 May 2016; Accepted: 03 October 2016
Communicated by Ljubica Velimirović
This work was supported by the National Natural Science Foundation of China (No. 11526080), Key Scientific Research Program in Universities of Henan Province (No. 16A110004), the Research Foundation for the Doctoral Program of Henan Normal University (No. qd14145) and the Youth Science Foundation of Henan Normal University (No. 2014QK01).
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for any vector fields X, Y, Z, then the manifold if of constant sectional curvature -1. According to Y. Wang and X. Liu [20, Theorem 1.1], we state that a three-dimensional $(k,\mu)'$-almost Kenmotsu manifold with Codazzi-type Ricci tensor, i.e.,

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{3}$$

for any vector fields X and Y, is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$. Some other results regarding the Ricci tensor on a three-dimensional (almost) Kenmotsu manifold can be seen in De and De [3], De and Tripathi [5] and De and Yildiz et al. [6, 23].

A. Ghosh [9] obtained that if the metric of a Kenmotsu manifold of dimension three represents a Ricci soliton, i.e.,

$$\frac{1}{2}\mathcal{L}_V g + S = \lambda g \tag{4}$$

holds for certian constant λ and a vector field V, then the manifold is of constant sectional curvature -1. Later, this was generalized by the present author and Liu in [21] on a special type of three-dimensional almost Kenmotsu manifolds. Very recently, the present author [19, pp. 84] proved that if a three-dimensional almost Kenmotsu manifold has a parallel Ricci tensor, then the manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$. Generalizing the above result, Wang and Liu [22] obtained some local classification results regarding the Ricci-semisymmetry condition, i.e.,

$$R \cdot Q = 0, \tag{5}$$

on a class of three-dimensional almost Kenmotsu manifold.

In the present paper, we consider the Ricci tensor and a Riemannian metric satisfying the so called Miao-Tam critical condition (defined in Section 3) on a three-dimensional $(k,\mu)'$-almost Kenmotsu manifold and prove that such manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$. We also consider such critical metric on an almost Kenmotsu manifold with conformal Reeb foliation and prove that either the manifold is of constant scalar curvature or it is Einstein. Some corollaries of our main results are also given.

2. Almost Kenmotsu Manifolds

On a $(2n+1)$-dimensional smooth differentiable manifold M^{2n+1} if there exist a triplet (ϕ, ξ, η) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1, \tag{6}$$

where id denotes the identity mapping, ϕ a $(1,1)$-type tensor field, ξ a global vector field and η a 1-form, then the triplet is called an *almost contact structure* and M^{2n+1} is called an *almost contact manifold*. If in addition there exists a Riemannian metric g on an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ which is *compatible* with the almost contact structure, i.e.,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{7}$$

for any vector fields X and $Y \in \mathfrak{X}(M)$, then M^{2n+1} is called an *almost contact metric manifold*, where $\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M^{2n+1}.

Let us consider the Riemannian product $M^{2n+1}\times\mathbb{R}$ of an almost contact manifold manifold and $\mathbb{R}$. We define on the product an almost complex structure J by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M^{2n+1}\times\mathbb{R}$. If the almost complex structure J is integrable, i.e., the Nijenhuis tensor of J vanishes, then the almost contact structure is said to be *normal*. By Blair [2], the normality of an almost contact structure is equivalent to

$$[\phi, \phi] = -2d\eta \otimes \xi,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ.

The *fundamental* 2*-form* Φ of an almost contact metric M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. An *almost Kenmotsu manifold* is defined as an almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$. Following Janssens and Vanhecke [11], a normal almost Kenmotsu manifold is called a *Kenmotsu manifold* [12]. On an almost Kenmotsu manifold M^{2n+1}, we consider three $(1, 1)$-type tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \phi$, $h' = h \circ \phi$ and $l = R(\cdot, \xi)\xi$, where R is the curvature tensor of g and $\mathcal{L}$ is the Lie derivative operator. Following Dileo and Pastore [7, 8], we see that h, h' and l are symmetric and satisfy the following relations:

$$h\xi = l\xi = 0, \ \mathrm{tr}(h) = \mathrm{tr}(h') = 0, \ h\phi + \phi h = 0, \tag{8}$$

$$\nabla \xi = h' + \mathrm{id} - \eta \otimes \xi, \tag{9}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{10}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{11}$$

$$\mathrm{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \mathrm{tr}h^2, \tag{12}$$

$$R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X \tag{13}$$

for any $X, Y \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of g, S the Ricci tensor, Q the Ricci operator with respect to g and tr the trace operator.

3. Almost Kenmotsu Manifolds Satisfying the Miao-Tam Critical Condition

In this paper by a $(k, \mu)'$-*almost Kenmotsu manifold* we mean an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ such that the characteristic vector field ξ satisfies the $(k, \mu)'$-nullity condition (see Dileo and Pastore [8]), that is,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{14}$$

for any vector fields X, Y, where both k and μ are constants on M^{2n+1}.

Replacing Y by ξ in (14) and using (8) we have

$$lX = k(X - \eta(X)\xi) + \mu h'X \tag{15}$$

for any vector field X and using (15) in (10) we get

$$h'^2 X = -(k + 1)X + (k + 1)\eta(X)\xi \tag{16}$$

for any vector field $X \in \mathfrak{X}(M)$. From (16) we observe that $h' = 0$ identically if and only if $k = -1$ and $h' \neq 0$ everywhere if and only if $k < -1$. From [8, Proposition 4.1] we remark that on any non-Kenmotsu $(k, \mu)'$-almost Kenmotsu manifold there holds $\mu = -2$. Moreover, a three-dimensional almost Kenmotsu manifold is a Kenmotsu manifold if and only if $h = 0$ (see [7, Proposition 3]).

By the symmetry of the Riemannian curvature tensor R, it follows directly from (14) that

$$R(\xi, X)Y = k(g(X, Y)\xi - \eta(Y)X) + \mu(g(h'X, Y)\xi - \eta(Y)h'X) \tag{17}$$

for any $X, Y \in \mathfrak{X}(M)$. Following [8] and using relation (14), in case of $k < -1$ we denote by $[\gamma]'$ and $[-\gamma]'$ the eigenspaces of h' corresponding two eigenvalues $\gamma > 0$ and $-\gamma$, respectively. Obviously, according to relation (14), on a non-Kenmotsu $(k, \mu)'$-almost Kenmotsu manifold we have

$$\gamma = \sqrt{-k - 1} > 0.$$

Lemma 3.1 ([22, Lemma 3.2]). *Let M^{2n+1} be a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold. Then the Ricci operator of M^{2n+1} is given by*

$$Q = -2n\mathrm{id} + 2n(k+1)\eta\otimes\xi - 2nh'. \tag{18}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Notice that any Einstein metric (i.e., $S=\rho g$, ρ is a constant) on a Riemannian manifold satisfies relations (1)-(3) and (5). For a Ricci soliton, if the potential vector field V is Killing or vanishes then the soliton becomes also an Einsten metric. Therefore, a Riemannian metric satisfying (1)-(5) is usually called an *Einstein-like metric*. In this paper, we shall study a new Einstein-like metric which is different from the above ones.

Definition 3.2. *On a Riemannian manifold (M,g) if there exists a non-zero smooth function λ such that*

$$\mathrm{Hess}\lambda - (\Delta\lambda)g - \lambda S = g, \tag{19}$$

where Δ denotes the Laplacian and Hess *is the Hessian operator with respect to the metric g and S is the Ricci tensor, then g is said to satisfy the Miao-Tam critical condition.*

Obviously, if in particular the potential function λ is a non-zero constant, then (19) is just an Einstein metric. A Riemannian metric satisfying relation (19) was introduced and deeply studied by Miao and Tam [14, 15]. They proved that (19) is a necessary and sufficient condition for a metric to be critical point of the volume functional restricted to the space of constant scalar curvature metrics on a given compact manifold with boundary.

Lemma 3.3 ([15, Theorem 7]). *If the metric of a connected and smooth Riemannian manifold satisfies the Miao-Tam critical condition, then the scalar curvature is a constant.*

The Miao-Tam critical conditions were recently studied by Patra and Ghosh in [17] on some classes of contact metric manifolds. In this paper, we study these metrics on (almost) Kenmotsu manifolds and obtain

Proposition 3.4. *If the metric of a three-dimensional Kenmotsu manifold satisfies the Miao-Tam critical condition, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$.*

Proof. J. Inoguchi in [10, Proposition 3.1] proved that a three-dimensional Kenmotsu manifold of constant scalar curvature is of constant sectional curvature -1. Then the proof follows directly from Lemma 3.3. □

Proposition 3.5. *The metric of a three-dimensional non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold does not satisfy the Miao-Tam critical condition.*

Proof. Taking the trace of relation (19) we have $\Delta\lambda = -\frac{1}{2}(r\lambda+3)$. Using this and Lemma 3.1 in (19) we obtain

$$\nabla_X D\lambda = \lambda QX + fX, \text{ where } f = (2-k)\lambda - \frac{1}{2} \tag{20}$$

for any vector field X, where D is the gradient operator and we have used Lemma 3.1. Taking the covariant derivative of (20) we have

$$\nabla_X\nabla_Y D\lambda = X(\lambda)QY + \lambda\nabla_X QY + X(f)Y + f\nabla_X Y$$

for any vector fields X, Y. Thus, it follows from the above relation that

$$\begin{aligned} &R(X,Y)D\lambda \\ =&\nabla_X\nabla_Y D\lambda - \nabla_Y\nabla_X D\lambda - \nabla_{[X,Y]}D\lambda \\ =&X(\lambda)QY - Y(\lambda)QX + \lambda\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + X(f)Y - Y(f)X \end{aligned} \tag{21}$$

for any vector fields X, Y, where we have used the following relation which is deduced directly from (20):

$$\nabla_{[X,Y]}D\lambda = \lambda Q[X,Y] + f[X,Y]$$

for any vector fields X, Y.

Taking the covariant derivative of (18) along arbitrary vector field and using (9) we get

$$\begin{aligned}(\nabla_X Q)Y =& 2(k+1)\eta(Y)(X + h'X) - 2(\nabla_X h')Y \\ &+ 2(k+1)\{g(X,Y) - 2\eta(X)\eta(Y) + g(h'X,Y)\}\xi\end{aligned}$$

for any vector fields X, Y. It follows from the above equation that

$$\begin{aligned}(\nabla_X Q)Y - (\nabla_Y Q)X =& -2\{(\nabla_X h')Y - (\nabla_Y h')X\} \\ &- 2(k+1)\{\eta(X)(Y + h'Y) - \eta(Y)(X + h'X)\}\end{aligned}$$

for any vector fields X, Y. Using the above relation in (21) and substituting X with ξ we get

$$\begin{aligned}R(\xi, Y)D\lambda =& \xi(\lambda)QY - 2kY(\lambda)\xi + \xi(f)Y - Y(f)\xi \\ &- 2\lambda\{(\nabla_\xi h')Y - (\nabla_Y h'\}\xi) - 2\lambda(k+1)(Y + h'Y - \eta(Y)\xi)\end{aligned} \tag{22}$$

for any vector field Y, where we have used Lemma 3.1. Taking the inner product of the above relation with ξ gives

$$g(R(\xi, Y)D\lambda, \xi) = 2k\xi(\lambda)\eta(Y) - 2kY(\lambda) + \xi(f)\eta(Y) - Y(f) \tag{23}$$

for any vector field Y. On the other hand, it follows from relations (14) and (17) that

$$\begin{aligned}g(R(\xi, Y)D\lambda, \xi) =& -g(R(\xi, Y)\xi, D\lambda) \\ =& -k\xi(\lambda)\eta(Y) + kY(\lambda) - 2g(h'D\lambda, Y)\end{aligned} \tag{24}$$

for any vector field Y. Subtracting (24) from (23) yields

$$2h'D\lambda = -3k\xi(\lambda)\xi + 3kD\lambda + Df - \xi(f)\xi.$$

For simplicity, using the second term of (20) in the above relation we obtain

$$h'D\lambda = -(k+1)\xi(\lambda)\xi + (k+1)D\lambda, \tag{25}$$

where $k < -1$ since that the almost Kenmotsu manifold is assumed to be non-Kenmotsu. Taking the action of h' on (25) and using relation (16) we have

$$(k+2)(D\lambda - \xi(\lambda)\xi) = 0, \tag{26}$$

where we have used $k < -1$.

Now let us assume that $D\lambda = \xi(\lambda)\xi$. Using this in relation (20) and using (9) gives

$$\lambda QX = \{X(\xi(\lambda)) - \xi(\lambda)\eta(X)\}\xi + \{\xi(\lambda) + (k-2)\lambda + \frac{1}{2}\}X + \xi(\lambda)h'X$$

for any vector field X. Comparing the above relation with (18) we obtain

$$\begin{cases} X(\xi(\lambda)) - \xi(\lambda)\eta(X) = 2(k+1)\lambda\eta(X), \\ \xi(\lambda) + (k-2)\lambda + \frac{1}{2} = -2\lambda, \\ \xi(\lambda) = -2\lambda \end{cases} \tag{27}$$

for any vector field X.

Using the third term of relation (27) in the second term of (27) gives that λ is a positive constant. Applying this in the third term of (27) or the first term of (27) we obtain $\lambda = 0$, a contradiction. Therefore, it follows from (26) that $k = -2$. However, if we further consider the $\xi^{\perp}$-component of relation (22) we obtain

$$4(2+k)\lambda = 2 + (2+k)^2.$$

Using $k = -2$ in this relation gives a contradiction. This completes the proof. □

Based on the above statements, we now give our main result as the following.

Theorem 3.6. *If the metric of a three-dimensional $(k,\mu)'$-almost Kenmotsu manifold satisfies the Miao-Tam critical condition, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$.*

Proof. According to relation (16), any $(k,\mu)'$-almost Kenmotsu manifold of dimension three becomes a Kenmotsu manifold if $k = -1$ or a non-Kenmotsu almost Kenmotsu manifold if $k < -1$. Then the proof follows from Propositions 3.4 and 3.5. □

Remark 3.7. *For the existences of the Miao-Tam critical metrics on a hyperbolic space we refer the reader to [14, 15].*

From now on we study the Miao-Tam critical metric on an almost Kenmotsu manifold with $h = 0$ without dimension restriction. Firstly, we give the following definition.

On an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, if the Ricci operator satisfies

$$Q = \alpha \mathrm{id} + \beta \eta \otimes \xi,$$

where both α and β are smooth functions, then M is called an *η-Einstein manifold*. It is easily seen that an η-Einstein manifold with $\beta = 0$ and α a constant is an Einstein manifold. Then an η-Einstein metric is also an Einstein-like metric.

Pastore and Saltarelli [16] prove that the Reeb foliation on an almost Kenmotsu manifold is conformal if and only if $h = 0$. Then we have

Lemma 3.8 ([16, Theorem 5.1]). *Let M^{2n+1} be an η-Einstein almost Kenmotsu manifold of dimension greater than 3 with conformal Reeb foliation, then either the Ricci operator is given by $Q = -2n\mathrm{id}$ or β is not a constant, $X(\beta) = 0$ for any vector field orthogonal to ξ, $\xi(\beta) = -2\beta$ and in this case the Ricci operator is given by $Q = -(2n+\beta)\mathrm{id} + \beta\eta\otimes\xi$, where β is locally given by $\beta = ce^{-2t}$ for some constant $c \neq 0$.*

Using the above lemma we now obtain the following

Theorem 3.9. *If the metric of a $(2n+1)$-dimensional almost Kenmotsu manifold with conformal Reeb foliation satisfies the Miao-Tam critical condition, then either the manifold is of constant scalar curvature $r = -2n(2n+1)$ or it is Einstein.*

Proof. Taking the trace of relation (19) we have $\Delta\lambda = -\frac{1}{2n}(r\lambda + 2n + 1)$. Using this and Lemma 3.1 in (19) we obtain

$$\nabla_X D\lambda = \lambda QX + fX, \text{ where } f = -\frac{1}{2n}(r\lambda + 1) \tag{28}$$

for any vector field X and (21) still holds in this context.

Since the Reeb foliation is conformal, using $h = 0$ in relation (13) gives

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{29}$$

for any vector fields X, Y, and from this we obtain from (12) that

$$Q\xi = -2n\xi. \tag{30}$$

Substituting X with ξ in (21) gives an equation, and taking the inner product of this equation with ξ and using (30) we obtain

$$g(R(\xi, Y)D\lambda, \xi) = -\left(2n + \frac{r}{2n}\right)\xi(\lambda)\eta(Y) + \left(2n + \frac{r}{2n}\right)Y(\lambda) \tag{31}$$

for any vector field Y, where we have used Lemma 3.3 and (28). On the other hand, using (29) we obtain

$$g(R(\xi, Y)D\lambda, \xi) = -g(R(\xi, Y)\xi, D\lambda) = \xi(\lambda)\eta(Y) - Y(\lambda)$$

for any vector field Y. Comparing the above relation with (31) gives

$$\left(2n + 1 + \frac{r}{2n}\right)(D\lambda - \xi(\lambda)\xi) = 0. \tag{32}$$

Next, we suppose that $r \neq -2n(2n + 1)$ and then from (32) we have $D\lambda = \xi(\lambda)\xi$. Using this and (9) in (28) gives

$$\lambda QX = \left(\xi(\lambda) + \frac{1}{2n}(r\lambda + 1)\right)X + \{X(\xi(\lambda)) - \xi(\lambda)\eta(X)\}\xi \tag{33}$$

for any vector field X. Taking into account $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$ for any vector fields X, Y and $D\lambda = \xi(\lambda)\xi$, we obtain directly from (33) that

$$\lambda QX = \left(\xi(\lambda) + \frac{1}{2n}(r\lambda + 1)\right)X + \{\xi(\xi(\lambda)) - \xi(\lambda)\}\eta(X)\xi \tag{34}$$

for any vector field X and this implies that the manifold is η-Einstein.

Next, we assume that the manifold is not Einstein. From Lemma 3.8 we see that the second case occurs, i.e.,

$$QX = -(2n + \beta)X + \beta\eta(X)\xi \text{ and } \mathrm{d}\beta = -2\beta\eta \tag{35}$$

for any vector field X, where β is locally given by $\beta = ce^{-2t}$ for some constant $c \neq 0$. Comparing the first term of (35) with (34) yields

$$\xi(\xi(\lambda)) = -\frac{1}{2n}(r\lambda + 1) - 2n\lambda. \tag{36}$$

Taking the trace of relation (34) we have

$$\xi(\xi(\lambda)) + \frac{1}{2n}(r\lambda + 1) + 1 + 2n\xi(\lambda) = 0.$$

Subtracting the above relation from (36) we obtain

$$\xi(\lambda) = \lambda - \frac{1}{2n}. \tag{37}$$

Finally, using (37) in relation (36) we obtain

$$(2n + 1 + \frac{r}{2n})\lambda = 0.$$

In view of λ being a non-zero function, it follows from the above equation that $r = -2n(2n + 1)$, a contradiction. Hence, according to Lemma 3.8 we conclude that the manifold is Einstein. This completes the proof. □

Remark 3.10. *Theorem 3.9 is in fact a generalization of Proposition 3.4 since that Lemma 3.8 holds even on a three-dimensional Kenmotsu manifold (see [16, Remark 5.1]).*

The Reeb foliation of any Kenmotsu manifold of dimension ≥ 3 must be conformal, but the converse is not necessarily true. Then from Theorem 3.9 we obtain

Corollary 3.11. *If the metric of a $(2n+1)$-dimensional Kenmotsu manifold satisfies the Miao-Tam critical condition, then either the manifold is of constant scalar curvature $r = -2n(2n+1)$ or it is Einstein.*

K. Kenmotsu in [12, Proposition 3] constructed a Kenmotsu structure on a warped product $L \times_{ce^t} K$ of a real line L and a Kähler manifold K, then the following corollary follows from Corollary 3.11.

Corollary 3.12. *If the metric of a $(2n+1)$-dimensional warped product $L \times_{ce^t} K$ satisfies the Miao-Tam critical condition, then either the manifold is of constant scalar curvature $r = -2n(2n+1)$ or it is Einstein.*

Acknowledgement. The authors would like to thank the anonymous referees for their valuable suggestions that have improved the original paper.

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