Filomat 31:16 (2017), 5177–5191
https://doi.org/10.2298/FIL1716177L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Perturbation Bound for the Generalized Drazin Inverse of an Operator in Banach Space

Xiaoji Liu[a], Yonghui Qin[a,b]

[a]*College of Science, Guangxi University for Nationalities, Nanning 530006, P.R. China.*
[b]*College of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin, 541004, P.R. China.*

Abstract. In this paper, we consider perturbation analysis for the generalized Drazin inverse of an operator in Banach space. An necessary and sufficient condition for the generalized Drazin invertible is given. The upper bound is given under some certain conditions, and a relative perturbation bound is also considered.

1. Introduction

The Drazin inverse arises in many fields, such as, differential equations, difference equations, Markov chains, and control theory [1, 2]. The perturbation analysis for the Drazin inverse is useful in computational mathematics [2, 7]. In recent years, many results on the perturbation bound for the Drazin inverse of a given matrix or operator have been considered in [3, 4, 24]. In [3], Castro and Koliha considered the perturbation for the Drazin inverse of a closed linear operator under some certain conditions, and also obtained an upper bound. In [5], Castro et al. studied perturbations for the Drazin inverse of a closed linear operator A, when the perturbing operator has the same spectral projection as A. In [20], Martínez and Castro considered the Drazin inverse of block matrices. Cvetković-Ilić investigated the generalized Drazin inverse with commutativity up to a factor in a Banach algebra in [8]. Xu et al. considered the stable perturbation of the Drazin inverse of the square matrices in [30]. In [10], Deng and Wei considered the perturbation for the generalized Drazin inverse of a bounded linear operator. They also gave an explicit generalized Drazin inverse expression for the perturbation under certain restrictions on the perturbing matrices. In [22], Rakočević and Wei investigated perturbation for the generalized Drazin inverse of a bounded linear operator over Banach space. Castro and Martínez studied additive properties for the generalized Drazin inverse in Banach algebra [6]. In [16], Huang et al. considered stable perturbations for outer inverses of linear operators in Banach space.

Let X, Y be Banach space and let $B(X,Y)$ be the set of all bounded linear operators from X to Y. If $X = Y$, then $B(X,Y) = B(X)$. For a bounded linear operator $A \in B(X)$, the symbols $R(A)$, $N(A)$, $\sigma(A)$ and $r(A)$ denote the range, the null space, the spectrum and the spectral radius of A, respectively. For any $A \in B(X)$, there exists B such that, for some $k \in \mathbb{N}$,

$$BAB = B,\ AB = BA,\ A^{k+1}X = A^k, \tag{1}$$

2010 *Mathematics Subject Classification*. 15A09, 15A23, 65F35
Keywords. Banach space, generalized Drazin inverse, perturbation bound
Received: 06 June 2015; Accepted: 07 June 2017
Communicated by Predrag Stanimirović
Research supported by the National Natural Science Foundation of China (11061005) and supported by open fund of Guangxi Key laboratory of hybrid computation and IC design analysis (HCIC201607).
Email addresses: xiaojiliu72@126.com. (Xiaoji Liu), yonghui1676@163.com (Yonghui Qin)

then B is called the Drazin inverse of A, and is denoted by $B = A^D$. The smallest nonnegative k in (1) is called the index of A, denoted by $\mathrm{Ind}(A)$. It is well-known that $A \in B(X)$ has a Drazin inverse if and only if the point $\lambda = 0$ is a pole of the resolvent $\lambda \mapsto (\lambda I - A)^{-1}$. The order of this pole is equal to $\mathrm{Ind}(A)$.

In [17], Koliha introduced the concept for a generalized Drazin (GD) inverse of an operator $A \in B(X)$. The GD inverse exists if and only if $0 \notin \mathrm{acc}\sigma(A)$. If $0 \notin \mathrm{acc}\sigma(A)$, then there exist open subsets U and V of $\mathbb{C}$, such that $\sigma(A) \setminus \{0\} \subset U$, $0 \in V$, and $U \cap V = \emptyset$. Define a function f as in [17] by

$$f(\lambda) = \begin{cases} 0 & \lambda \in V \\ \dfrac{1}{\lambda} & \lambda \in U \end{cases}.$$

The GD inverse of A is defined by $f(A) = A^{\mathrm{d}}$. If A is GD-invertible, then the spectral idempotent of A corresponding to 0 is denoted by $A^\pi = I - AA^{\mathrm{d}}$. Let $X = N(A^\pi) \oplus R(A^\pi)$, where $A \in B(X)$ is GD-invertible. If A is GD-invertible, then A has the following form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} N(A^\pi) \\ R(A^\pi) \end{bmatrix} \longrightarrow \begin{bmatrix} N(A^\pi) \\ R(A^\pi) \end{bmatrix}, \tag{2}$$

where A_1 is invertible and A_2 is quasinilpotent. Also, the GD inverse A^{d} is given by

$$A^{\mathrm{d}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N(A^\pi) \\ R(A^\pi) \end{bmatrix} \longrightarrow \begin{bmatrix} N(A^\pi) \\ R(A^\pi) \end{bmatrix}. \tag{3}$$

The motivation of the results in this paper is from those given in [10], which is considered the perturbation of the GD inverse in the following cases:

- $A_1 + Q_{11}$ is invertible and $\dim[R(A^\pi)]$ is finite,
- $A_1 + Q_{11}$ is invertible and $Q_{22}A_2 = 0$,
- $A_1 + Q_{11}$ is invertible and $Q_{22}A_2 = A_2Q_{22}$,

where Q is the perturbing operator of A and given by $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix}$ with respect to the decomposition $X = N(A^\pi) \oplus R(A^\pi)$.

Some results on the perturbation for the GD inverse with suitable certain conditions are given in [4, 13, 19, 23–29] as follows:

- $\|I + A_1^{-1}Q_{11}\| < 1$, $A_2 + Q_{22}$ is quasinilpotent, $Q_{12} = 0$.
- $A_1 + Q_{11}$ is invertible, Q_{22} is quasinilpotent, $Q_{22}A_2 = A_2Q_{22}$, $Q_{12} = 0$.
- $Q_{11} = 0$, Q_{22} is quasinilpotent, $Q_{22}A_2 = 0$.
- $\|I + A_1^{-1}Q_{11}\| < 1$, $Q_{12} = 0$, $Q_{22} = 0$.

In this paper, we consider perturbation bound for $\|(A + Q)^{\mathrm{d}} - A^{\mathrm{d}}\|$ with $A_1 + Q_{11}$ being invertible and one of the conditions being given as bellow:

(i) $A_2Q_{22} = 0$, $Q_{22}^2 = 0$; (ii) $A_2Q_{22}^2 = 0$, $A_2^2 = 0$; (iii) $Q_{22}^2 = Q_{22}$.

Also, we give a relative perturbation bound for $\frac{\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|}$.

Lemma 1.1. [14] *Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ be a bounded linear operator on $X \oplus Y$. If $\sigma(A) \cap \sigma(C)$ has no-interior point, then $\sigma(M) = \sigma(A) \cup \sigma(C)$.*

Lemma 1.2. [10] *Let $A, Q \in B(X)$ and let $\sigma_\epsilon(A) = \{\lambda : \mathrm{dist}(\lambda, \sigma(A)) < \epsilon\}$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma(A + Q) \subset \sigma_\epsilon(A)$, whenever $\|Q\| < \delta$.*

The following lemma is given by Theorem 5.1 in [12] and Lemma 2.2 in [13] (see [15] and [21] for the finite dimensional case).

Lemma 1.3. *If $A \in B(X)$ and $B \in B(Y)$ are GD-invertible, then*

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \quad N = \begin{bmatrix} B & 0 \\ C & A \end{bmatrix}$$

are GD-invertible and

$$M^{\mathrm{d}} = \begin{bmatrix} A^{\mathrm{d}} & S \\ 0 & B^{\mathrm{d}} \end{bmatrix}, \ N^{\mathrm{d}} = \begin{bmatrix} B^{\mathrm{d}} & 0 \\ S & A^{\mathrm{d}} \end{bmatrix}, \tag{4}$$

where $S = \sum_{n=0}^{\infty}(A^{\mathrm{d}})^{n+2}CB^nB^{\pi} + \sum_{n=0}^{\infty} A^{\pi}A^nC(B^{\mathrm{d}})^{n+2} - A^{\mathrm{d}}CB^{\mathrm{d}}$.

Lemma 1.4. [18] *Let $A \in B(X)$ be Drazin invertible. If $r(A) > 0$, then*

$$\mathrm{dist}(0, \sigma(A)\backslash\{0\}) = (r(A^{\mathrm{d}}))^{-1}.$$

Lemma 1.5. [11] *Let $A, Q \in B(X)$ be GD-invertible and $AQ = 0$, then $A + Q$ is GD-invertible and*

$$(A + Q)^{\mathrm{d}} = Q^{\pi} \sum_{n=0}^{\infty} Q^n(A^{\mathrm{d}})^{n+1} + \sum_{n=0}^{\infty}(Q^{\mathrm{d}})^{n+1}A^nA^{\pi}.$$

Lemma 1.6. [9] *Let $A, Q \in B(X)$ be GD-invertible and $AQ = QA$. Then $A + Q$ is GD-invertible if and only if $1 + A^{\mathrm{d}}Q$ is GD-invertible. In this case, we have*

$$(A + Q)^{\mathrm{d}} = A^{\mathrm{d}}(1 + A^{\mathrm{d}}Q)^{\mathrm{d}}QQ^{\mathrm{d}} + Q^{\pi} \sum_{n=0}^{\infty}(-Q)^n(A^{\mathrm{d}})^{n+1} + \sum_{n=0}^{\infty}(Q^{\mathrm{d}})^{n+1}(-A)^nA^{\pi}$$

and

$$(A + Q)(A + Q)^{\mathrm{d}} = (AA^{\mathrm{d}} + QA^{\mathrm{d}})(1 + A^{\mathrm{d}}Q)QQ^{\mathrm{d}} + Q^{\pi}AA^{\mathrm{d}} + QQ^{\mathrm{d}}A^{\pi}.$$

2. Perturbation Bound for the GD Inverse of an Operator

In this section, we consider the perturbation for the GD inverse of an operator $A \in B(X)$. We present a necessary and sufficient condition for the GD invertible under certain conditions. An upper bound for $\|(A + Q)^{\mathrm{d}} - A^{\mathrm{d}}\|$ is given, and a relative perturbation bound for $\frac{\|(A+Q)^{\mathrm{d}}-A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|}$ is also considered, where Q is perturbing operator.

Theorem 2.1. *Let $A \in B(X)$ be GD-invertible and $Q \in B(X)$. Assume that $A^2Q = 0$, then $A + Q$ is GD-invertible if and only if $(A + Q)A^{\pi}$ is GD-invertible. If AQ is GD-invertible and $Q^2 = 0$, then*

(i) *$A + Q$ is GD invertible and*

$$(A + Q)^{\mathrm{d}} = \sum_{j=0}^{\infty} \left(Q(AQ)^{\pi}(AQ)^jA^{\mathrm{d}} + (AQ)^{\pi}(AQ)^j\right)(A^{\mathrm{d}})^{2j+1} + \sum_{j=0}^{\infty} \left(Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A\right)A^{2j}A^{\pi}.$$

(ii) *Further, if $\|AQ\|\|A^{\mathrm{d}}\|^2 < 1$, $\|(AQ)^{\mathrm{d}}\|\|A^2\| < 1$, then*

$$\frac{\|(A + Q)^{\mathrm{d}} - A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|} \le \|Q(AQ)^{\pi}A^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}AQ\| + \frac{\|(AQ)^{\pi}\|\|AQ\|\|A^{\mathrm{d}}\|^2}{1 - \|(AQ)\|\|A^{\mathrm{d}}\|^2}\left(\|Q\|\|A^{\mathrm{d}}\| + 1\right) + \frac{\|(AQ)^{\mathrm{d}}\|\|A\|}{\mathcal{K}(A)(1 - \|(AQ)^{\mathrm{d}}\|\|A^2\|)}\left(\|Q\|\|A^{\pi}\| + \|AA^{\pi}\|\right),$$

where $\mathcal{K}(A) = \|A\|\|A^{\mathrm{d}}\|$.

Proof. Suppose that A, A^{d} are given by (2), (3), respectively. Thus, we have

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} : \begin{bmatrix} R(A^{\pi}) \\ N(A^{\pi}) \end{bmatrix} \longrightarrow \begin{bmatrix} R(A^{\pi}) \\ N(A^{\pi}) \end{bmatrix}. \tag{5}$$

Since $A^2Q = 0$ and A_1 is invertible, we have

$$Q_{11} = 0,\ Q_{12} = 0,\ A_2^2Q_{21} = 0,\ A_2^2Q_{22} = 0. \tag{6}$$

Thus, we derive

$$A + Q = \begin{bmatrix} A_1 & 0 \\ Q_{21} & A_2 + Q_{22} \end{bmatrix}.$$

By Lemma 1.3, we can prove that $(A + Q)^{\mathrm{d}}$ exists if and only if $(A_2 + Q_{22})^{\mathrm{d}}$ exists. i.e. $(A + Q)^{\mathrm{d}}$ exists if and only if $(A + Q)A^{\pi}$ is GD-invertible.

From $Q^2 = 0$, we get

$$Q_{22}Q_{21} = 0,\ Q_{22}^2 = 0. \tag{7}$$

By (6) and (7), we have

$$(A_2Q_{22})(Q_{22}A_2) = (Q_{22}A_2)(A_2Q_{22}) = 0. \tag{8}$$

From the GD-invertibility of AQ, then $(A_2Q_{22})^{\mathrm{d}}$ exists. Hence, $(Q_{22}A_2)^{\mathrm{d}}$ exits. By Lemma 1.6, we get

$$(A_2Q_{22} + Q_{22}A_2)^{\mathrm{d}} = (A_2Q_{22})^{\mathrm{d}} + (Q_{22}A_2)^{\mathrm{d}}. \tag{9}$$

Since $A_2^2(A_2Q_{22} + Q_{22}A_2) = 0$ and A_2 is quasinilpotent. By Lemma 1.5 and (9), we have that $A_2 + Q_{22}$ is GD-invertible and

$$\begin{aligned}
[(A_2 + Q_{22})^2]^{\mathrm{d}} &= (A_2^2 + A_2Q_{22} + Q_{22}A_2)^{\mathrm{d}} \\
&= \sum_{n=0}^{\infty}[(A_2Q_{22} + Q_{22}A_2)^{\mathrm{d}}]^{n+1}A_2^{2n} \\
&= \sum_{n=0}^{\infty}\left((A_2Q_{22})^{\mathrm{d}} + (Q_{22}A_2)^{\mathrm{d}}\right)^{n+1}A_2^{2n}. \qquad (10)
\end{aligned}$$

Similarly, we obtain

$$[(A_2Q_{22})^{\mathrm{d}} + (Q_{22}A_2)^{\mathrm{d}}]^{n+1} = [(A_2Q_{22})^{\mathrm{d}}]^{n+1} + [(Q_{22}A_2)^{\mathrm{d}}]^{n+1},\ n \in \mathbb{N}.$$

By (10) and Cline's formula $(Q_{22}A_2)^{\mathrm{d}} = Q_{22}\Big((A_2Q_{22})^2\Big)^{\mathrm{d}} A_2$, we obtain

$$
\begin{aligned}
(A_2+Q_{22})^{\mathrm{d}} &= \Big((A_2+Q_{22})^{\mathrm{d}}\Big)^2 (A_2+Q_{22}) \\
&= (A_2+Q_{22})\sum_{n=0}^{\infty}\Big((A_2Q_{22})^{\mathrm{d}} + (Q_{22}A_2)^{\mathrm{d}}\Big)^{n+1} A_2^{2n} \\
&= (A_2+Q_{22})\sum_{n=0}^{\infty}((A_2Q_{22})^{\mathrm{d}})^{n+1} + ((Q_{22}A_2)^{\mathrm{d}})^{n+1}A_2^{2n} \\
&= \sum_{n=0}^{\infty}\Big(Q_{22}((A_2Q_{22})^{\mathrm{d}})^{n+1} + A_2((Q_{22}A_2)^{\mathrm{d}})^{n+1}\Big)A_2^{2n} \\
&= \sum_{n=0}^{\infty}\Big(Q_{22}((A_2Q_{22})^{\mathrm{d}})^{n+1} + A_2[Q_{22}((A_2Q_{22})^{\mathrm{d}})^2A_2]^{n+1}\Big)A_2^{2n} \\
&= \sum_{n=0}^{\infty}\Big(Q_{22}((A_2Q_{22})^{\mathrm{d}})^{n+1} + ((A_2Q_{22})^{\mathrm{d}})^{n+1}A_2\Big)A_2^{2n} \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
(A_2+Q_{22})^{\pi} &= (A_2Q_{22})^{\pi} - \sum_{n=1}^{\infty} A_2Q_{22}\Big((A_2Q_{22})^{\mathrm{d}}\Big)^{n+1} A_2^{2n} \\
&\quad - \sum_{n=0}^{\infty} Q_{22}\Big((A_2Q_{22})^{\mathrm{d}}\Big)^{n+1} A_2^{2n+1}. \qquad (12)
\end{aligned}
$$

By Lemma 1.3, we get that $A+Q$ is GD-invertible and

$$
(A+Q)^{\mathrm{d}} = \begin{bmatrix} A_1^{-1} & 0 \\ R & (A_2+Q_{22})^{\mathrm{d}} \end{bmatrix}, \qquad (13)
$$

where

$$
R = \sum_{n=0}^{\infty}(A_2+Q_{22})^{\pi}(A_2+Q_{22})^{n}Q_{21}(A_1^{-1})^{n+2} - (A_2+Q_{22})^{\mathrm{d}}Q_{21}A_1^{-1}.
$$

Thus, for $n \geq 2$, we obtain

$$
(A_2+Q_{22})^{n}Q_{21} = \begin{cases} (Q_{22}A_2)^{n/2}Q_{21}, & \text{if } n \text{ is even} \\ A_2(Q_{22}A_2)^{(n-1)/2}Q_{21}, & \text{if } n \text{ is odd} \end{cases}. \qquad (14)
$$

By (6) and (12), we have

$$
\begin{aligned}
&(A_2+Q_{22})^{\pi}Q_{21} = (I - Q_{22}(A_2Q_{22})^{\mathrm{d}}A_2)Q_{21}, \\
&(A_2+Q_{22})^{\pi}Q_{22} = Q_{22}(A_2Q_{22})^{\pi}, \\
&(A_2+Q_{22})^{\pi}A_2Q_{22} = (A_2Q_{22})^{\pi}A_2Q_{22}, \\
&(A_2+Q_{22})^{\pi}A_2Q_{21} = (A_2Q_{22})^{\pi}A_2Q_{21}. \qquad (15)
\end{aligned}
$$

Note that

$$(AQ)^{\pi}A^{\mathrm{d}} = \begin{bmatrix} A_1^{-1} & 0 \\ -(A_2Q_{22})^{\mathrm{d}}A_2Q_{21}A_1^{-1} & 0 \end{bmatrix},$$

$$Q(AQ)^{\pi}(A^{\mathrm{d}})^2 = \begin{bmatrix} 0 & 0 \\ Q_{21}(A_1^{-1})^2 - Q_{22}(A_2Q_{22})^{\mathrm{d}}A_2Q_{21}(A_1^{-1})^2 & 0 \end{bmatrix},$$

$$\sum_{j=1}^{\infty}\Big(Q(AQ)^{\pi}(AQ)^{j}A^{\mathrm{d}} + (AQ)^{\pi}(AQ)^{j}\Big) \tag{16}$$

$$= \begin{bmatrix} 0 & 0 \\ \begin{matrix}\sum_{j=1}^{\infty}\Big(Q_{22}(A_2Q_{22})^{\pi}(A_2Q_{22})^{j-1}A_2Q_{21}A_1^{-1} \\ +(A_2Q_{22})^{\pi}(A_2Q_{22})^{j-1}A_2Q_{21}\Big)\end{matrix} & \sum_{j=1}^{\infty}(A_2Q_{22})^{\pi}(A_2Q_{22})^{j} \end{bmatrix}.$$

From (13) – (16), we have

$$R = \Big(I - Q_{22}(A_2Q_{22})^{\mathrm{d}}A_2\Big)Q_{21}(A_1^{-1})^2 + \sum_{j=1}^{\infty} Q_{22}(A_2Q_{22})^{\pi}(A_2Q_{22})^{j-1}A_2Q_{21}(A_1^{-1})^{2j+2}$$

$$+ \sum_{j=1}^{\infty}(A_2Q_{22})^{\pi}(A_2Q_{22})^{j-1}A_2Q_{21}(A_1^{-1})^{2j+1} - (A_2Q_{22})^{\mathrm{d}}A_2Q_{21}A_1^{-1}.$$

Hence,

$$\begin{bmatrix} A_1^{-1} & 0 \\ R & 0 \end{bmatrix} = \sum_{j=0}^{\infty}\Big(Q(AQ)^{\pi}(AQ)^{j}A^{\mathrm{d}} + (AQ)^{\pi}(AQ)^{j}\Big)(A^{\mathrm{d}})^{2j+1}. \tag{17}$$

Now, by (13) and (17), we have

$$(A+Q)^{\mathrm{d}} = \sum_{j=0}^{\infty}(Q(AQ)^{\pi})(AQ)^{j}A^{\mathrm{d}} + (AQ)^{\pi}(AQ)^{j})(A^{\mathrm{d}})^{2j+1}$$

$$+ \sum_{j=0}^{\infty}(Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A)A^{2j}A^{\pi}. \tag{18}$$

i.e. (i) holds.

By (18), we obtain

$$\begin{aligned}(A+Q)^{\mathrm{d}} - A^{\mathrm{d}} &= \sum_{j=0}^{\infty}\Big(Q(AQ)^{\pi}(AQ)^{j}A^{\mathrm{d}} + (AQ)^{\pi}(AQ)^{j}\Big)(A^{\mathrm{d}})^{2j+1} \\ &+ \sum_{j=0}^{\infty}\Big(Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A\Big)A^{2j}A^{\pi} - A^{\mathrm{d}} \\ &= Q(AQ)^{\pi}(A^{\mathrm{d}})^2 - (AQ)^{\mathrm{d}}(AQ)A^{\mathrm{d}} \\ &+ \sum_{j=1}^{\infty}\Big(Q(AQ)^{\pi}(AQ)^{j}A^{\mathrm{d}} + (AQ)^{\pi}(AQ)^{j}\Big)(A^{\mathrm{d}})^{2j+1} \\ &+ \sum_{j=0}^{\infty}\Big(Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A\Big)A^{2j}A^{\pi}.\end{aligned} \tag{19}$$

By (19), it leads to

$$\begin{aligned}
\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\| &\leq \|Q(AQ)^{\pi}(A^{\mathrm{d}})^2\| + \|(AQ)^{\mathrm{d}} AQA^{\mathrm{d}}\| \\
&+ \sum_{j=1}^{\infty} \|(AQ)^{\pi}\| \left(\|AQ\|\|A^{\mathrm{d}}\|^2\right)^j \left(\|Q\|\|A^{\mathrm{d}}\|^2 + \|A^{\mathrm{d}}\|\right) \\
&+ \sum_{j=0}^{\infty} \|Q\|\|(AQ)^{\mathrm{d}}\| \left(\|(AQ)^{\mathrm{d}}\|\|A^2\|\right)^j \|A^{\pi}\| \\
&+ \sum_{j=0}^{\infty} \|(AQ)^{\mathrm{d}}\| \left(\|(AQ)^{\mathrm{d}}\|\|A^2\|\right)^j \|AA^{\pi}\|.
\end{aligned}$$

If $\|AQ\|\|A^{\mathrm{d}}\|^2 < 1$ and $\|(AQ)^{\mathrm{d}}\|\|A^2\| < 1$, then

$$\begin{aligned}
\frac{\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|} &\leq \|Q(AQ)^{\pi}A^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}AQ\| \\
&+ \frac{\|(AQ)^{\pi}\|\|AQ\|}{1 - \|AQ\|\|A^{\mathrm{d}}\|^2}\left(\|Q\|\|A^{\mathrm{d}}\|^3 + \|A^{\mathrm{d}}\|^2\right) \\
&+ \frac{\|(AQ)^{\mathrm{d}}\|\|A\|}{\mathcal{K}(A)(1 - \|(AQ)^{\mathrm{d}}\|\|A^2\|)}\left(\|Q\|\|A^{\pi}\| + \|AA^{\pi}\|\right),
\end{aligned} \tag{20}$$

where $\mathcal{K}(A) = \|A\|\|A^{\mathrm{d}}\|$. Thus, (ii) holds. □

Corollary 2.2. *Let $A, Q \in B(X)$ and Q be GD-invertible. Assume that $AQ^2 = 0$, then $A + Q$ is GD-invertible if and only if $Q^{\pi}(A + Q)$ is GD-invertible. If $Q^{\pi}AQ$ is GD-invertible and $A^2 = 0$, then*

(i) *$A + Q$ is invertible and*

$$\begin{aligned}
(A+Q)^{\mathrm{d}} = &\sum_{j=0}^{\infty} (Q^{\mathrm{d}})^{2j+1} \left[(AQ)^{\pi}(AQ)^j + Q^{\mathrm{d}}(AQ)^{\pi}(AQ)^j\right] \\
&+ \sum_{j=0}^{\infty} Q^{\pi}Q^{2j} \left[Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A\right].
\end{aligned}$$

(ii) *Further, if $\|AQ\|\|Q^{\mathrm{d}}\|^2 < 1$, $\|(AQ)^{\mathrm{d}}\|\|Q^2\| < 1$, then*

$$\begin{aligned}
\frac{\|(A+Q)^{\mathrm{d}} - Q^{\mathrm{d}}\|}{\|Q^{\mathrm{d}}\|} \leq &\|(AQ)^{\pi}Q^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}AQ\| \\
&+ \frac{\|(AQ)^{\pi}\|\|AQ\|}{1 - \|AQ\|\|A^{\mathrm{d}}\|^2}\left[\|Q^{\mathrm{d}}\|^3 + \|Q^{\mathrm{d}}\|^2\right] \\
&+ \frac{\|(AQ)^{\mathrm{d}}\|\|Q\|}{\mathcal{K}(Q)(1 - \|(AQ)^{\mathrm{d}}\|\|A^2\|)}\left[\|A\|\|Q^{\pi}\| + \|Q^{\pi}Q\|\right],
\end{aligned}$$

where $\mathcal{K}(Q) = \|Q\|\|Q^{\mathrm{d}}\|$.

Theorem 2.3. *Let $A \in B(X)$ be GD-invertible and any $Q \in B(X)$. Assume that $A^{\pi}Q(I - A^{\pi}) = 0$ and $A^2A^{\pi}Q = 0$, then there exists a constant $\delta > 0$ such that $A + Q$ is GD-invertible, when $\|Q\| < \delta$, if and only if $((A + Q)A^{\pi})^{\mathrm{d}}$ exists. If $A^{\pi}Q$ is GD-invertible and $A^{\pi}Q^2 = 0$, then*

(i) $(A+Q)^{\mathrm{d}} = v + w - vQw + \left\{\sum_{n=0}^{\infty} v^{n+2}AA^{\mathrm{d}}Q[A^{\pi}(A+Q)]^n\right\} \times [I - A^{\pi}(A+Q)w]$,

(ii) *Further, if* $\|A^{\mathrm{d}}QAA^{\mathrm{d}}\| < 1$ *and* $\|(AQ)^{\mathrm{d}}\|\|A^2\| < 1$, *then*

$$\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\| \le \delta_2(1+\delta_1\|Q\|) + \frac{\delta_1}{\|A^{\mathrm{d}}Q\|} + \delta_3,$$

$$\frac{\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|} \le \frac{\mathcal{K}(A)\delta_2\|A^{\mathrm{d}}Q\|(1+\|A^{\pi}(A+Q)\|_2)}{(1-\|A^{\mathrm{d}}Q\|)^2}\sum_{n=0}^{\infty}\left(\frac{\delta_1\|A^{\pi}(A+Q)\|}{\|A^{\mathrm{d}}Q\|}\right)^n + \delta_2\left(\frac{\|A\|}{\mathcal{K}(A)} + \frac{\|A^{\mathrm{d}}Q\|\|Q\|}{1-\|A^{\mathrm{d}}Q\|}\right) + \frac{\|A^{\mathrm{d}}\|}{1-\|A^{\mathrm{d}}Q\|} + \frac{\delta_2\|A^{\mathrm{d}}\|\|Q\|}{1-\|A^{\mathrm{d}}Q\|},$$

where $\mathcal{K}(A) = \|A^{\mathrm{d}}\|\|A\|$ *and*

$$v = (I + A^dQ)^{-1}A^{\mathrm{d}},\ \delta_1 = \frac{\|A^{\mathrm{d}}\|\|A^{\mathrm{d}}Q\|}{1-\|A^{\mathrm{d}}Q\|},$$

$$w = \sum_{j=0}^{\infty}\left[Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A\right]A^{2j}A^{\pi}(AA^{\pi})^{\pi},$$

$$\delta_2 = \frac{\|A^{\pi}\|\|(AA^{\pi})^{\pi}\|\left[\|Q(AQ)^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}\|\|A\|\right]}{1-\|(AQ)^{\mathrm{d}}\|\|A^2\|},$$

$$\delta_3 = \frac{\delta_1^2(1+\|A^{\pi}(A+Q)\|_2\delta_2)\|A\|}{\|A^{\mathrm{d}}Q\|}\sum_{n=0}^{\infty}\left(\frac{\delta_1\|A^{\pi}(A+Q)\|}{\|A^{\mathrm{d}}Q\|}\right)^n + \frac{\delta_1\delta_2\|Q\|}{\|A^{\mathrm{d}}Q\|}.$$

Proof. Suppose that A, A^{d} and Q are given by (2), (3), and (5), respectively. If A is invertible when $\|Q\| < \delta$, then $\mathrm{Ind}(A) = 0$ and $A^{\pi} = 0$. From Lemma 1.2, there exists a constant $\delta > 0$ such that $A+Q$ is invertible. Thus, $(A+Q)^{-1} = (I + A^{-1}Q)^{-1}A^{-1}$ and $\mathrm{Ind}(A+Q) = 0$. Thus, we get the results. If A is quasinilpotent, then $A^{\pi} = 1$, $A^2A^{\pi}Q = A^2Q = 0$, and $A^{\pi}Q^2 = Q^2 = 0$. Thus, (i) holds by using Theorem 2.1.

Next, we consider nontrivial cases. By $A^{\pi}Q(I - A^{\pi}) = 0$, we have $Q_{21} = 0$ and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix},\ A+Q = \begin{bmatrix} A_1 + Q_{11} & Q_{12} \\ 0 & A_2 + Q_{22} \end{bmatrix} \tag{21}$$

with respect to the decomposition $X = N(A^{\pi}) \oplus R(A^{\pi})$. Let A be GD-invertible with $\mathrm{ind}(A) > 0$ and $\sigma(A) \neq \{0\}$, i.e. A is neither invertible nor quasinilpotent. By Lemma 1.1, $\sigma(A) = \sigma(A_1) \cup \{0\}$ and by Lemma 1.4, we have

$$\mathrm{dist}(0, \sigma(A)\backslash\{0\}) = (r(A^{\mathrm{d}}))^{-1} > 0.$$

Now, we can conclude that there exist two disjoint closed subsets M_1 and M_2 such that $\sigma_{\epsilon}(A_1) = \{\lambda : \mathrm{dist}(\lambda, \sigma(\mathrm{A}_1)) < \epsilon\} \subset \mathrm{M}_1$ and $\sigma_{\epsilon}(A_2) = \{\lambda : \mathrm{dist}(\lambda, \sigma(\mathrm{A}_2)) < \epsilon\} \subset \mathrm{M}_2$ for small enough $\epsilon > 0$. Applying Lemma 1.2 and (21), for some constant $\delta > 0$ and $\|Q\| < \delta$, we get

$$\sigma(A_1 + Q_{11}) \subset \sigma_{\epsilon}(A_1) \subset M_1,\ \ \sigma(A_2 + Q_{22}) \subset \sigma_{\epsilon}(A_2) \subset M_2.$$

It shows that $\|Q_{11}\| < \delta$ and $\|Q_{22}\| < \delta$. Note that

$$\sigma(A_1 + Q_{11}) \cap \sigma(A_2 + Q_{22}) = \emptyset.$$

By Lemma 1.2, we conclude that

$$\sigma(A+Q) = \sigma(A_1 + Q_{11}) \cup \sigma(A_2 + Q_{22})$$

and there always exists a $\delta > 0$ such that $A_1 + Q_{11}$ is invertible. By using (21) and Lemma 1.3, we note that $(A+Q)^{\mathrm{d}}$ exists if and only if $(A_2 + Q_{22})^{\mathrm{d}}$ exists. i.e. $(A^{\pi}(A+Q))^{\mathrm{d}}$ exists.

Now, we give the proof of (i). Assume that $A^2A^{\pi}Q = 0$, implies $A_2^2Q_{22} = 0$. Since $A^{\pi}Q^2 = 0$, we get that $Q_{22}^2 = 0$, $Q_{22}^{\mathrm{d}} = 0$. From the existence of $(A^{\pi}AQ)^{\mathrm{d}}$, we have that $(A_2Q_{22})^{\mathrm{d}}$ exists. By Cline's formula, we have $(Q_{22}A_2)^{\mathrm{d}} = Q_{22}[(A_2Q_{22})^2]^{\mathrm{d}}A_2$. If $A^{\pi}Q$ is GD-invertible and $A_2^2Q_{22} = 0$, $Q_{22}^2 = 0$, by Theorem 2.1, we get

$$(A_2 + Q_{22})^{\mathrm{d}} = \sum_{j=0}^{\infty}(Q_{22}((A_2Q_{22})^{\mathrm{d}})^{j+1} + ((A_2Q_{22})^{\mathrm{d}})^{j+1}A_2)A_2^{2j}A_2^{\pi}. \tag{22}$$

Using Lemma 1.3, we have that $B = A + Q$ is GD-invertible and

$$(A+Q)^{\mathrm{d}} = \begin{bmatrix} (A_1 + Q_{11})^{-1} & Y \\ 0 & (A_2 + Q_{22})^{\mathrm{d}} \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} Y &= \sum_{n=0}^{\infty}(A_1 + Q_{11})^{-(n+2)}Q_{12}(A_2 + Q_{22})^n(A_2 + Q_{22})^{\pi} \\ &\quad -(A_1 + Q_{11})^{-1}Q_{12}(A_2 + Q_{22})^{\mathrm{d}}. \end{aligned}$$

Note that

$$(A_1 + Q_{11})^{-1} \oplus 0 = (I + A_1^{-1}Q_{11})A_1^{-1} \oplus 0 = (I + A^{\mathrm{d}}Q)^{-1}A^{\mathrm{d}} = v. \tag{24}$$

Thus, we have

$$\begin{aligned} [A^{\pi}(A+Q)]^{\mathrm{d}} &= 0 \oplus (A_2 + Q_{22})^{\mathrm{d}} \\ &= 0 \oplus \sum_{j=0}^{\infty}(Q_{22}((A_2Q_{22})^{\mathrm{d}})^{j+1} + ((A_2Q_{22})^{\mathrm{d}})^{j+1}A_2)A_2^{2j}A_2^{\pi} \\ &= \sum_{j=0}^{\infty}\left[Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1}A\right]A^{2j}A^{\pi}(AA^{\pi})^{\pi} \\ &= w \end{aligned} \tag{25}$$

and

$$\begin{aligned} \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}_p &= \left\{\sum_{n=0}^{\infty} v^{n+2}AA^{\mathrm{d}}Q[A^{\pi}(A+Q)]^n\right\} \\ &\times \quad [I - A^{\pi}(A+Q)w] - vQw. \end{aligned} \tag{26}$$

From (23)–(26), we get (i).

By $\|A^{\mathrm{d}}QAA^{\mathrm{d}}\| < 1$, we obtain

$$\|A_1^{-1}Q_{11}\| < 1. \tag{27}$$

Obviously, $\sigma(A_1^{-1}Q_{11}) \cup \{0\} = \sigma(Q_{11}A_1^{-1}) \cup \{0\}$, which implies that $\|Q_{11}A_1^{-1}\| < 1$. By (27), we obtain

$$\begin{aligned} v &= (I + A^{\mathrm{d}}Q)^{\mathrm{d}}A^{\mathrm{d}} = (I + A_1^{-1}Q_{11})^{-1}A_1^{-1} \\ &= \sum_{n=0}^{\infty}(A_1^{-1}Q_{11})^nA_1^{-1} = \sum_{n=0}^{\infty}(A^{\mathrm{d}}Q)^nA^{\mathrm{d}}. \end{aligned} \tag{28}$$

From (27) and (i), we obtain

$$\begin{aligned}(A+Q)^{\mathrm{d}} - A^{\mathrm{d}} &= \left\{\sum_{n=0}^{\infty} v^{n+2} AA^{\mathrm{d}} Q[A^{\pi}(A+Q)]^{n}\right\} \times [I - A^{\pi}(A+Q)w] \\ &+ v + w - vQw - A^{\mathrm{d}} \\ &= \left\{\sum_{n=0}^{\infty} v^{n+2} AA^{\mathrm{d}} Q[A^{\pi}(A+Q)]^{n}\right\} \times [I - A^{\pi}(A+Q)w] \\ &+ \sum_{n=0}^{\infty} (A^{\mathrm{d}}Q)^{n} A^{\mathrm{d}} + w - vQw - A^{\mathrm{d}} \\ &= \left\{\sum_{n=1}^{\infty} v^{n+2} AA^{\mathrm{d}} Q[A^{\pi}(A+Q)]^{n}\right\} \times [I - A^{\pi}(A+Q)w] \\ &+ \sum_{n=0}^{\infty} (A^{\mathrm{d}}Q)^{n} A^{\mathrm{d}} + w - vQw. \end{aligned} \tag{29}$$

Note that

$$\left\|\sum_{n=1}^{\infty} (A^{\mathrm{d}}Q)^{n} A^{\mathrm{d}}\right\| \leq \frac{\|A^{\mathrm{d}}\|\|A^{\mathrm{d}}Q\|}{1 - \|A^{\mathrm{d}}Q\|} = \delta_1. \tag{30}$$

If $\|(AQ)^{\mathrm{d}}\|\|A^2\| < 1$, then

$$\sum_{j=0}^{\infty} \left(\|(AQ)^{\mathrm{d}}\|\|A^2\|\right)^{j} = \frac{1}{1 - \|(AQ)^{\mathrm{d}}\|\|A^2\|}. \tag{31}$$

From (25), (26), and (31), we obtain

$$\begin{aligned}\|w\| &= \left\|\sum_{j=0}^{\infty} \left[Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1} A\right] A^{2j} A^{\pi} (AA^{\pi})^{\pi}\right\| \\ &\leq \left\{\sum_{j=0}^{\infty} \|Q((AQ)^{\mathrm{d}})^{j+1}\|\|A^2\|^{j} + \sum_{j=0}^{\infty} \|((AQ)^{\mathrm{d}})^{j+1} A\|\|A^2\|^{j}\right\} \|A^{\pi}\|\|(AA^{\pi})^{\pi}\| \\ &\leq \sum_{j=0}^{\infty} \|Q(AQ)^{\mathrm{d}}\| \left(\|(AQ)^{\mathrm{d}}\|\|A^2\|\right)^{j} \|A^{\pi}\|\|(AA^{\pi})^{\pi}\| \\ &+ \sum_{j=0}^{\infty} \|(AQ)^{\mathrm{d}}\| \left(\|(AQ)^{\mathrm{d}}\|\|A^2\|\right)^{j} \|A\|\|A^{\pi}\|\|(AA^{\pi})^{\pi}\| \\ &\leq \sum_{j=0}^{\infty} \left(\|(AQ)^{\mathrm{d}}\|\|A^2\|\right)^{j} \|A^{\pi}\|\|(AA^{\pi})^{\pi}\| \left[\|Q(AQ)^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}\|\|A\|\right] \\ &\leq \frac{\|A^{\pi}\|\|(AA^{\pi})^{\pi}\| \left[\|Q(AQ)^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}\|\|A\|\right]}{1 - \|(AQ)^{\mathrm{d}}\|\|A^2\|} = \delta_2, \end{aligned} \tag{32}$$

$$\begin{aligned}
\|Y\| &= \left\| \left\{ \sum_{n=0}^{\infty} v^{n+2} AA^{\mathrm{d}} Q [A^{\pi}(A+Q)]^n \right\} \times [I - A^{\pi}(A+Q)w] - vQw \right\| \\
&\le \|[I - A^{\pi}(A+Q)w]\|_2 \|AA^{\mathrm{d}}Q\| \sum_{n=0}^{\infty} \left(\frac{\delta_1}{\|A^{\mathrm{d}}Q\|} \right)^{n+2} \|[A^{\pi}(A+Q)]^n\| + \|vQw\| \\
&\le \frac{\delta_1^2 (1 + \|A^{\pi}(A+Q)\|_2 \delta_2) \|A\|}{\|A^{\mathrm{d}}Q\|} \sum_{n=0}^{\infty} \left(\frac{\delta_1 \|A^{\pi}(A+Q)\|}{\|A^{\mathrm{d}}Q\|} \right)^n + \frac{\delta_1 \delta_2 \|Q\|}{\|A^{\mathrm{d}}Q\|} \\
&= \delta_3. \qquad (33)
\end{aligned}$$

The proof is completed. □

Theorem 2.4. *Let $A \in B(X)$ be GD-invertible and any $Q \in B(X)$. Assume $A^{\pi}Q(I - A^{\pi}) = 0$ and $AA^{\pi}Q^2 = 0$, then there exists $\delta > 0$ such that $A + Q$ is GD-invertible, when $\|Q\| < \delta$, if and only if $A^{\pi}(A + Q)$ is GD-invertible. If $A^{\pi}AQ$ is GD-invertible and $A^{\pi}A^2 = 0$, then*

(i) $(A+Q)^{\mathrm{d}} = v + w - vQw + \left\{ \sum_{n=0}^{\infty} v^{n+2} AA^{\mathrm{d}} Q [A^{\pi}(A+Q)]^n \right\} \times [I - A^{\pi}(A+Q)w]$.

(ii) *Further, if $\|A^{\mathrm{d}}QAA^{\mathrm{d}}\| < 1$, $\|AQ\|\|(Q^{\mathrm{d}})^2\| < 1$, and $\|(AQ)^{\mathrm{d}}\|\|Q^2\| < 1$, then*

$$\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\| \le \bar{\delta}_2 (1 + \delta_1 \|Q\|) + \frac{\delta_1}{\|A^{\mathrm{d}}Q\|} + \delta_3,$$

$$\begin{aligned}
\frac{\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|} &\le \frac{\mathcal{K}(A) \|A^{\mathrm{d}}Q\| (1 + \|A^{\pi}(A+Q)\|_2 \bar{\delta}_2)}{(1 - \|A^{\mathrm{d}}Q\|)^2} \sum_{n=0}^{\infty} \left(\frac{\delta_1 \|A^{\pi}(A+Q)\|}{\|A^{\mathrm{d}}Q\|} \right)^n \\
&\quad + \bar{\delta}_2 \left(\frac{\|A\|}{\mathcal{K}(A)} + \frac{\|A^{\mathrm{d}}Q\|\|Q\|}{1 - \|A^{\mathrm{d}}Q\|} \right) + \frac{\|A^{\mathrm{d}}\|}{1 - \|A^{\mathrm{d}}Q\|} + \frac{\bar{\delta}_2 \|A^{\mathrm{d}}\|\|Q\|}{1 - \|A^{\mathrm{d}}Q\|},
\end{aligned}$$

where $\mathcal{K}(A) = \|A^{\mathrm{d}}\|\|A\|$ and

$$\begin{aligned}
v &= (I + A^{\mathrm{d}}Q)^{-1} A^{\mathrm{d}}, \ \delta_1 = \frac{\|A^{\mathrm{d}}\|\|A^{\mathrm{d}}Q\|}{1 - \|A^{\mathrm{d}}Q\|}, \\
\bar{w} &= A^{\pi} \sum_{j=0}^{\infty} (Q^{\mathrm{d}})^{2j+1} \left[(AQ)^{\pi} (AQ)^j + Q^{\mathrm{d}} (AQ)^{\pi} (AQ)^j \right] \\
&\quad + \sum_{j=0}^{\infty} (A^{\pi}Q)^{\pi} Q^{2j} \left[Q((AQ)^{\mathrm{d}})^{j+1} + ((AQ)^{\mathrm{d}})^{j+1} A \right] A^{\pi}, \\
\bar{\delta}_2 &= \frac{\|A^{\pi}\|\|(AQ)^{\pi}\| \left[\|Q^{\mathrm{d}}\| + \|Q^{\mathrm{d}}\|^2 \right]}{1 - \|(Q^{\mathrm{d}})^2\|\|AQ\|} + \frac{\|A^{\pi}\|\|(AA^{\pi})^{\pi}\| \left[\|Q(AQ)^{\mathrm{d}}\| + \|(AQ)^{\mathrm{d}}\|\|A\| \right]}{1 - \|(AQ)^{\mathrm{d}}\|\|A^2\|}, \\
\delta_3 &= \frac{\delta_1^2 (1 + \bar{\delta}_2 \|A^{\pi}(A+Q)\|_2) \|A\|}{\|A^{\mathrm{d}}Q\|} \sum_{n=0}^{\infty} \left(\frac{\delta_1 \|A^{\pi}(A+Q)\|}{\|A^{\mathrm{d}}Q\|} \right)^n + \frac{\delta_1 \bar{\delta}_2 \|Q\|}{\|A^{\mathrm{d}}Q\|}.
\end{aligned}$$

Theorem 2.5. *Let $A \in B(X)$ be GD-invertible and any $Q \in B(X)$. Assume that $A^{\pi}Q(I - A^{\pi}) = 0$ and $A^{\pi}Q^2 = A^{\pi}Q$, then there exists a constant $\delta > 0$ such that $A + Q$ is GD-invertible, when $\|Q\| < \delta$, if and only if $A^{\pi}(A + Q)$ is GD-invertible. If $(A^{\pi}Q)^{\mathrm{d}}$ is GD invertible and $PA^{\pi}A(1 - P) = 0$, then*

(i) $(A+Q)^{\mathrm{d}} = \sum_{n=0}^{\infty} (A^{\mathrm{d}}Q)^n A^{\mathrm{d}} + w + t$.

(ii) *Further, if $\|A^{\mathrm{d}}QAA^{\mathrm{d}}\| < 1$, $\|A^{\pi}Q(A^{\pi}Q)^{\mathrm{d}}A\| < 1$, then*

$$\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\| \le \delta_1 + \delta_2 + \delta_3$$

$$\frac{\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\|}{\|A^{\mathrm{d}}\|} \le \frac{\|QA^{\mathrm{d}}\|}{1 - \|QA^{\mathrm{d}}\|} + \frac{\|A\|}{\mathcal{K}(A)} (\delta_2 + \delta_3)$$

where $\mathcal{K}(A) = \|A\|\|A^{\mathrm{d}}\|$, *P is some idempotent operator, and*

$$\begin{aligned}
v &= A^{\mathrm{d}}(I + QA^{\mathrm{d}})^{\mathrm{d}} = \sum_{n=0}^{\infty} A^{\mathrm{d}}(QA^{\mathrm{d}})^n, \delta_1 = \frac{\|QA^{\mathrm{d}}\|\|A^{\mathrm{d}}\|}{1 - \|QA^{\mathrm{d}}\|},\\
t &= \sum_{n=0}^{\infty} v^{n+2}Q(A+Q)^n A^{\pi}(I - (A+Q)w) - vQw,\\
w &= \sum_{n=0}^{\infty} A^n (A^{\pi}Q)^{\pi}(A+Q)\mathcal{T}^{n+2} + \mathcal{T},\\
\mathcal{T} &= [(A^{\pi}Q)(A^{\pi}Q)^{\mathrm{d}}((A+Q))]^{\mathrm{d}} = \sum_{n=0}^{\infty}(A^{\pi}Q(A^{\pi}Q)^{\mathrm{d}}A)^n,\\
\delta_2 &= \frac{\|(A^{\pi}Q)^{\pi}(A+Q)\|}{(1 - \|A^{\pi}Q(A^{\pi}Q)^{\mathrm{d}}A\|)^2} \sum_{n=0}^{\infty}\left(\frac{\|A\|}{1 - \|A^{\pi}Q(A^{\pi}Q)^{\mathrm{d}}A\|}\right)^n + \frac{1}{1 - \|A^{\pi}Q(A^{\pi}Q)^{\mathrm{d}}A\|},\\
\delta_3 &= \frac{\|Q\|A^{\pi}\|(1 + \|(A+Q)\|_2\delta_2)}{(1 - \|QA^{\mathrm{d}}\|)^2} \sum_{n=0}^{\infty}\left(\frac{\|A+Q\|}{1 - \|QA^{\mathrm{d}}\|}\right)^n + \frac{\delta_2\|Q\|}{1 - \|QA^{\mathrm{d}}\|}.
\end{aligned}$$

Proof. Suppose that A, A^{d}, and Q are given by (2), (3), and (5), respectively. From $A^{\pi}Q(I - A^{\pi}) = 0$, we get that Q and B are presented by (21). If A is invertible or quasinilpotent, the proof follows as in the Theorem 2.3.

Suppose that A is GD-invertible with index$(a) > 0$ and $\sigma(A) \neq \{0\}$, i.e. A is neither invertible nor quasinilpotent. Similarly as in the proof of Theorem 2.3, we have that $A_1 + Q_{11}$ is invertible and

$$(A_1 + Q_{11})^{-1} \oplus 0 = (I + A^{\mathrm{d}}Q)^{\mathrm{d}}A^{\mathrm{d}} = v. \tag{34}$$

According to (21) and (34), we obtain that $(A+Q)^{\mathrm{d}}$ exists if and only if $(A_2 + Q_{22})^{\mathrm{d}}$ exists, i.e., $[A^{\pi}(A+Q)]^{\mathrm{d}}$ exists.

Next, we consider the GD-invertibility of $A_2 + Q_{22}$. By $A^{\pi}Q^2 = A^{\pi}Q$, we get $Q_{22}^2 = Q_{22}$. It follows that $\sigma(Q_{22}) \subseteq \{1, 0\}$. If $\boldsymbol{Q} = Q_{22}Q_{22}^{\mathrm{d}} = Q_{22}Q_{22}$, we get that Q_{22} is represented by

$$Q_{22} = \begin{bmatrix} I & 0 \\ X_1 & 0 \end{bmatrix} : \begin{bmatrix} R(I - \boldsymbol{Q}) \\ N(I - \boldsymbol{Q}) \end{bmatrix} \longrightarrow \begin{bmatrix} R(I - \boldsymbol{Q}) \\ N(I - \boldsymbol{Q}) \end{bmatrix}. \tag{35}$$

From the above decomposition, A_2 has the form

$$A_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{36}$$

Since $PA^{\pi}A(I - P) = 0$, we obtain $A_{12} = 0$ and

$$A_2 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}. \tag{37}$$

From (35)-(37), we have

$$A_2 + Q_{22} = \begin{bmatrix} A_{11} + I & 0 \\ A_{21} + X_1 & A_{22} \end{bmatrix}. \tag{38}$$

Since Q is an unit in Banach algebra $B(N(A^\pi))$ and A_2 is quasinilpotent, we obtain that A_{11}, A_{22} are both quasinilpotent and $A_{11} + I$ is invertible. By lemma 1.3,

$$\begin{aligned} w &= 0 \oplus (A_2 + Q_{22})^{\mathrm{d}} \\ &= 0 \oplus \begin{bmatrix} (A_{11}+I)^{-1} & 0 \\ \mathcal{X} & 0 \end{bmatrix} \\ &= \sum_{n=0}^{\infty} A^n (A^\pi Q)^\pi (A+Q) \mathcal{T}^{n+2} + \mathcal{T}, \end{aligned} \tag{39}$$

where

$$\mathcal{X} = \sum_{n=0}^{\infty} A_{22}^n (A_{21} + X_1) \left[(A_{11}+I)^{-1} \right]^{n+2},$$

$$\mathcal{T} = [(A^\pi Q)(A^\pi Q)^{\mathrm{d}}((A+Q))]^{\mathrm{d}} = \sum_{n=0}^{\infty} (A^\pi Q (A^\pi Q)^{\mathrm{d}} A)^n.$$

By the invertibility of $A_1 + Q_{11}$, Lemma 1.3, (39), and (34), we obtain

$$\begin{aligned} (A+Q)^{\mathrm{d}} &= \begin{bmatrix} (A_1+Q_{11})^{-1} & Z \\ 0 & (A_2+Q_{22})^{\mathrm{d}} \end{bmatrix} \\ &= \begin{bmatrix} (A_1+Q_{11})^{-1} & Z_{11} & Z_{12} \\ 0 & (A_{11}+I)^{-1} & 0 \\ 0 & \mathcal{X} & 0 \end{bmatrix} \\ &= v + w + t, \end{aligned} \tag{40}$$

where

$$z = \sum_{n=0}^{\infty} (A_1+Q_{11})^{n+2} Q_{12} (A_2+Q_{22})^n (A_2+Q_{22})^\pi - (A_1+Q_{11})^{-1} Q_{12} (A_2+Q_{22})^{\mathrm{d}},$$

$$t = \sum_{n=0}^{\infty} v^{n+2} Q (A+Q)^n A^\pi (I - (A+Q)w) - vQw.$$

Similarly, for the second equation of (39), we have the same result.

If A is quasinilpotent, then $A^\pi Q^2 = Q^2 = 0$. It means that the proof of $A + Q$ is similar to the section of the depiction of $A_2 + Q_{22}$ in this case. Now we completed the proof of (i).

If $\|A^{\mathrm{d}} Q A A^{\mathrm{d}}\| < 1$, then $\|A_1^{-1} Q_{11}\| < 1$. Thus, it proves that $\|Q_{11} A_1^{-1}\| < 1$ (it is equivalent to $\|QA^{\mathrm{d}}\| < 1$) from

$$\sigma(A^{\mathrm{d}} Q A A^{\mathrm{d}}) \cup \{0\} = \sigma(QA^{\mathrm{d}}) \cup \{0\}.$$

From (34), we obtain

$$\begin{aligned} v &= (I + A^{\mathrm{d}} Q)^{\mathrm{d}} A^{\mathrm{d}} = (A_1 + Q_{11})^{-1} \oplus 0 \\ &= \sum_{n=0}^{\infty} A_1^{-1} (Q_{11} A_1^{-1})^n = \sum_{n=0}^{\infty} A^{\mathrm{d}} (QA^{\mathrm{d}})^n. \end{aligned} \tag{41}$$

Using the result in (i), we obtain

$$\begin{aligned} (A+Q)^{\mathrm{d}} - A^{\mathrm{d}} &= \sum_{n=0}^{\infty} (A^{\mathrm{d}} Q)^n A^{\mathrm{d}} + w + t - A^{\mathrm{d}} \\ &= \sum_{n=1}^{\infty} (A^{\mathrm{d}} Q)^n A^{\mathrm{d}} + t + w. \end{aligned} \tag{42}$$

From (41) and $\|QA^{\mathrm{d}}\| = \|Q_{11}A_1^{-1}\| < 1$, we obtain

$$\left\|\sum_{n=1}^{\infty} A^{\mathrm{d}}(QA^{\mathrm{d}})^n\right\| \leq \frac{\|QA^{\mathrm{d}}\|\|A^{\mathrm{d}}\|}{1-\|QA^{\mathrm{d}}\|} = \delta_1. \tag{43}$$

By (39) and $\|A^\pi Q(A^\pi Q)^{\mathrm{d}} A\| < 1$, we get

$$\begin{aligned}\|w\| &= \left\|\sum_{n=0}^{\infty} A^n (A^\pi Q)^\pi (A+Q)\mathcal{T}^{n+2} + \mathcal{T}\right\| \\ &\leq \left\|\sum_{n=0}^{\infty} A^n (A^\pi Q)^\pi (A+Q)\mathcal{T}^{n+2}\right\| + \|\mathcal{T}\| \\ &\leq \sum_{n=0}^{\infty} \|A^n\|\|(A^\pi Q)^\pi (A+Q)\|\|\mathcal{T}\|^{n+2}\| + \|\mathcal{T}\| \\ &\leq \frac{\|(A^\pi Q)^\pi (A+Q)\|}{(1-\|A^\pi Q(A^\pi Q)^{\mathrm{d}} A\|)^2} \sum_{n=0}^{\infty}\left(\frac{\|A\|}{1-\|A^\pi Q(A^\pi Q)^{\mathrm{d}} A\|}\right)^n + \frac{1}{1-\|A^\pi Q(A^\pi Q)^{\mathrm{d}} A\|} = \delta_2. \end{aligned} \tag{44}$$

In order to complete the proof of (ii), we do some calculations as bellows

$$\begin{aligned}\|t\| &= \left\|\sum_{n=0}^{\infty} v^{n+2} Q(A+Q)^n A^\pi (I-(A+Q)w) - vQw\right\| \\ &\leq \left\|\sum_{n=0}^{\infty} v^{n+2} Q(A+Q)^n A^\pi (I-(A+Q)w)\right\| + \|vQw\| \\ &\leq \frac{\|Q\|A^\pi\|(1+\|(A+Q)\|_2\delta_2)}{(1-\|Q^{\mathrm{d}}A\|)^2} \sum_{n=0}^{\infty}\left(\frac{\|A+Q\|}{1-\|QA^{\mathrm{d}}\|}\right)^n + \frac{\delta_2\|Q\|}{1-\|QA^{\mathrm{d}}\|} = \delta_3. \end{aligned} \tag{45}$$

According to (42)–(45), we have

$$\begin{aligned}\|(A+Q)^{\mathrm{d}} - A^{\mathrm{d}}\| &= \left\|\sum_{n=1}^{\infty} A^{\mathrm{d}}(QA^{\mathrm{d}})^n + w + t\right\| \\ &\leq \left\|\sum_{n=1}^{\infty} A^{\mathrm{d}}(QA^{\mathrm{d}})^n\right\| + \|w\| + \|t\| \\ &\leq \delta_1 + \delta_2 + \delta_3. \end{aligned} \tag{46}$$

From (46), the proof is completed. □

Acknowledgements

We would like to thank the anonymous referees for their valuable comments and suggestions which improve the manuscript. This work was supported by the National Natural Science Foundation of China (11061005) and supported by open fund of Guangxi Key laboratory of hybrid computation and IC design analysis (HCIC201607).

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