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Caputo Fractional Differential Equations with Non-Instantaneous Impulses and Strict Stability by Lyapunov Functions

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Abstract. In this paper the statement of initial value problems for fractional differential equations with non-instantaneous impulses is given. These equations are adequate models for phenomena that are characterized by impulsive actions starting at arbitrary fixed points and remaining active on finite time intervals. Strict stability properties of fractional differential equations with non-instantaneous impulses by the Lyapunov approach is studied. An appropriate definition (based on the Caputo fractional Dini derivative of a function) for the derivative of Lyapunov functions among the Caputo fractional differential equations with non-instantaneous impulses is presented. Comparison results using this definition and scalar fractional differential equations with non-instantaneous impulses are presented and sufficient conditions for strict stability and uniform strict stability are given. Examples are given to illustrate the theory.

1. Introduction

One of the main properties studied in the qualitative theory of differential equations is stability. The usual stability concepts do not give any information concerning the rate of decay of the solutions, and hence are not strict concepts. As a result, strict stability was defined and criteria for such notions was discussed (see, for example, [30], [33], [42], [47]).

Fractional derivatives arise in modeling various physical phenomena and are used in linear viscoelasticity acoustics, rheology, polymeric chemistry (see for example the book [36] and the cited therein references). The stability of fractional order systems is quite recent. There are several approaches in the literature to study stability, one of which is the Lyapunov approach. Results on stability in the literature via Lyapunov functions could be divided into two main groups:

- continuously differentiable Lyapunov functions (see, for example, the papers [10], [16], [26], [34]). Different types of stability are discussed using the Caputo derivative of Lyapunov functions which depends significantly of the unknown solution of the fractional equation.

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- continuous Lyapunov functions (see, for example, the papers [14], [31], [32]) in which the authors use the derivative of a Lyapunov function which is similar to the Dini derivative of Lyapunov functions.

In this paper fractional differential equations with impulses is studied. The impulses start abruptly at some points and their action continue on given finite intervals. As a motivation for the study of these systems we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. The model of this situation is the so called non-instantaneous impulsive differential equation.

In the literature there are two popular types of impulses involved in the differential equations:

- *instantaneous impulses*- the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, for example, [2], [23], [24], [25], the monographs [22], [29] and the cited references therein);
- *non-instantaneous impulses* - an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan ([21]) introduced this new class of abstract differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. For recent work we refer the reader to [17], [37], [38], [41]. Note stability for fractional equations with non-instantaneous impulses is studied in ([6]).

In the paper a system of nonlinear Caputo fractional differential equations with non-instantaneous impulses is set up and strict stability is defined and studied using Lyapunov functions. The Caputo fractional Dini derivative of a Lyapunov function is defined in an appropriate way among the nonlinear Caputo fractional differential equations with non-instantaneous impulses. This type of derivative was introduced in [5] and used to study stability and asymptotic stability ([5]), strict stability ([3]), practical stability ([4]) and practical stability with initial time difference ([9]) of Caputo fractional differential equations, and for stability of Caputo fractional differential equations with non-instantaneous impulses ([6]). In this paper comparison results using this definition and scalar fractional differential equations with non-instantaneous impulses are presented. Several sufficient conditions for strict stability and uniform strict stability are obtained. Some examples are provided to illustrate the theory.

2. Notes on Fractional Calculus

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [13, 15, 31, 39, 40] and there are several definitions of fractional derivatives and fractional integrals. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0,1)$.

1: The Riemann–Liouville (RL) fractional derivative of order $q \in (0,1)$ of $m(t)$ is given by (see, for example, Section 1.4.1.1 [13], or [39])

$$ {}_{t_0}^{RL}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \qquad t \geq t_0. $$

where $\Gamma(.)$ denotes the Gamma function.

2: The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see, for example, Section 1.4.1.3 [13])

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \qquad t \geq t_0. \tag{1} $$

Note the Caputo derivative of a constant is zero, whereas the Riemann-Liouville derivative of a constant C is ${}_{t_0}^{RL}D^q C = \frac{C(t-t_0)^{-q}}{\Gamma(1-q)}$. The properties of the Caputo derivative are quite similar to those of ordinary derivatives and initial value problems for fractional differential equations with the Caputo derivative has a clear physical meaning so as a result it is usually used in real applications.

If both the Caputo derivative and Riemann-Liouville derivative of $m(t)$ exist (for example, if $m(t)$ is absolutely continuous function) then from (2.4.2) [27] we have that ${}_{t_0}^{c}D^q m(t) = {}_{t_0}^{RL}D^q[m(t) - m(t_0)] = {}_{t_0}^{RL}D^q m(t) - \frac{m(t_0)(t-t_0)^{-q}}{\Gamma(1-q)}$ holds (see Lemma 3.4 in [15]).

3: The Grunwald−Letnikov fractional derivative is given by (see, for example, Section 1.4.1.2 [13])

$$ {}_{t_0}^{GL}D^q m(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \quad t \ge t_0, $$

and the Grunwald−Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \quad t \ge t_0, \tag{2} $$

where $\binom{q}{r} = \frac{q(q-1)(q-2)\dots(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

From Section 2.3.7 [39] if the function $m(t) \in C[t_0, a]$ and $m(t)$ is integrable in $[t_0, a]$, then for every q, $0 < q < 1$ both the Riemann- Liouville derivative, ${}_{t_0}^{RL}D^q m(t)$, and the Grunwald-Letnikov derivative, ${}_{t_0}^{GL}D^q m(t)$, exist and coincide. Also, if $m(t) \in C[t_0, a]$ then ${}_{t_0}^{GL}D^q m(t) = {}_{t_0}^{GL}D_+^q m(t)$.

Note ${}_{t_0}^{GL}D_+^q 1 = {}_{t_0}^{RL}D^q 1 = \frac{(t-t_0)^{-q}}{\Gamma(1-q)}$.

Proposition 1. *(Theorem 2.25 [15]). Let $m \in C^1[t_0, b]$. Then, for $t \in (t_0, b]$, ${}_{t_0}^{GL}D^q m(t) = {}_{t_0}^{RL}D^q m(t)$.*

From the relation between the Caputo fractional derivative and the Grunwald−Letnikov fractional derivative using (2) we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{GL}D_+^q [m(t) - m(t_0)], $$

i.e.

$$ {}_{t_0}^{c}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \Big[m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{q}{r} \big(m(t-rh) - m(t_0) \big) \Big]. \tag{3} $$

Definition 1. *(Definition 2.6 [14]) We say $m \in C^q([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{c}D^q m(t)$ exists and satisfies (1) for $t \in [t_0, T]$.*

Remark 1. *Definition 1 could be extended to any interval $I \subset \mathbb{R}_+$.*

Remark 2. *If $m \in C^q([t_0, T], \mathbb{R}^n)$ then ${}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{GL}D_+^q \big(m(t) - m(t_0)\big) = {}_{t_0}^{GL}D^q \big(m(t) - m(t_0)\big) = {}_{t_0}^{RL}D^q \big(m(t) - m(t_0)\big) = {}_{t_0}^{c}D^q m(t)$.*

3. Non-Instantaneous Impulses in Fractional Differential Equations

The main goal of the paper is study types of stability properties of a nonlinear system of fractional differential equations with non-instantaneous impulses. From a practical point of view, the independent variable t is modeling the time which is nonnegative, i.e. we will consider the half real line $\mathbb{R}_+ = [0, \infty)$. Let $\mathbb{Z}_+$ denote the set of all natural numbers and $\mathbb{Z}_0$ the set of all nonnegative integers.

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $s_0 = 0 < t_i \leq s_i < t_{i+1}$, $i \in \mathbb{Z}_+$, and $\lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ be a given arbitrary point. Define the sequence of points $\{\tau_k\}$ by

$$\tau_k = \begin{cases} t_0, & \text{for } k \in \mathbb{Z}_0 : \ t_0 \in [s_k, t_{k+1}) \\ s_k, & \text{if } k \in \mathbb{Z}_+ : \ s_k > t_0. \end{cases} \tag{4}$$

Consider the IVP for the system of Caputo *fractional differential equations* (FrDE)

$$ {}_{\tau}^{c}D^q x = f(t, x) \quad \text{for } \ t \in [\tau, t_{k+1}] \quad \text{with } \ x(\tau) = \tilde{x}_0 \tag{5}$$

with $x \in \mathbb{R}^n$ and $\tau \in [\tau_k, t_{k+1})$. Assume (5) has a solution $x(t) = x(t; \tau, \tilde{x}_0) \in C^q([\tau, t_{k+1}], \mathbb{R}^n)$. Some sufficient conditions for global existence of solutions of (5) are given in [11], [31].

If $x \in C^q([\tau, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the IVP for FrDE (5) then it also satisfies the fractional integral equation (see (2.8) [14])

$$x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} f(s, x(s)) ds \quad \text{for } \ t \in [\tau, t_{k+1}]. \tag{6}$$

Consider the initial value problem (IVP) for the system of *non-instantaneous impulsive fractional differential equations* (NIFrDE) with a Caputo derivative for $0 < q < 1$,

$$\begin{aligned} &{}_{t_0}^{c}D^q x = f(t, x) \text{ for } t \in [t_0, \infty) \cap (\tau_k, t_{k+1}], \\ &x(t) = \phi_k(t, x(t_k - 0)) \quad \text{for } t \in [t_0, \infty) \cap (t_k, \tau_k], \\ &x(t_0) = x_0 \end{aligned} \tag{7}$$

where $x, x_0 \in \mathbb{R}^n$, $f : \cup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_i : [t_i, s_i] \times \mathbb{R}^n \to \mathbb{R}^n$, $(i = 1, 2, 3, \dots)$.

Remark 3. *The intervals $(t_k, s_k]$, $k = 1, 2, \dots$ are called intervals of non-instantaneous impulses and the functions $\phi_k(t, x)$, $k = 1, 2, \dots,$ are called non-instantaneous impulsive functions.*

We will give a brief description of the solution of IVP for NIFrDE (7). Without loss of generality we assume $t_0 \in [0, t_1)$. The solution $x(t; t_0, x_0)$, $t \geq t_0$ of (7) is given by

$$x(t; t_0, x_0) = \begin{cases} X_k(t) \quad \text{for } \ t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, \\ \phi_k(t, X_{k-1}(t_k - 0)) \quad \text{for } \ t \in (t_k, s_k], \ \ k = 1, 2, \dots, \end{cases}$$

where

- $X_0(t)$ is the solution of IVP for FrDE (5) for $k = 0$, $\tau = t_0$, $t \in [t_0, t_1]$, $\tilde{x}_0 = x_0$ and $X_0(t)$ satisfies (6) on $[t_0, t_1]$;

- $X_1(t)$ is the solution of IVP for FrDE (5) for $k = 1$, $\tau = s_1$, $t \in [s_1, t_2]$, $\tilde{x}_0 = \phi_1(s_1, X_0(t_1 - 0))$, and $X_1(t)$ satisfies (6) on $[s_1, t_2]$;

- $X_2(t)$ is the solution of IVP for FrDE (5) for $k = 2$, $\tau = s_2$, $t \in [s_2, t_3]$, $\tilde{x}_0 = \phi_2(s_2, X_1(t_2 - 0))$, and $X_2(t)$ satisfies (6) on $[s_2, t_3]$;

and so on.

Also, the solution $x(t) = x(t; t_0, x_0)$, $t \geq t_0$ of (7) satisfies the integral-algebraic equations

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [t_0, t_1], \\ \phi_1(t, x(t_1 - 0)) & \text{for } t \in (t_1, s_1], \\ \phi_1(s_1, x(t_1 - 0)) + \frac{1}{\Gamma(q)} \int_{s_1}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [s_1, t_2], \\ \phi_2(t, x(t_2 - 0)) & \text{for } t \in (t_2, s_2], \\ \phi_2(s_2, x(t_2 - 0)) + \frac{1}{\Gamma(q)} \int_{s_2}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [s_2, t_3], \\ \dots\dots\dots & \\ \phi_k(t, x(t_k - 0)) & \text{for } t \in (t_k, s_k], \\ \phi_k(s_k, x(t_k - 0)) + \frac{1}{\Gamma(q)} \int_{s_k}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [s_k, t_{k+1}], \\ \dots\dots\dots & \end{cases}$$

Remark 4. *If $t_k = s_k$, $k = 1, 2, \dots$ then the IVP for NIFrDE (7) reduces to an IVP for impulsive fractional differential equations (see, for example, [1], [12], [43]). In this case at any point of instantaneous impulse t_k the amount of jump of the solution $x(t)$ is given by $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0) = \Phi_k(x(t_k - 0)) = \phi_k(t_k, x(t_k - 0)) - x(t_k - 0)$.*

Remark 5. *In the case $q = 1$ the IVP for NIFrDE (7) reduces to an IVP for non-instantaneous impulsive differential equations studied in [41], [44].*

Remark 6. *In the case $q = 1$, $t_k = s_k$, $k = 1, 2, \dots$ the IVP for NIFrDE (7) reduces to an IVP for impulsive differential equations (see, for example, the books [22], [29] and the cited references therein).*

Remark 7. *According to the above description any solution of (7) could have a discontinuity at points t_k, $k = 1, 2, \dots$.*

We will use the following conditions:

(H1) The function $f \in C(\cup_{k=0}^{\infty}(s_k, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ is such that $f(t, 0) = 0$, $t \in \cup_{k=0}^{\infty}(s_k, t_{k+1}]$ and for any initial point $(\tau, \tilde{x}_0) \in [s_k, t_{k+1}) \times \mathbb{R}^n$, $k = 0, 1, 2, \dots$ the IVP for the system of FrDE (5) has a solution $\tilde{x}(t; \tau, \tilde{x}_0)$.

(H2) The function $\phi_k \in C([t_k, s_k] \times \mathbb{R}^n, \mathbb{R}^n)$ and $\phi_k(t, 0) = 0$, $t \in [t_k, s_k]$.

Let $J \subset \mathbb{R}_+$ be a given interval and the set $\Delta \subset \mathbb{R}^n$. We introduce the following class of functions

$$\begin{aligned} PC^q(J, \Delta) \;=\; & \{u \in C^q\big(J \cap \big(\cup_{k=0}^{\infty} (s_k, t_{k+1}]\big), \Delta\big) \bigcup C\big(J \cap \big(\cup_{k=1}^{\infty} (t_k, s_k]\big), \Delta\big) : \\ & u(t_k) = u(t_k - 0) = \lim_{t \uparrow t_k} u(t) < \infty \text{ and } u(t_k + 0) = \lim_{t \downarrow t_k} u(t) < \infty \text{ for } k: \; t_k \in J, \\ & u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) = u(s_k + 0) = \lim_{t \downarrow s_k} u(t) \text{ for } k: \; t_k \in J\}. \end{aligned}$$

Example 1. Let $t_0 \in [0, t_1)$ and consider the IVP for the scalar NIFrDE

$$\begin{aligned} & {}_{t_0}^{c}D^q x = Ax \text{ for } t \in (\tau_k, t_{k+1}], \; k \in \mathbb{Z}_0 \\ & x(t) = \Xi_k(t, x(t_k - 0)) \text{ for } \; t \in (t_k, s_k], \; k \in \mathbb{Z}_+, \\ & x(t_0) = x_0, \end{aligned} \tag{8}$$

where $x, x_0 \in \mathbb{R}$, A is a constant, $\Xi : \; [t_k, s_k] \times \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}_+$.

The solution $x(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R})$ of (8) is given by

$$x(t) = \begin{cases} x_0 E_q(A(t - t_0)^q) & \text{for } t \in [t_0, t_1] \\ \Xi_k(t, x(t_k - 0)) & \text{for } t \in (t_k, s_k], \; k \in \mathbb{Z}_+ \\ \Xi_k\big(s_k, x(t_k - 0)\big) E_q\big(A(t - s_k)^q\big) & \text{for } t \in [s_k, t_{k+1}], \; k \in \mathbb{Z}_+ \end{cases}$$

where the Mittag-Leffler function (with one parameter) is defined by $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$.

Let $\Xi_k(t,x) = a_k(t)x,\ a_k : [t_k, s_k] \to \mathbb{R},\ k = 1,2,3,\dots$. Then the solution of NIFrDE (8) is given by

$$x(t) = \begin{cases} x_0 E_q(A(t-t_0)^q) & \text{for } t \in [t_0, t_1] \\ x_0 E_q(AC_k^q) a_k(t) \prod_{j=1}^{k-1} \Big(a_j(s_j) E_q(AC_j^q)\Big) & \text{for } t \in (t_k, s_k],\ k \in \mathbb{Z}_+ \\ x_0 \prod_{j=1}^{k} \Big(a_j(s_j) E_q(AC_j^q)\Big) E_q(A(t-s_k)^q) & \text{for } t \in (s_k, t_{k+1}],\ k \in \mathbb{Z}_+ \end{cases}$$

where $C_1 = t_1 - t_0$ and $C_{k+1} = t_{k+1} - s_k \geq 0,\ k \in \mathbb{Z}_+$.

□

4. Strict Stability and Lyapunov Functions

The study of fractional differential equations with non-instantaneous impulses is quite recent and only some qualitative properties are investigated ([18], [19], [20], [28], [35], [46]). The goal of the paper is to study strict stability of the NIFrDE (7). We will define strict stability for fractional equations following the idea for ordinary differential equations (see, for example, [33]).

Definition 2. *The zero solution of the system NIFrDE (7) is said to be*

- strictly stable *if for given $\epsilon_1 > 0$ and $t_0 \in \cup_{k=0}^{\infty} [s_k, t_{k+1})$ there exists $\delta_1 = \delta_1(t_0, \epsilon_1) > 0$ such that for any initial point $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta_1$ implies $\|x(t; t_0, x_0)\| < \epsilon_1,\ t \geq t_0$, and for any $\delta_2 = \delta_2(t_0, \epsilon_1),\ \delta_2 \in (0, \delta_1]$ there exists $\epsilon_2 = \epsilon_2(t_0, \delta_2),\ \epsilon_2 \in (0, \delta_2]$ such that the inequality $\delta_2 < \|x_0\|$ implies $\epsilon_2 < \|x(t; t_0, x_0)\|$ for $t \geq t_0$ where $x(t; t_0, x_0)$ is a solution of the IVP for the NIFrDE (7);*
- uniformly strictly stable *if for any given $\epsilon_1 > 0$ there exists $\delta_1 = \delta_1(\epsilon_1) > 0$ such that for any initial time $t_0 \in \cup_{k=0}^{\infty} [s_k, t_{k+1})$ and any initial point $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta_1$ implies $\|x(t; t_0, x_0)\| < \epsilon_1,\ t \geq t_0$, and for any $\delta_2 \in (0, \delta_1]$ there exists $\epsilon_2 \in (0, \delta_2],\ \varepsilon_2 = \varepsilon_2(\delta_2)$, such that the inequality $\delta_2 < \|x_0\|$ implies $\epsilon_2 < \|x(t; t_0, x_0)\|$ for $t \geq t_0$ where $x(t; t_0, x_0)$ is a solution of the IVP for the NIFrDE (7).*

Example 2. *(Strict stability of NIFrDE).* Consider the following scalar IVP for FrDE ${}_{t_0}^{c}D^q x = 0,\ t \geq t_0,\ x(t_0) = x_0$ with an arbitrary $t_0 \in \mathbb{R}_+$. Its solution $x(t) = x_0$ is uniformly strictly stable.

Now let $s_0 = 0, s_k = 2k,\ t_k = 2k - 1,\ k \in \mathbb{Z}_+$ and $t_0 \in \cup_{k=0}^{\infty} [s_k, t_{k+1})$ be a given initial time. Let $p \in \mathbb{Z}_0$ be such that $t_0 \in [2p, 2p+1)$.

Consider the IVP for NIFrDE

$$\begin{aligned} &{}_{t_0}^{c}D^q x = 0 \text{ for } t \in [t_0, 2p] \bigcup \cup_{k=p}^{\infty} (2k, 2k+1], \\ &x(t) = \Xi_k(t, x(t_k - 0)) \text{ for } \ t \in (2k-1, 2k],\ k = p+1, p+2, \dots, \\ &x(t_0) = x_0, \end{aligned} \tag{9}$$

where $x, x_0 \in \mathbb{R}$ and $\Xi_k(t,x) = a_k(t)x,\ a_k : [2k-1, 2k] \to \mathbb{R},\ k = p+1, p+2, \dots$. Then the solution of NIFrDE (9) is given by

$$x(t) = \begin{cases} x_0 & \text{for } t \in [t_0, 2p+1] \\ x_0 a_{2k+1}(t) \prod_{j=p}^{k-1} a_{2j+1}(2j+2) & \text{for } t \in (2k+1, 2k+2],\ k = p, p+1, p+2, \dots \\ x_0 \prod_{j=p}^{k} a_{2j+1}(2j+2) & \text{for } t \in (2k, 2k+1],\ k = p+1, p+2, \dots. \end{cases}$$

The type of the non-instantaneous impulsive functions $\Xi_k(t,x)$, i.e. $a_k(t)$ has an influence on the behavior of the solution of NIFrDE.

Let $a_k(t) = \frac{t}{t+1}$. Then $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{2i}{2i+1} = 0$ (see the graph of $\prod_{i=1}^{n} \frac{2i}{2i+1}$ in Figure 1). Thus the zero solution of (9) is asymptotically stable (see Figure 2).

Let $a_k(t) = 1 + \frac{(-1)^k}{t}$ for $k = 1,2,3,\dots$. Then $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{(-1)^i}{2i} \neq 0$ (see Figure 3). Thus the solution of (9) is not asymptotically stable but it is strictly stable (see Figure 4).

□

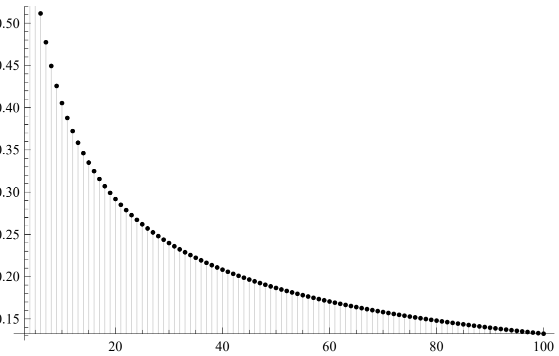

Figure 1. Example 2.Graph of the product of impulsive function $\prod \frac{2k}{2k+1}$

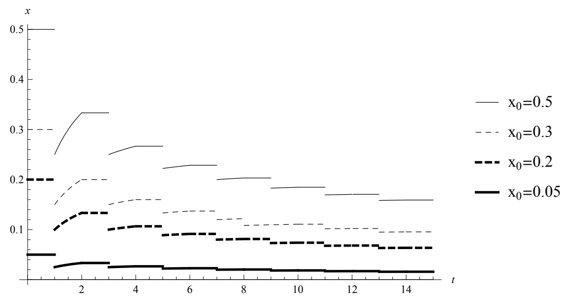

Figure 2. Graph of the solutions of (9) for $t_0 = 0$ *and various* x_0*.*

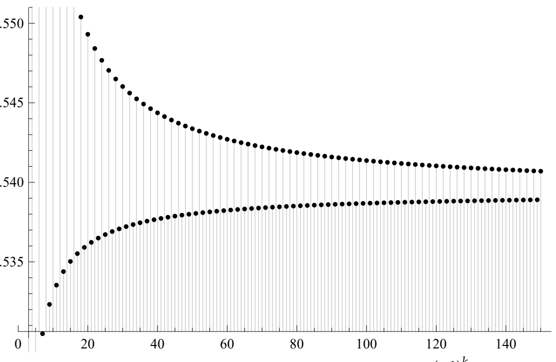

Figure 3. Example 2. Graph of $\prod(1 + \frac{(-1)^k}{2k})$*.*

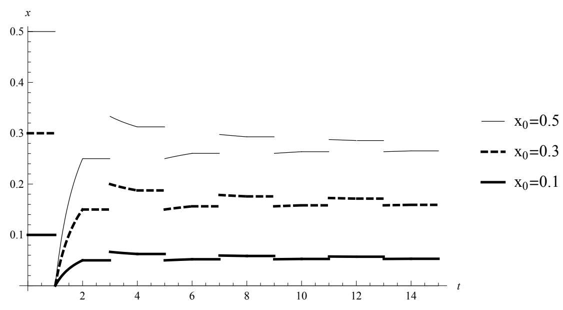

Figure 4. Graph of the solutions of (9) for $t_0 = 0$ *and various* x_0*.*

In this paper we will use the followings sets:

$$\mathcal{K} = \{a \in C[\mathbb{R}_+,\mathbb{R}_+] : a(u) \text{ is strictly increasing and } a(0) = 0\},$$
$$\mathcal{M} = \{a \in \mathcal{K} \text{ and } \lim_{s\to\infty} a(s) = \infty\},$$
$$B(\lambda) = \{x \in \mathbb{R}^n : \|x\| \le \lambda\}, \quad \lambda = const > 0.$$

We will use comparison results for initial value problem for scalar Caputo fractional differential equation with non-instantaneous impulses (NIFrDE) of the type

$$\begin{aligned} &{}^c_{t_0}D^q u = g(t,u) \text{ for } t \in \cup_{k=0}^{\infty}(\tau_k, t_{k+1}], \\ &u(t) = \Xi_k(t, u(t_k - 0)) \text{ for } t \in (t_k, s_k], \quad k \in \mathbb{Z}_+ \\ &u(t_0) = u_0 \end{aligned} \tag{10}$$

where $u \in \mathbb{R}$, $g : \cup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R} \to \mathbb{R}$, $g(t,0) \equiv 0$, $\Xi_i : [t_i, s_i] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Xi_i(t,0) = 0$, $(i = 1,2,3,\dots)$.

We also will use the corresponding IVP for the scalar Caputo *fractional differential equations* (FrDE)

$$ {}^c_{\tau}D^q u = g(t,u) \quad \text{for } t \in [\tau, t_{k+1}] \quad \text{with } u(\tau) = \tilde{u}_0 \tag{11}$$

where $\tau \in [\tau_k, t_{k+1})$.

We will use minimal/maximal solutions of the IVP for FrDE (11). Some sufficient conditions for global existence of solutions of (11) are given in [11], [15], [31].

We will use the following conditions:

(H3) The function $g \in C(J \times \mathbb{R}, \mathbb{R})$, $J \subset \cup_{k=0}^{\infty}(s_k, t_{k+1}]$ is such that $g(t,0) = 0,\ t \in J$ and for any initial point $(\tau, \tilde{u}_0):\ \tau \in [s_k, t_{k+1}) \cap J$, $k = 0,1,2,\dots$, and $\tilde{u}_0 \in \mathbb{R}$ the IVP for FrDE (11) has a maximal solution $\tilde{u}(t;\tau,\tilde{u}_0)$ defined on $[\tau, t_{k+1}]$.

(H4) The function $g \in C(J \times \mathbb{R}, \mathbb{R})$, $J \subset \cup_{k=0}^{\infty}(s_k, t_{k+1}]$ is such that $g(t,0) = 0,\ t \in J$ and for any initial point $(\tau, \tilde{u}_0):\ \tau \in [s_k, t_{k+1}) \cap J$, $k = 0,1,2,\dots$, and $\tilde{u}_0 \in \mathbb{R}$ the IVP for FrDE (11) has a minimal solution $\tilde{u}(t;\tau,\tilde{u}_0)$ defined on $[\tau, t_{k+1}]$.

(H5) The function $\Xi_k \in C([t_k, s_k]\times\mathbb{R}, \mathbb{R})$, $\Xi_k(t,0) = 0$ for $t \in [t_k, s_k]$ and $\Xi_k(t,u) \le \Xi_k(t,v)$ for $u \le v,\ t \in [t_k, s_k]$.

Definition 3. *Let p be a natural number and $T \in (s_p, t_{p+1}]$ be a given number. The function $u^*(t)$ will be called a maximal solution (minimal solution) of the IVP for NIDE (10) on the interval $[t_0, T]$ if*

- *it is a solution of the IVP for NIDE(10) on $[t_0, T]$;*
- *for any $k = 0,1,2,\dots,p+1$ and any solution $u(t) \in C^1([s_k, t_{k+1}], \mathbb{R})$ of IVP for FrDE (11) with $\tau = s_k$, $\tilde{u}_0 = u^*(s_k)$ the inequalities*

$$u^*(t) \ge (\le) u(t) \quad \text{for } t \in [s_k, t_{k+1}] \cap [t_0, T]$$

and

$$\Xi_k(t, u^*(t_k - 0)) \ge (\le) \Xi_k(t, u(t_k)) \quad \text{for } t \in (t_k, s_k]$$

hold.

Lemma 1. *Let:*

1. Condition (H3) be satisfied for $J = \cup_{k=0}^{p}(s_k, t_{k+1}]$ where $p \le \infty$ is a positive integer.

2. Condition (H5) be satisfied for all $k = 1,2,\dots,p$.

Then there exist a maximal solution of IVP for NIDE (10) on the interval $[s_0, t_{p+1}]$.

P r o o f: We will use induction to prove the claim.

Let $t \in [s_0, t_1]$. According to condition (H3) there exists a maximal solution $u_0^*(t)$ of IVP for FrDE (11) with $\tau = s_0$ and $\tilde{u}_0 = u_0$.

Let $t \in (t_1, s_1]$. According to condition (H5) for the function $\Xi_1(t,u)$ the inequality $\Xi_1(t, u_0^*(t_1)) \ge \Xi_1(t, u(t_1))$ for $t \in (t_1, s_1]$ holds where $u(t)$ is any solution of IVP for FrDE (11) with $\tau = s_0$, $\tilde{u}_0 = u_0$ which exists on $[s_0, t_1]$.

Let $t \in (s_1, t_2]$. According to condition (H3) there exists a maximal solution $u_1^*(t)$ of IVP for FrDE (11) with $\tau = s_1$ and $\tilde{u}_0 = \Xi_1(s_1, u_0^*(t_1))$.

Let $t \in (t_2, s_2]$. According to condition (H5) for Ξ_2 the inequality $\Xi_2(t, u_1^*(t_2)) \geq \Xi_2(t, u(t_2))$ for $t \in (t_2, s_2]$ holds where $u(t)$ is any solution of IVP for FrDE (11) with $\tau = s_1$, $\tilde{u}_0 = \Xi_1(s_1, u_0^*(t_1)) = u_1^*(t_2)$ which exists on $[s_1, t_2]$.

Following the same idea we construct the function

$$u^*(t; t_0, u_0) = \begin{cases} u_k^*(t) & \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, p \\ \Xi_{k-1}(t, u_{k-1}^*(t_k - 0)) & \text{for } t \in (t_k, s_k],\ k = 1, 2, \dots, p, \end{cases}$$

where $u_k^*(t)$ is the maximal solution of the IVP for FrDE (11) on $[s_k, t_{k+1}]$ with $\tau = s_k$ and $\tilde{u}_0 = \Xi_{k-1}(s_k, u_{k-1}^*(t_k))$ (in the case $k = 0$ it is denoted $\Xi_{-1}(s_0, u_{-1}^*(t_{-1})) = u_0$).

According to Definition 3 the function $u^*(t; t_0, u_0)$ is a maximal solution of IVP for NIDE (10).

□

Lemma 2. *Let:*

1. Condition (H4) be satisfied for $J = \cup_{k=0}^{p} (s_k, t_{k+1}]$ *where* $p \leq \infty$ *is a positive integer.*

2. Condition (H5) be satisfied for all $k = 1, 2, \dots, p$.

Then there exist a minimal solution of IVP for NIDE (10) on the interval $[s_0, t_{p+1}]$.

We will use the following type of a couple of Caputo fractional differential equations with non-instantaneous impulses

$$\begin{aligned} &{}_{t_0}^{c}D^q u = g_1(t, u), && {}_{t_0}^{c}D^q v = g_2(t, v), && t \in (\tau_k, t_{k+1}], \\ &u(t) = \Phi_k(t, u(t_k - 0)), && v(t) = \Psi_k(t, v(t_k - 0)), && t \in (t_k, \tau_k] \\ &u(t_0) = u_0, && v(t_0) = v_0, && \end{aligned} \tag{12}$$

where $u, v \in \mathbb{R}$, $g_1, g_2 : \cup_{k=0}^{\infty} [s_k, t_{k+1}] \times \mathbb{R} \to \mathbb{R}$, $g_j(t, 0) \equiv 0, (j = 1, 2)$, $\Phi_i, \Psi_i : [t_i, s_i] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_i(t, 0) = 0$, $\Psi_i(t, 0) = 0$, $(i = 1, 2, 3, \dots)$.

We will introduce the strict stability of the couple of Caputo fractional differential equations as follows:

Definition 4. *The zero solution of the couple of NIFrDE (12) is said to be*

- strictly stable in couple *if for given* $\epsilon_1 > 0$ *and* $t_0 \in \cup_{k=0}^{\infty} [s_k, t_{k+1})$ *there exists* $\delta_1 = \delta_1(t_0, \epsilon_1) > 0$ *and for any* $\delta_2 = \delta_2(t_0, \epsilon_1)$, $\delta_2 \in (0, \delta_1]$ *there exists* $\epsilon_2 = \epsilon_2(t_0, \delta_2)$, $\epsilon_2 \in (0, \delta_2]$ *such that the inequalities* $|u_0| < \delta_1$ *and* $\delta_2 < |v_0|$ *imply* $|u(t; t_0, u_0)| < \epsilon_1$ *and* $\epsilon_2 < |v(t; t_0, v_0)|$ *for* $t \geq t_0$ *where the couple of functions* $\big(u(t; t_0, u_0), v(t; t_0, u_0)\big)$ *is a solution of the IVP for the couple of NIFrDE (12).*

- uniformly strictly stable in couple *if for any given* $\epsilon_1 > 0$ *there exists* $\delta_1 = \delta_1(\epsilon_1) > 0$ *and for any* $\delta_2 \in (0, \delta_1]$ *there exists* $\epsilon_2 \in (0, \delta_2]$, $\varepsilon_2 = \varepsilon_2(\delta_2)$, *such that for any initial time* $t_0 \in \cup_{k=0}^{\infty} [s_k, t_{k+1})$ *the inequalities* $|u_0| < \delta_1$ *and* $\delta_2 < |v_0|$ *imply* $|u(t; t_0, u_0)| < \epsilon_1$ *and* $\epsilon_2 < |v(t; t_0, v_0)|$ *for* $t \geq t_0$ *where the couple of functions* $\big(u(t; t_0, u_0), v(t; t_0, u_0)\big)$ *is a solution of the IVP for the couple of NIFrDE (12).*

Remark 8. *Note if the zero solution of the couple of NIFrDE (12) is strictly stable,* $|v(t)| \leq |u(t)|, t \geq t_0$ *then according to Definition 3 the inequalities* $\delta_2 < |v_0| \leq |u_0| < \delta_1$ *guarantees* $\epsilon_2 < |v(t)| \leq |u(t)| < \epsilon_1$ *for* $t \geq t_0$, *i.e. the solutions remain in an appropriate tube.*

Remark 9. *If* $g_1(t, x) \equiv g_2(t, x)$, $\Psi_k(t, x) \equiv \Phi_k(t, x)$, $k = 1, 2, \dots$ *in (12) , then the strict stability (uniform strict stability) in a couple given by Definition 3 reduces to strict stability (uniform strict stability) of the zero solution of a scalar NIFrDE defined by Definition 2.*

Example 3. *(Uniform strict stability in couple).* Let $s_0 = 0, s_k = 2k,\ t_k = 2k-1$ for $k \in \mathbb{Z}_+$. Consider the couple of Caputo fractional differential equations with noninstantaneous impulses

$$\begin{aligned} &{}_0^cD^q u = Au, && {}_0^cD^q v = -Bv, && t \in (2k, 2k+1],\ k \in \mathbb{Z}_0 \\ &u(t) = \frac{b_k}{E_q(A)} u(t_k - 0), && v(t) = \frac{c_k}{E_q(-B)} v(t_k - 0)), && t \in (2k+1, 2k+2],\ k \in \mathbb{Z}_0 \\ &u(0) = u_0, && v(0) = v_0, \end{aligned} \tag{13}$$

where $u, v \in \mathbb{R}, A, B > 0, b_k,\ |b_k| \le 1, c_k,\ |c_k| \ge 1$ are given constants such that $\prod_{i=1}^{\infty} b_i \le M$ and $\prod_{i=1}^{\infty} c_i \ge N > 0$ with $N \le \frac{1}{E_q(-B)}$.

The solution of (13) is given by

$$u(t) = \begin{cases} u_0\Big(\prod_{j=1}^{k} b_j\Big) & \text{for } t \in (t_k, s_k],\ k = 1, 2, \dots, \\ u_0\Big(\prod_{j=1}^{k} b_j\Big) E_q(A(t-s_k)^q) & \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots \end{cases}$$

and

$$v(t) = \begin{cases} v_0\Big(\prod_{j=1}^{k} c_j\Big) & \text{for } t \in (t_k, s_k],\ k = 1, 2, \dots, \\ v_0\Big(\prod_{j=1}^{k} c_j\Big) E_q(-B(t-s_k)^q) & \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots \end{cases}$$

where the Mittag-Leffler function (with one parameter) is defined by $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$.

Let $\epsilon_1 > 0$ be arbitrary. Choose $\delta_1 = \frac{\epsilon_1}{ME_q(A)}$ and let $|u_0| < \delta_1$. Then from $1 \le E_q(A(t-s_k)^q) \le E_q(A)$ for $t \in (s_k, t_{k+1}]$ we obtain $|u(t)| \le |u_0| \Big(\prod_{j=1}^{k} b_j\Big) E_q(A) < \epsilon_1$. For any $\delta_2 \in (0, \delta_1]$ we choose $\epsilon_2 = \delta_2 E_q(-B) N \le \delta_2$. Then for $|v_0| > \delta_2$ using $1 \ge E_q(-B(t-s_k)^q) \ge E_q(-B)$ we obtain $|v(t)| \ge |v_0| \Big(\prod_{j=1}^{k} c_j\Big) E_q(-B) > \epsilon_2$. Therefore, the zero solution of the couple of FrDE (13) is uniformly strictly stable in couple.

Note the above conclusion is true if $b_k = 1 - \frac{1}{2^k}, c_k = 1 + \frac{1}{2^k}, M = 0.288, N = 2.38$ (see Figure 5 for $q = 0.2$, $A = B = 1$ and various initial conditions).

The conclusion of Example 3 is true also if $A = B = 0$.

□

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the strict stability of the zero solution of the system NIFrDE (7).

Definition 5. *Let $J \in \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t, x) : J \times \Delta \to \mathbb{R}_+$, $V(t, 0) \equiv 0$ belongs to the class $\Lambda(J, \Delta)$ if*

1. *The function $V(t, x)$ is continuous on $J/\{t_k \in J\} \times \Delta$ and it is locally Lipschitzian with respect to its second argument;*
2. *For each $t_k \in Int(J)$ and $x \in \Delta$ there exist finite limits*

$$V(t_k - 0, x) = \lim_{t \uparrow t_k} V(t, x) < \infty, \quad \text{and} \ \ V(t_k + 0, x) = \lim_{t \downarrow t_k} V(t, x) < \infty$$

and the following equalities are valid

$$V(t_k - 0, x) = V(t_k, x).$$

We now give a brief overview of the literature on derivatives of Lyapunov functions to fractional differential equations of type (5):

- some authors (see, for example, [34]) used the so called Caputo fractional derivative of the Lyapunov function ${}_{t_0}^cD^q V(t, x(t))$ where $x(t)$ is the unknown solution of the studied fractional differential equation. This approach requires the function to be smooth enough (at least continuously differentiable) and also some conditions involved are quite restrictive. Additionally, the Caputo fractional derivative ${}_{t_0}^cD^q V(t, x(t))$ in some cases is very difficult to obtain.

- other authors used the so called Dini fractional derivative of the Lyapunov function ([31], [32]). This is based on the Dini derivative of the Lyapunov function $V(t,x)$ among the ordinary differential equation $x' = f(t,x)$ given by

$$DV(t,x) = \limsup_{h\to 0} \frac{1}{h}\Big[V(t,x) - V(t-h, x - hf(t,x))\Big]. \tag{14}$$

The authors generalized (14) to the *Dini fractional derivative* along the FrDE ${}_{t_0}^{c}D^q x = f(t,x),\ t \geq t_0$ by

$${}^{c}D_{+}^{q}V(t,x) = \limsup_{h\to 0} \frac{1}{h^q}\Big[V(t,x) - V(t-h, x - h^q f(t,x))\Big]. \tag{15}$$

This definition requires only the continuity of the Lyapunov function. However the fractional Dini derivative (15) is a local operator, so it is not acceptable as a fractional derivative.

Example 4. Consider the IVP for FrDE (5) with $\tau_k = t_0,\ t_{k+1} = \infty$, where t_0 is given. Let $V(t,x) = m(t)x^2$ where the function $m \in C^1([t_0,\infty), \mathbb{R}_+)$.

- Consider the Caputo fractional derivative of the Lyapunov function ${}_{t_0}^{c}D^q V(t,x(t)) = {}_{t_0}^{c}D^q\Big(m(t)x^2(t)\Big)$ where $x(t)$ is the unknown solution of (5). To use the product rule ($m(t) \not\equiv 1$) for Caputo fractional derivatives is complicated.

- Apply formula (15) to get the Dini fractional derivative and we obtain

$${}^{c}D_{+}^{q}V(t,x) = 2xm(t)f(t,x). \tag{16}$$

The Dini fractional derivative (15) does not depend on the order of the differential equation q. However the behavior of solutions of fractional differential equations depends significantly on the order q. For example, let $t_0 = 0, x_0 = 0$ and $f(t,x) = 1 - x(t)$. Then the solution of (5) is given by $x(t) = t^q E_{q,1+q}(-t^q)$. Note $lim_{t\to\infty} x(t) = a$ varies for different values of the order q of the fractional differential equation. The derivative ${}^{c}D_{+}^{q}V(t,x) = 2x(1-x)m(t)$ which is often used to study the behaviour of the solution for t approaching infinity does not depend on q.

Consider the integer case, i.e. $q = 1$ and the corresponding ordinary differential equation $x' = f(t,x)$. Use (14) to obtain the Dini derivative

$$D_{+}V(t,x) = 2x\, m(t)f(t,x) + \frac{d}{dt}\Big[x^2 m(t)\Big]. \tag{17}$$

From (16) and (17) it follows that the Dini fractional derivative gives a quite different result compared with the ordinary case ($q = 1$).

□

In [3], [8], [4], [5] another type of derivative of Lyapunov functions is introduced, the so called *Caputo fractional Dini derivative* and it is based on the Caputo fractional Dini derivative of a function $m(t)$ given by (3). This derivative is given by

$$\begin{aligned} {}_{(5)}^{\ c}D_{+}^{q}V(t,x;t_0,x_0) = \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ & V(t,x) - V(t_0,x_0) \\ - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r} & \Big[V(t - rh, x - h^q f(t,x)) - V(t_0,x_0)\Big]\Big\} \\ & \text{for } t \in (t_0, T), \end{aligned} \tag{18}$$

where $\tau_k = t_0,\ t_{k+1} = T \leq \infty$, and $t_0 \geq 0, t > t_0, x, x_0 \in \mathbb{R}^n$.

Example 5. Consider as in Example 4 the IVP for FrDE (5) with $\tau_k = t_0,\ t_{k+1} = \infty$, where t_0 is given. Let $V(t,x) = m(t)x^2$ where the function $m \in C^1([t_0,\infty),\mathbb{R}_+)$.

For any points $t_0 \ge 0, t > t_0, x, x_0 \in \mathbb{R}^n$ we use (18) to obtain the Caputo fractional Dini derivative of the function V:

$$\begin{aligned}
&{}^{c}_{(5)}D^q_+ V(t,x;t_0,x_0)\\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big[m(t)\Big(x^2 - (x - h^q f(t,x))^2\Big) + \sum_{r=0}^{[\frac{t-t_0}{h}]} m(t_0)(x_0)^2 + (x - h^q f(t,x))^2 \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh)\Big]\\
&= \lim_{h\to 0^+}\sup \Big[m(t)f(t,x)\Big(2x + h^q f(t,x)\Big) + \frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r}\Big(m(t-rh)x^2 - m(t_0)(x_0)^2\Big)\\
&\quad + (h^q f(t,x)^2 - 2xf(t,x))\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh)\Big].
\end{aligned} \tag{19}$$

Now using $m \in C^1([t_0,\infty),\mathbb{R}_+)$, $\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh) = \Big(\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r}(m(t-rh)-m(t))\Big) + m(t)\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} = h\Big(\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m'(\xi_h)\Big) + m(t)\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r}$ where $\xi_h \in (t-rh,t)$, and $\lim_{h\to 0}\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} = 0$ we see the final sum in (19) is 0.

Using $m \in C^1([t_0,\infty),\mathbb{R}_+)$, $W(t) = m(t)x^2 - m(t_0)(x_0)^2 \in C^1([t_0,\infty),\mathbb{R}_+)$ for any fixed $t_0 \ge 0, x, x_0 \in \mathbb{R}^n$, $\lim_{h\to 0^+}\sup \frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} W(t-rh) = {}^{GL}_{t_0}D^q_+ W(t) = {}^{GL}_{t_0}D^q W(t) = {}^{RL}_{t_0}D^q W(t)$ and we obtain

$${}^{c}_{(5)}D^q_+ V(t,x;t_0,x_0) = 2xm(t)f(t,x) + {}^{RL}_{t_0}D^q\Big(m(t)x^2 - m(t_0)(x_0)^2\Big). \tag{20}$$

Note the Caputo fractional Dini derivative ${}^{c}_{(5)}D^q_+ V(t,x;t_0,x_0)$ depends significantly not only on the order q of the fractional differential equation but also on the initial data.

From (17) and (20) it seems both formulas are similar, with the ordinary derivative replaced by a fractional derivative. Therefore, the Caputo fractional Dini derivative given by formula (22) seems to be the natural generalization of the derivative of Lyapunov functions for ordinary differential equations.

□

In this paper we will use piecewise continuous Lyapunov functions from the above defined class $\Lambda(J,\Delta)$. We introduce in an appropriate way the derivative of the Lyapunov function among a fractional differential equation with noninstantaneous impulses.

We define the *generalized Caputo fractional Dini derivative* of the Lyapunov-like function $V(t,x) \in \Lambda(J,\Delta)$ along trajectories of solutions of IVP for the system NIFrDE (7). It is based on the Caputo fractional Dini derivative of a function $m(t)$ given by (3) and it is based on Eq. (18). The Caputo fractional Dini derivative along trajectories of solutions of IVP for the system NIFrDE (7) is given by:

$$\begin{aligned}
{}^{c}_{(7)}D^q_+ V(t,x;t_0,x_0) &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{V(t,x) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\Big[V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)\Big]\Big\}\\
&\text{for } t \in (s_k,t_{k+1})\cap J,\ k = 0,1,2,\ldots,
\end{aligned} \tag{21}$$

where $x, x_0 \in \Delta$, $t_0 \in J$, and for any $t \in (s_k,t_{k+1})\cap J$ there exists $h_t > 0$ such that $t - h \in (s_k,t_{k+1})\cap J$, $x - h^q f(t,x) \in \Delta$ for $0 < h \le h_t$.

Remark 10. *The Caputo fractional Dini derivative of continuous Lyapunov functions was introduced and used to study stability properties of the zero solution of Caputo fractional differential equations in [3], [5]. Also, the generalized*

Caputo fractional Dini derivative w.r.t. to initial time difference along trajectories of solutions of a nonlinear system of FrDE was defined in [9] and used to study stability of FrDE with initial time difference.

Formula (21) could be reduced to

$$
\begin{aligned}
{}_{(7)}^{\;c}D_{+}^{q}V(t,x;t_0,x_0) &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{V(t,x)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}V(t-rh,x-h^q f(t,x))\Big\} \\
&\quad - V(t_0,x_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \quad \text{for } t\in(s_k,t_{k+1})\cap J,\ k=0,1,2,\dots.
\end{aligned}
\tag{22}
$$

Remark 11. *Note, that if in (21) (respectively (22)) instead of a point $x\in\mathbb{R}^n$ we use the value $y(t)$ of the function $y:\Big(\cup_{k=0}^{\infty}(s_k,t_{k+1})\Big)\cap J\to\mathbb{R}^n:\ y(t_0)=x_0$, then*

$$
\begin{aligned}
{}_{(7)}^{\;c}D_{+}^{q}V(t,y(t);t_0,x_0) &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{V(t,y(t))-V(t_0,x_0) \\
&\quad -\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big[V(t-rh,y(t)-h^q f(t,y(t)))-V(t_0,x_0)\Big]\Big\}
\end{aligned}
\tag{23}
$$

Example 6. Consider the scalar case, i.e. let $V(t,x)=x^2$ where $x\in\mathbb{R}$.
Then for any $t\in\Big(\cup_{k=0}^{\infty}(s_k,t_{k+1})\Big)\cap J$ and $x\in\mathbb{R}$ we have from (21)

$$
\begin{aligned}
{}_{(7)}^{\;c}D_{+}^{q}V(t,x;t_0,x_0) &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big[\big(x^2-x_0^2\big)-(x-h^q f(t,x))^2+(x_0)^2\Big] \\
&\quad -\limsup_{h\to 0^+}\frac{1}{h^q}\Big[(x-h^q f(t,x))^2-(x_0)^2\Big]\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\} \\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\Big[h^q f(t,x)(2x-h^q f(t,x))\Big] \\
&\quad -\limsup_{h\to 0^+}\frac{1}{h^q}\Big[-h^q f(t,x)\big(2x-h^q f(t,x)\big)+x^2-(x_0)^2\Big]\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\} \\
&= f(t,x)\limsup_{h\to 0^+}(2x-h^q f(t,x))+f(t,x)\limsup_{h\to 0^+}\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\} \\
&\quad -(x^2-(x_0)^2)\limsup_{h\to 0^+}\frac{1}{h^q}\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\} \\
&= 2xf(t,x)-(x^2-(x_0)^2)\,{}_{t_0}^{GL}D_{+}^{q}1 = 2xf(t,x)-(x^2-(x_0)^2)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned}
\tag{24}
$$

Now consider a scalar function $y(t)$; $y(t_0)=x_0$, defined for any $t\in\Big(\cup_{k=0}^{\infty}(s_k,t_{k+1})\Big)\cap J$. Then for any fixed $t\in\Big(\cup_{k=0}^{\infty}(s_k,t_{k+1})\Big)\cap J$ we have from (23)

$$
\begin{aligned}
{}_{(7)}^{c}D_{+}^{q}V(t,y(t);t_0,x_0) &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big[\Big((y(t))^2-x_0^2\Big)-(y(t)-h^q f(t,y(t)))^2+(x_0)^2\Big]\\
&-\limsup_{h\to 0^+}\frac{1}{h^q}\Big[(y(t)-h^q f(t,y(t)))^2-(x_0)^2\Big]\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\}\\
&= f(t,y(t))\limsup_{h\to 0^+}(2x-h^q f(t,y(t)))+f(t,y(t))\limsup_{h\to 0^+}\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\}\\
&-((y(t))^2-(x_0)^2)\limsup_{h\to 0^+}\frac{1}{h^q}\Big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big\}\\
&= 2y(t)f(t,y(t))-((y(t))^2-(x_0)^2)\,{}_{t_0}^{GL}D_{+}^{q}1\\
&= 2y(t)f(t,y(t))-((y(t))^2-(x_0)^2)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned}
\tag{25}
$$

□

5. Comparison Results for the Scalar NIFrDE

Let $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ so without loss of generality we assume $t_0 \in [s_0, t_1)$. We will obtain some comparison results for NIFrDE (7) using definition (22) for a generalized Caputo fractional Dini derivative of Lyapunov-like function.

We will use the following comparison result, which is proved and used in [3] for studying stability properties of Caputo fractional differential equations

$$
\begin{aligned}
&{}_{t_0}^{c}D^q x = f(t,x) \text{ for } t \geq t_0,\\
&x(t_0) = x_0
\end{aligned}
\tag{26}
$$

where $x, x_0 \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 3. *[3](Comparison result for FrDE.) Assume the following conditions are satisfied:*

1. The function $x^*(t) = x(t; t_0, x_0)$, $x^* \in C^q([t_0, T], \Delta)$, is a solution of the FrDE (26) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, t_0, $T \in \mathbb{R}_+$, $t_0 < T$ are given constants, $x_0 \in \Delta$.

2. The function $g \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$.

3. The function $V \in \Lambda([t_0, T], \Delta)$ and for any $t \in (t_0, T]$ the inequality

$$
{}_{(26)}^{c}D_{+}^{q}V(t, x^*(t); t_0, x_0) \leq (\geq) g(t, V(t, x^*(t)))
$$

holds.

4. The function $u^*(t) = u(t; t_0, u_0)$, $u^* \in C^q([t_0, T], \mathbb{R})$, is the maximal solution (minimal solution) of the initial value problem ${}_{t_0}^{c}D^q u = g(t, u)$ for $t \in [t_0, T]$ with $u(t_0) = u_0$.

Then the inequality $V(t_0, x_0) \leq (\geq) u_0$ implies $V(t, x^*(t)) \leq (\geq) u^*(t)$ for $t \in [t_0, T]$.

If $g(t, x) \equiv 0$ in Lemma 1 we obtain the following result:

Corollary 1. *[3]. Assume the following conditions are satisfied:*

1. The function $x^*(t) = x(t; t_0, x_0)$, $x^* \in C^q([t_0, T], \Delta)$, is a solution of the FrDE (26) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.

2. The function $V \in \Lambda([t_0, T], \Delta)$ and for any $t \in (t_0, T]$ the inequality

$$
{}_{(26)}^{c}D_{+}^{q}V(t, x^*(t); t_0, x_0) \leq (\geq) 0
$$

holds.

Then for $t \in [t_0, T]$ the inequality $V(t, x^*(t)) \leq (\geq) V(t_0, x_0)$ holds.

The result of Lemma 1 is also true on the half line.

Corollary 2. *[3]. Assume the following conditions are satisfied:*

1. The function $x^*(t) = x(t; t_0, x_0)$, $x^* \in C^q([t_0, \infty), \Delta)$, is a solution of the FrDE (26) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.

2. The function $g \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$

3. The function $V \in \Lambda([t_0, \infty), \Delta)$ and for any $t > t_0$ the inequality

$$ {}^{c}_{(26)}D^q_+ V(t, x^*(t); t_0, x_0) \le (\ge) g(t, V(t, x^*(t))) $$

holds.

4. The function $u^*(t) = u(t; t_0, u_0)$, $u^* \in C^q([t_0, \infty), \mathbb{R})$, is the maximal solution (minimal solution) of the initial value problem ${}^{c}_{t_0}D^q u = g(t, u)$ for $t \ge t_0$ with $u(t_0) = u_0$.

Then the inequality $V(t_0, x_0) \le (\ge) u_0$ implies $V(t, x^*(t)) \le (\ge) u^*(t)$ for $t \ge t_0$.

Now we will prove some comparison results for non-instantaneous impulsive fractional differential equations.

Lemma 4. *(Comparison result for NIFrDE). Let:*

1. *The function $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \Theta]], \Delta)$ is a solution of the NIFrDE (7) where $t_0 \in [0, s_0)$, $\Delta \subset \mathbb{R}^n$, $\Theta \in (t_p, s_p]$, p is a natural number.*
2. *For any natural number $k :\ t_k \in (t_0, \Theta)$ the condition (H5) is satisfied.*
3. *The condition (H3) (respectively (H4)) are satisfied on $[t_0, \Theta]$.*
4. *The function $V \in \Lambda([t_0, \Theta], \Delta)$ and*
   (i) for any $k = 0, 1, 2, \dots, p-1$ the inequality

$$ {}^{c}_{(7)}D^q_+ V(t, x^*(t); \tau_k, y_0) \le (\ge) g(t, V(t, x^*(t))) \quad \text{for} \quad t \in (\tau_k, t_{k+1}],\ y_0 = x^*(\tau_k) $$

   holds;

   (ii) for any $k = 0, 1, 2, \dots, p-1$ the inequalities

$$ V(t, \phi_k(t, x^*(t_k - 0))) \le (\ge) \Xi_k(t, V(t_k - 0, x^*(t_k - 0))) \ \text{for}\ t \in [t_0, \Theta] \bigcap (t_k, s_k] $$

   hold.

Then the inequality $V(t_0, x_0) \le (\ge) u_0$ implies $V(t, x^*(t)) \le (\ge) u^*(t)$ on the interval $[t_0, \Theta]$ where $u^*(t) = u(t; t_0, u_0) \in PC^q([t_0, \Theta], \mathbb{R})$ is the maximal solution (minimal solution) of (10) .

Remark 12. *Note the case of $\le$ and $g(t, u) \equiv 0$ is studied in [6].*

P r o o f: Consider the case of $\ge$ in Conditions 4(i), 4(ii). According to Lemma 2 there exists a minimal solution $u^*(t) = u(t; t_0, u_0) \in PC^q([t_0, \Theta], \mathbb{R})$ of the IVP for NIFrDE (10). Without loss of generality we assume $t_0 \in [s_0, t_1)$. Let $V(t_0, x_0) \ge u_0$. We use induction to prove Lemma 4.

Let $t \in [t_0, t_1] \cap [t_0, \Theta]$. The function $x^*(t) \in C^q([t_0, T], \mathbb{R}^n)$, satisfies FrDE (26) with initial time t_0 and $T = \min\{t_1, \Theta\}$. From condition 4 with $k = 0$, $y_0 = x_0$ it follows ${}^{c}_{(26)}D^q_+ V(t, x^*(t); t_0, x_0) = {}^{c}_{(7)}D^q_+ V(t, x^*(t); t_0, x_0) \ge g(t, V(t, x^*(t)))$for $t \in (\tau_0, t_1]$. Therefore, condition 3 of Lemma 3 is satisfied. Also, the function $u^*(t)$ is a minimal solution of ${}^{c}_{t_0}D^q u = g(t, u)$ for $t \in [t_0, T]$ with $u(t_0) = u_0$. Therefore, all conditions of Lemma 3 are satisfied on the interval $[t_0, t_1]$ and therefore the inequality $V(t, x^*(t)) \ge u^*(t)$ holds on $[t_0, t_1] \cap [t_0, \Theta]$.

Let $\Theta > t_1$ and $t \in (t_1, s_1] \cap [t_0, \Theta]$. Then $x^*(t) = \phi_1(t, x^*(t_1 - 0))$. From conditions 2, 4(ii) and the above we get $V(t, x^*(t)) = V(t, \phi_1(t, x^*(t_1 - 0))) \ge \Xi_1(t, V(t_1 - 0, x^*(t_1 - 0)) \ge \Xi_1(t, u^*(t_1 - 0)) = u^*(t)$, $t \in (t_1, s_1] \cap [t_0, \Theta]$.

Let $\Theta > s_1$ and $t \in (s_1, t_2] \cap [t_0, \Theta]$. Consider the function $\overline{x}_1(t) = x^*(t)$ for $t \in (s_1, t_2]$ and $\overline{x}_1(s_1) = x^*(s_1) = \phi_1(s_1, x^*(t_1 - 0))$. The function $\overline{x}_1(t) \in C^q([s_1, T], \mathbb{R}^n)$ and satisfies IVP for FrDE (26) with initial time

$t_0 = s_1$, initial value $x_0 = x^*(s_1)$, and $T = \min\{t_2, \Theta\}$. From condition 4(i) for $k = 1$ and $y_0 = x(s_1)$ it follows ${}^{c}_{(26)}D^q_+ V(t, x^*(t); t_1, x(s_1)) = {}^{c}_{(7)}D^q_+ V(t, x^*(t); t_1, x(s_1)) \geq g(t, V(t, x^*(t)))$ for $t \in (s_1, t_2]$. Therefore, condition 3 of Lemma 3 is satisfied.Also, the function $u^*(t)$ is a minimal solution of ${}^{c}_{s_1}D^q u = g(t, u)$ for $t \in [s_1, t_2]$ with $u(s_1) = u^*(s_1)$. Therefore, all conditions of Lemma 3 are satisfied on the interval $[s_1, t_2]$. According to Lemma 3 for the function $\bar{x}_1(t)$ and the above we obtain $V(t, x^*(t)) = V(t, \bar{x}_1(t)) \geq u^*(t)$ for $t \in (s_1, t_2] \cap [t_0, \Theta]$.

Continue this process and an induction argument proves the claim of Lemma 4 is true for $t \in [t_0, \Theta]$.

The proof of Lemma 4 in the case of $\leq$ in Condition 4 and $u^*(t) = u(t; t_0, u_0) \in PC^q([t_0, \Theta], \mathbb{R})$ is the maximal solution of the IVP for NIFrDE (10)(it exsts according to Lemma 1) is similar and we omit it.

□

Based on Corollary 2 the following comparison results for non-instantaneous impulsive fractional differential equations on an infinite interval is true:

Corollary 3. *(Comparison result for NIFrDE on a half plane). Let:*

1. *The function $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \Delta)$ is a solution of the NIFrDE (7) where $t_0 \in [0, s_0]$, $\Delta \subset \mathbb{R}^n$.*
2. *For any natural number k the condition (H5) is satisfied.*
3. *The condition (H3) (respectively (H4)) is satisfied for $[t_0, \infty)$.*
4. *The function $V \in \Lambda([t_0, \infty), \Delta)$ and*
   - *(i) for any $k = 0, 1, 2 \ldots$ the inequality*
     $${}^{c}_{(7)}D^q_+ V(t, x^*(t); \tau_k, y_0) \leq (\geq) g(t, V(t, x^*(t))) \quad for \quad t \in (\tau_k, t_{k+1}], \ y_0 = x^*(\tau_k)$$
     holds;
   - *(ii) for any $k = 0, 1, 2, \ldots$ the inequalities*
     $$V(t, \phi_k(t, x^*(t_k - 0))) \leq (\geq) \Xi_k(t, V(t_k - 0, x^*(t_k - 0))) \ for \ t \in [t_0, \infty) \bigcap (t_k, s_k]$$
     hold.

Then the inequality $V(t_0, x_0) \leq (\geq) u_0$ implies $V(t, x^*(t)) \leq (\geq) u^*(t)$ for $t \geq t_0$.

6. Main Results

We obtain sufficient conditions for strict stability of the system NIFrDE (7). Again we assume $0 < q < 1$.

Theorem 1. *(Strict stability of NIFrDE). Let the following conditions be satisfied:*

1. *The conditions (H1) and (H3) for $g(t, u) = g_1(t, u)$, and (H3) for $g(t, u) = g_2(t, u)$ are satisfied and for any natural number k the condition (H5) holds for functions $\Xi_k = \Phi_k$ and $\Xi_k = \Psi_k$.*
2. *There exists a function $V_1 \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that $V_1(t, 0) \equiv 0$ for $t \in \mathbb{R}_+$ and*
   - *(i) for any $k = 0, 1, 2, \ldots$ and $x_0 \in \mathbb{R}^n$ the inequality*
     $${}^{c}_{(7)}D^q_+ V_1(t, x; s_k, x_0) \leq g_1(t, V_1(t, x)) \quad for \ t \in (s_k, t_{k+1}), \ x \in \mathbb{R}^n$$
     holds;
   - *(ii) for any natural number k the inequality*
     $$V_1(t, \phi_k(t, x)) \leq \Phi_k(t, V_1(t_k - 0, x)) \ for \ t \in (t_k, s_k], \ x \in \mathbb{R}^n$$
     holds;

(iii) $a(\|x\|) \le V_1(t,x)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $a \in \mathcal{K}$.

3. *There exists a function* $V_2 \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*

(iv) for any $k = 0,1,2,\dots$ *and* $x_0 \in \mathbb{R}^n$ *the inequality*

$$ {}_{(7)}^{\,c}D_+^q V_2(t,x;s_k,x_0) \ge g_2(t, V_2(t,x)) \quad \text{for } t \in (s_k, t_{k+1}),\ x \in \mathbb{R}^n $$

holds;

(v) for any natural number k the inequality

$$ V_2(t, \phi(t,x)) \ge \Psi_k(t, V_2(t_k - 0, x)) \ \text{for } t \in (t_k, s_k],\ x \in \mathbb{R}^n $$

holds;

(vi) $c(\|x\|) \le V_2(t,x) \le b(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $b, c \in \mathcal{M}$.

4. *The zero solution of NIFrDE (12) is strictly stable in couple.*

Then the zero solution of the system NIFrDE (7) is strictly stable.

P r o o f: Let $\varepsilon_1 > 0$ and $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ be arbitrary. Without loss of generality we assume $t_0 \in [s_0, t_1)$. From condition 4 there exists $\delta_1 = \delta_1(t_0, \varepsilon_1) \ge 0$ and for any $\delta_2 \in (0, \delta_1]$ there exists $\varepsilon_2 \in (0, \delta_2]$ such that $|u_0| < \delta_1$ and $|v_0| > \delta_2$ imply

$$ |u(t;t_0,u_0)| < a(\varepsilon_1) \quad \text{for } t \ge t_0, \tag{27} $$

$$ |v(t;t_0,v_0)| > \varepsilon_2 \quad \text{for } t \ge t_0, \tag{28} $$

where the couple $\big(u(t;t_0,u_0), v(t;t_0,v_0)\big)$ is a solution of (12).

Since $V_1(t_0, 0) = 0$ there exists $\delta_3 = \delta_3(t_0, \varepsilon)$, $\delta_3 \in (0, \delta_1)$ such that $V_1(t_0, x) < \delta_1$ for $\|x\| < \delta_3$.

Let $\delta_4 \in (0, \delta_3]$ be an arbitrary number. Then there exists $\delta_5 \in (0, \delta_1]$ such that $c(\delta_4) > \delta_5$. According to the above for $\delta_5 \in (0, \delta_1]$ there exists $\varepsilon_3 \in (0, \delta_5]$ such that $|v_0| > \delta_5$ implies

$$ |v(t;t_0,v_0)| > \varepsilon_3, \quad t \ge t_0. \tag{29} $$

Choose $\varepsilon_4 > 0$ such that $\varepsilon_4 < \min\{b^{-1}(\varepsilon_3), \delta_4\}$.

Choose $x_0 \in \mathbb{R}^n$ with $\delta_4 < \|x_0\| < \delta_3$ and let $x^*(t) = x(t;t_0,x_0)$ be a solution of the IVP for NIFrDE (7) for the initial data (t_0, x_0).

Let $u_0 = V_1(t_0, x_0)$ and $v_0 = V_2(t_0, x_0)$. Let $\big(u^*(t), v^*(t)\big)$ be the solution of the IVP for the couple of NIFrDE (12) with initial values (u_0, v_0) where $u^*(t) = u(t;t_0,u_0)$, $v^*(t) = v(t;t_0,v_0)$ are the maximal solution and the minimal solution of the first and second equation of (12), respectively.

From condition 2(i) of Theorem 1 for the function $V_1(t,x)$ and Remark 11 it follows that for any $k = 0,1,2,\dots$ for $t_0 = \tau_k$, $x = x^*(t)$ and $x_0 = x^*(\tau_k)$ the inequality ${}_{(7)}^{\,c}D_+^q V_1(t, x^*(t); \tau_k, x_0) \le g_1(t, V_1(t,x))$ for $t \in (\tau_k, t_{k+1}]$ holds, i.e. condition 4(i) of Corollary 3 is satisfied with inequality $\le$.

From Condition 2(ii) of Theorem 1 for the function $V_1(t,x)$ with $x = x^*(t_k - 0)$ we have Condition 4(ii) of Corollary 3 with inequality $\le$. According to Corollary 3 the inequality $V_1(t, x^*(t)) \le u^*(t)$, $t \ge t_0$ holds.

Similarly from conditions 3(i) and 3(ii) of Theorem 1 for the function $V_2(t,x)$ and Remark 11 we have that Conditions 4(i) and 4(ii) of Corollary 3 are satisfied with inequality $\ge$. According to Corollary 3 the inequality $V_2(t, x^*(t)) \ge v^*(t)$, $t \ge t_0$ holds.

From the choice of x_0 it follows that $|u_0| < \delta_1$. Therefore, the function $u^*(t)$ satisfies the inequality (27). From condition 2(iii) and above we obtain $a(\|x^*(t)\|) \le V(t, x^*(t)) \le u^*(t) < a(\varepsilon_1)$, $t \ge t_0$. Therefore $\|x^*(t)\| < \varepsilon_1$ for $t \ge t_0$.

From the choice of x_0 and condition 3(iv) it follows that $|v_0| = V_2(t_0, x_0) \ge c(\|x_0\|) > c(\delta_4) > \delta_5$. Therefore, the function $v^*(t)$ satisfies the inequality (29).

From condition 3(vi) and above we obtain $b(\|x^*(t)\|) \ge V(t, x^*(t)) \ge v^*(t) > \varepsilon_3$, $t \ge t_0$. Therefore $\|x^*(t)\| \ge b^{-1}(\varepsilon_3) > \varepsilon_4$ for $t \ge t_0$.

Since δ_4 is an arbitrary, from the above we have the strict stability of the zero solution of NIFrDE (7).

□

Remark 13. *If all solutions of IVP for NIFrDE (7) satisfy* $\|x(t)\| \leq (\geq)\|x_0\|$ *then the claim of Theorem 1 is true if the conditions 2, 3 are satisfied only for points* $x, x_0 \in \mathbb{R}^n$ *such that* $\|x\| \leq (\geq)\|x_0\|$.

Theorem 2. *(Uniform strict stability of NIFrDE). Let the following conditions be satisfied:*

1. *The condition 1 of Theorem 1 is satisfied.*
2. *There exists a function* $V_1 \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*
   (i) for any $k = 0, 1, 2, \dots$ *and* $x_0 \in B(\lambda)$ *the inequality*
   $${}_{(7)}^{c}D_+^q V_1(t, x; s_k, x_0) \leq g_1(t, V_1(t, x)) \quad \text{for } t \in (s_k, t_{k+1}],\ x \in B(\lambda)$$
   holds where $\lambda > 0$ *is a given number;*
   (ii) for any natural number k the inequality
   $$V_1(t, \phi(t, x)) \leq \Phi_k(t, V_1(t_k - 0, x)) \ \text{for } t \in (t_k, s_k],\ x \in B(\lambda)$$
   holds;
   (iii) $a(\|x\|) \leq V_1(t, x) \leq b(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $a, b \in \mathcal{K}$.
3. *For any* $\eta \in (0, \lambda)$ *there exists a function* $V_\eta \in \Lambda(\mathbb{R}_+, \mathbb{R}^n$ *such that*
   (iv) for any $k = 0, 1, 2, \dots$ *and* $x_0 \in B(\lambda)$ *the inequality*
   $${}_{(7)}^{c}D_+^q V_\eta(t, x; s_k, x_0) \geq g_2(t, V_\eta(t, x)) \quad \text{for } t \in (s_k, t_{k+1}],\ x \in B(\lambda),\ \|x\| \geq \eta$$
   holds;
   (v) for any natural number k the inequality
   $$V_\eta(t, \phi(t, x)) \geq \Psi_k(t, V_\eta(t_k - 0, x)) \ \text{for } t \in (t_k, s_k],\ x \in B(\lambda)$$
   holds;
   (vi) $c(\|x\|) \leq V_\eta(t, x) \leq d(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in B(\lambda)$, *where* $d, c \in \mathcal{K}$
4. *The zero solution of the couple of NIFrDE (12) is uniformly strict stable in couple.*

Then the zero solution of the system NIFrDE (7) is uniformly strictly stable.

P r o o f: Let $\varepsilon_1 \in (0, \lambda]$ be an arbitrary number. From condition 4 there exists $\delta_1 = \delta_1(\epsilon_1) > 0$ and for any $\delta_2 \in (0, \delta_1]$ there exists $\epsilon_2 \in (0, \delta_2]$ such that for any $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ the inequalities $|u_0| < \delta_1$ and $\delta_2 < |v_0|$ imply

$$|u(t; t_0, u_0)| < a(\epsilon_1), \qquad t \geq t_0 \tag{30}$$

and

$$\epsilon_2 < |v(t; t_0, v_0)|, \qquad t \geq t_0 \tag{31}$$

where the couple of functions $\big(u(t; t_0, u_0), v(t; t_0, u_0)\big)$ is a solution of the IVP for NIFrDE (12).

Let $\delta_3 \in (0, \lambda)$ be such that $b(\delta_3) < \delta_1$. Choose $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta_3$ and $x^*(t) = x(t; t_0, x_0)$ be the solution of the IVP for NIFrDE (7) for the initial data (t_0, x_0).

Let $u_0 = V_1(t_0, x_0)$. Let $u^*(t) = u(t; t_0, u_0)$ be the maximal solution of the first equation of the couple of NIFrDE (12). According to condition 2*(iii)* and the choice of x_0 we obtain $u_0 = V_1(t_0, x_0) \leq b(\|x_0\|) < b(\delta_3) < \delta_1$. Therefore the function $u^*(t)$ satisfies (30).

Assume inequality

$$\|x^*(t)\| < \varepsilon_1 \quad \text{for } t \ge t_0 \tag{32}$$

is not true. There are three possible cases.

Case 1. There exists a point $t^* > t_0,\ t \neq s_k,\ k = 0, 1, 2\ldots$ such that

$$\|x^*(t)\| < \varepsilon_1 \quad \text{for } t \in [t_0, t^*) \quad \text{and} \quad \|x^*(t^*)\| = \varepsilon_1. \tag{33}$$

According to Lemma 2 for $T = t^*$ and $\Delta = B(\lambda)$ we obtain $V_1(t, x^*(t)) \le u^*(t)$ for $t \in [t_0, t^*]$. Then from conditions 1, 2(ii) we get $a(\|x^*(t^*)\|) \le V_1(t, x^*(t^*)) \le u^*(t^*) < a(\varepsilon_1)$. We obtain a contradiction.

Case 2 There exists an integer $k \ge 0$ such that

$$\|x^*(t)\| < \varepsilon_1 \quad \text{for } t \in [t_0, s_k) \quad \text{and} \quad \|x^*(s_k - 0)\| = \varepsilon_1. \tag{34}$$

As in Case 1 with replacing t^* by s_k we obtain a contradiction.

Case 3.. There exists an integer $k \ge 0$ such that

$$\|x^*(t)\| < \varepsilon_1 \quad \text{for } t \in [t_0, s_k] \quad \text{and} \quad \|x^*(s_k + 0)\| \ge \varepsilon_1. \tag{35}$$

According to Lemma 2 for $T = s_k$ and $\Delta = B(\lambda)$ we obtain $V_1(t, x^*(t)) \le u^*(t)$ for $t \in [t_0, s_k]$.

Then $x^*(s_k + 0) = \phi_k(s_k, x^*(s_k - 0))$ and according to conditions 2 (ii) and 2(iii) we get

$$\begin{aligned} a(\varepsilon_1) &\le a(\|x^*(s_k + 0)\|) = a(\|\phi_k(s_k, x^*(s_k - 0))\|) \\ &\le V_1(s_k, \phi_k(s_k, x^*(s_k - 0))) \le \Psi_k(s_k, V_1(s_k - 0, x^*(s_k - 0))) \\ &\le \Psi_k(s_k, u(s_k - 0)) = u(s_k + 0) < a(\varepsilon_1). \end{aligned} \tag{36}$$

The contradictions above prove inequality (32) is true.

Let $\delta_4 \in (0, \delta_3]$ be an arbitrary number. Then there exists $\delta_5 \in (0, \delta_4]$ such that $c(\delta_4) > \delta_5$. Let the initial value $x_0 \in \mathbb{R}^n$ additionally satisfy $\|x_0\| > \delta_4$. From condition 3(iv) for $\eta = \delta_4$ there exists a function $V_\eta(t, x)$ and $V_\eta(t_0, x_0) \ge c(\|x_0\|) > c(\delta_4) > \delta_5$. Let $v_0 = V_\eta(t_0, x_0)$ and $v^*(t) = v(t; t_0, v_0)$ be the minimal solution of the second equation of (12). According to condition 4 for δ_5 there exists $\varepsilon_2^* \in (0, \delta_5]$ such that $|v_0| > \delta_5$ implies the inequality (31) with $\varepsilon_2 = \varepsilon_2^*$.

Choose $\varepsilon_3 \in (0, \delta_4]$ such that $d(\varepsilon_3) \le \varepsilon_2^*$. Therefore, $\varepsilon_3 < d^{-1}(\varepsilon_2^*) \le d^{-1}(\delta_5) \le d^{-1}(c(\delta_4))$, and $c(\delta_4) < c(\|x_0\|) \le V_\eta(t_0, x_0) \le d(\|x_0\|)$, i.e. $\|x_0\| > \varepsilon_3$.

We will prove the inequality

$$\|x^*(t)\| > \varepsilon_3 \quad \text{for } t \ge t_0. \tag{37}$$

Assume (37) is not true. There are three possible cases.

Case 1. There exists a point $t^* > t_0,\ t \neq s_k,\ k = 0, 1, 2\ldots$ such that

$$\|x^*(t)\| > \varepsilon_3 \quad \text{for } t \in [t_0, t^*) \quad \text{and} \quad \|x^*(t^*)\| = \varepsilon_3. \tag{38}$$

According to Lemma 2 for $T = t^*$, V_η, v^* and $\Delta = \{x : \|x\| \ge \varepsilon_3\}$ we obtain $V_\eta(t, x^*(t)) \ge v^*(t)$ for $t \in [t_0, t^*]$. Then from condition 3(vi) we get $d(\varepsilon_3) = d(\|x^*(t^*)\|) \ge V_\eta(t, x^*(t^*)) \ge v^*(t^*) > \varepsilon_2^* > d(\varepsilon_3)$. We obtain a contradiction.

Case 2 There exists an integer $k \ge 0$ such that

$$\|x^*(t)\| > \varepsilon_3 \quad \text{for } t \in [t_0, s_k) \quad \text{and} \quad \|x^*(s_k - 0)\| = \varepsilon_3. \tag{39}$$

As in Case 1 with replacing t^* by s_k we obtain a contradiction.

Case 3.. There exists an integer $k \ge 0$ such that

$$\|x^*(t)\| > \varepsilon_3 \quad \text{for } t \in [t_0, s_k] \quad \text{and} \quad \|x^*(s_k + 0)\| \le \varepsilon_3. \tag{40}$$

According to Lemma 2 for $T = s_k$, V_η, v^*, and $\Delta = \{x : \ \|x\| \geq \varepsilon_3\}$ we obtain $V_\eta(t, x^*(t)) \leq v^*(t)$ for $t \in [t_0, s_k]$.

Then $x^*(s_k + 0) = \phi_k(s_k, x^*(s_k - 0))$ and according to conditions 3 (v) and 3(vi) we get

$$\begin{aligned} d(\varepsilon_3) &\geq d(\|x^*(s_k + 0)\|) = d(\|\phi_k(s_k, x^*(s_k - 0))\|) \\ &\geq V_\eta(s_k, \phi_k(s_k, x^*(s_k - 0))) \geq \Psi_k(s_k, V_\eta(s_k - 0, x^*(s_k - 0))) \\ &\geq \Psi_k(s_k, v^*(s_k - 0)) = v^*(s_k + 0) > \varepsilon_2^*. \end{aligned} \tag{41}$$

The contradictions above prove inequality (37) is true.

□

Remark 14. *Note in the case of FrDE if conditions 2 and 3 of Theorem 2 are satisfied with $g_i(t, x) \equiv 0, \ i = 1, 2$ then the zero solution of FrDE is uniformly strictly stable (see [3]).*

In the case of non-instantaneous impulses the condition $g_i(t, x) \equiv 0, \ i = 1, 2$ is not enough for strict stability (see Example 2).

Sufficient conditions for strict stability could be obtained in the case of one Lyapunov function.

Theorem 3. *Let the following conditions be fulfilled:*

1. *The condition 1 of Theorem 1 is satisfied and the inequalities $g_2(t, u) \leq g_1(t, u)$ for $t \in (t_0, \infty) \bigcap \cup_{k=0}^{\infty}(s_k, t_{k+1}), u \in \mathbb{R}$ and $\Psi_k(t, u) \leq \Phi_k(t, u)$ for $t \in (t_k, s_k], \ u \in \mathbb{R}, \ k = 1, 2, \ldots$ hold.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   - *(i) for any $k = 0, 1, 2, \ldots$ and $x_0 \in \mathbb{R}^n$ the inequality*

$$g_2(t, V(t, x)) \leq {}_{(7)}^{\ c}D_+^q V(t, x; s_k, x_0) \leq g_1(t, V(t, x)) \quad \text{for } t \in (s_k, t_{k+1}], \ x \in \mathbb{R}^n$$

   holds;
   - *(ii) for any natural number k the inequality*

$$\Psi_k(t, V(t_k - 0, x)) \leq V(t, \phi(t, x)) \leq \Phi_k(t, V(t_k - 0, x)) \ \text{ for } t \in (t_k, s_k], \ x \in \mathbb{R}^n$$

   holds;
   - *(iii) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$ for $t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n$, where $a, b \in \mathcal{M}$.*
3. *The zero solution of the couple of NIFrDE (12) is strictly stable (uniformly strictly stable) in couple.*

Then the zero solution of the system NIFrDE (7) is strictly stable (uniformly strictly stable).

The result of Theorem 3 is a special case of Theorem 1 and Theorem 2.

Remark 15. *The results are generalizations of those for impulsive differential equations. If $q = 1$ and $s_i = 0, i = 1, 2, \ldots$ then problem (7) reduces to impulsive differential equation, the set PC^q reduces to PC^1 and our results reduce to results for impulsive differential equations.*

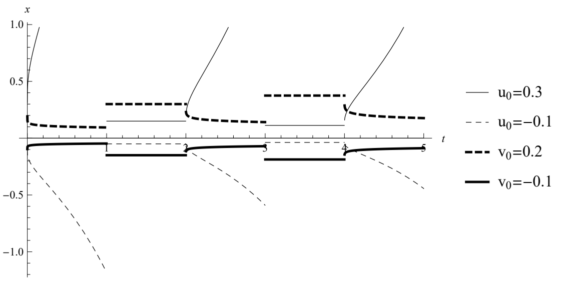

Figure 5. Example 3. Graph of the solution $(u(t), v(t))$ for various initial values.

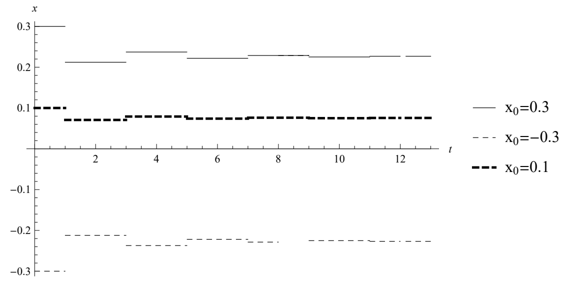

Figure 6. Example 7 . Graph of the solutions of (42) for various initial values x_0.

We will discuss of some of our sufficient conditions.

Example 7. *(Strict stability of NIFrDE).* Let $s_0 = 0, s_k = 2k,\ t_k = 2k-1$ for $k \in \mathbb{Z}_+$. Consider the Caputo fractional differential equation with non-instantaneous impulses

$$\begin{aligned} &{}_0^cD^q x(t) = 0, && t \in (2k, 2k+1],\ k \in \mathbb{Z}_0 \\ &x(t) = a_k(t_k - 0), && t \in (2k+1, 2k+2],\ k \in \mathbb{Z}_0 \\ &x(0) = x_0, && v(0) = v_0, \end{aligned} \tag{42}$$

where $x_0 \in \mathbb{R}$, $a_k = \sqrt{1 - \frac{1}{2^k}}$ for k odd and $a_k = \sqrt{1 + \frac{1}{2^k}}$ for k even.

Note the IVP for NIFrDE (42) has a solution for which $|x(t)| \le |x_0|,\ t \ge t_0$ (see Figure 6 for $q = 0.2$ and various initial values).

Let $V_1(t, x) = x^2$ for $t \in \mathbb{R}_+,\ x \in \mathbb{R}$. Let $x, x_0 \in \mathbb{R}$ with $|x| \le |x_0|$ and $t_0 \in \cup_{k=0}^{\infty}[2k, 2k+1)$. According to Example 5 and (20) we obtain

$$ {}_{(42)}^{c}D_+^q V_1(t, x; s_k, x_0) = \Big(x^2 - (x_0)^2\Big)\frac{1}{(t-s_k)^q\Gamma(1-q)} \le 0. \tag{43}$$

For k odd we get $(\sqrt{1 - \frac{1}{2^k}}x)^2 \le (1 + \frac{1}{2^k})x^2$.

For k even we get $(\sqrt{1 + \frac{1}{2^k}}x)^2 = (1 + 2^k)V_1(t, x)$. Therefore condition 2 of Theorem 1 is satisfied.

Let for any η the function $V_\eta(t, x) = V(t, x) = (2 - E_q(-(t-s_k)^q))x^2$ for $t \in (s_k, s_{k+1}],\ x \in \mathbb{R}$. Let $x, x_0 \in \mathbb{R}$ with $|x| \le |x_0|$. According to Example 5 and Eq. (20) with $m(t) = 2 - E_q(-(t-s_k)^q),\ t \in (s_k, s_{k+1}],\ k = 0, 1, 2, \ldots$ and ${}_{s_k}^{RL}D^q E_q(-(t-s_k)^q) = \frac{1}{(t-s_k)^k\Gamma(1-q)} - E_q(-(t-s_k)^q)$ we get

$$\begin{aligned} &{}_{(42)}^{c}D_+^q V(t, x; s_k, x_0) \\ &= x^2\, {}_{s_k}^{RL}D^q\Big((2 - E_q(-(t-s_k)^q)\Big) + (x^2 - x_0)(2 - E_q(-(t-s_k)^q))\frac{(t-s_k)^{-q}}{\Gamma(1-q)} \\ &= x^2\Big(\frac{2}{(t-s_k)^q\Gamma(1-q)} - \frac{1}{(t-s_k)^q\Gamma(1-q)} + E_q(-(t-s_k)^q)\Big) - x_0^2\frac{1}{(t-s_k)^q\Gamma(1-q)} \end{aligned} \tag{44}$$

For k odd we get $V(t, \sqrt{1 - \frac{1}{2^k}}x) = (2 - E_q(-t^q))(\sqrt{1 - \frac{1}{2^k}}x)^2 = (1 - \frac{1}{2^k})(2 - E_q(-t^q))x^2 = (1 - \frac{1}{2^k})V(t, x)$.

For k even we get $V(t, \sqrt{1 + \frac{1}{2^k}}x) = (2 - E_q(-t^q))(\sqrt{1 + 2^k\frac{1}{2^k}}x)^2 \ge (1 - \frac{1}{2^k})V(t, x)$.

Therefore condition 3 of Theorem 1 is satisfied.

According to Example 3 with $A = B = 0$ and Theorem 1 the zero solution of (42) is strictly stable (see Figure 6).

□

7. Conclusions

In this paper we investigate stability for non-instantaneous impulsive Caputo fractional differential equations. This type of stability guarantees that solutions remain in an appropriate tube. More precisely, the main contributions of this paper are:

- the statement of the initial value problem for Caputo fractional differential equations with non-instantaneous impulses is given and the solution is discussed;
- the appropriate definition of the so called *generalized Caputo fractional Dini derivative* of the Lyapunov functions along the Caputo fractional differential equation with non-instantaneous impulses is given. With examples we show the advantages of our approach;
- several comparison results for generalized Caputo fractional Dini derivative of Lyapunov functions and scalar Caputo fractional differential equation with non-instantaneous impulses are obtained;
- strict stability of the zero solution is defined and some sufficient conditions for strict stability and for uniform strict stability are obtained.

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