



Dynamics of a Delayed Competitive System Affected by Toxic Substances with Imprecise Biological Parameters

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Abstract. In this paper we have studied the dynamical behaviours of a delayed two-species competitive system affected by toxicant with imprecise biological parameters. We have proposed a method to handle these imprecise parameters by using parametric form of interval numbers. We have discussed the existence of various equilibrium points and stability of the system at these equilibrium points. In case of toxic stimulatory system, the delay model exhibits a stable limit cycle oscillation. Computer simulations are carried out to illustrate our analytical findings.

1. Introduction

Environmental pollution is a burning issue for the industrialized society in the world today because it leads to damage the both terrestrial and aquatic environments. Various kind of industrial discharges and chemical wastes are polluting the air and contaminating the streams, lakes, rivers and oceans with varieties of toxicant and chemicals such as arsenic, lead, cadmium, zinc, copper, iron, mercury etc. [21, 28]. There are many species which becomes extinct and several others are at the verge of extinction because of uncontrolled effects of toxicant to the environment. Therefore, the study of the effects of the toxic substances on ecological communities are becoming quite important from both environmental and conservation point of view.

The problem of estimating qualitatively the effect of toxic substances on the species by using mathematical models is a very effective way. The deterministic dynamic models with the effect of toxic substances on various ecosystems was analysed by Hallam et al.[16, 17], Hallam and De Luna [18], De Luna and Hallam [8], Freedman and Shukla [10], Ghosh et al. [11], He and Wang [20], Das et al.[7] and many others. Another important observation made by researchers is that the increased population of one species might affect the growth of another species or of several other species through the production of allelopathic toxins or stimulators, thus influencing seasonal succession [38]. The toxin produced by the unicellular green alga, *Chlorella Vulgaris*, is an autotoxin that limits the size of its own population [36, 37] as well as inhibits the growth of the planktonic algae *Asterionella formosa* and *Nitzschia frustrum* (Bacillareae)[39]. Several researchers have also observed toxic inhibition of phytoplankton by other phytoplankton as well as some of these algae produce auxin which stimulate the growth of the other algae [1]. Such allelopathic stimulators

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and inhibitors certainly affect algal succession, blooms and pulses by causing stimulated (inhibited) species to have a selective advantage (disadvantage) in competition [2, 38]. It has been suggested by Rice [38] that 'all meaningful' functional ecological models will eventually have to include a category on allelopathic and allelochemic effects.

In nature, time delays occur in almost every biological situation [26] and assume to be one of the causes of regular fluctuations on population biomass. In population dynamics, a time delay is introduced when the rate of change of population biomass is not only a function of the present population biomass but also depends on the past population biomass. Therefore, time delay can be incorporated in the mathematical population model due to various ways such as maturation time, capturing time and other reasons. Moreover, existence of time delays is frequently a source of instability in some way. Many researchers [4–6, 22–25, 33, 34, 42, 44] have introduced time delay in their respective biological models to make it more realistic.

Most of the researches in theoretical ecology have considered models based on the assumption that the biological parameters are precisely known. But in real world, the values of all the biological parameters can not be known precisely for the lack of information, lack of data, mistakes done in the measurement process and determining the initial conditions. To overcome these difficulties, imprecise model is more realistic in the field of mathematical biology. The impreciseness of the bio-mathematical model is also occurring due to environmental fluctuations or due to imprecise biological phenomenon. There are several approaches to handle such models having imprecise parameters such as stochastic approach, fuzzy approach, fuzzy stochastic approach etc. In stochastic approach, the imprecise parameters are replaced by random variables with known probability distributions. In fuzzy approach, the imprecise parameters are replaced by fuzzy sets with known membership functions. In fuzzy stochastic approach, some parameters are as fuzzy in nature and rest of the parameters are taken as random variables. However it is very difficult to construct a suitable membership function or a suitable probability distribution for each of the imprecise biological parameters. Some researchers have introduced fuzzy models in predator-prey system such as Bassanezi et al. [3], Peixoto et al. [35], Guo et al. [14], Pal et al. [29–32], Sharma and Samanta [41] have presented an optimal harvesting predator-prey system with imprecise biological parameters and they have also exercised about bio-economic equilibrium and optimal harvesting policy.

In this paper, we have considered a delayed two species competitive system which is affected by toxicant [27]. To make the delay model more realistic, we have considered imprecise biological parameters as interval numbers. We present the interval numbers in parametric function form and study the parametric model. The dynamical behaviour of the parametric model is investigated for different values of the parameter $p \in [0, 1]$. In section 2, we discuss some basic definitions on interval numbers. In section 3, we present mathematical form of two species competitive system with time delay. Section 4 briefly describes the delayed two species competitive system with imprecise parameters. Then parametric form of the system is formulated for the study of different behaviours of the model. Section 5 deals with the existence of equilibrium points of the system and stability. Our important analytical results are numerically verified in section 7. Finally, section 8 contains the conclusions of the paper.

2. Basic Definitions

In this section we discuss some basic definitions of the interval number and interval-valued function which have been used to study the imprecise competition model.

Definition 1 : (*Interval number*) An interval number A is represented by closed interval $[a_l, a_u]$ and defined by $A = [a_l, a_u] = \{x : a_l \leq x \leq a_u, x \in \mathbb{R}\}$, where \mathbb{R} is the set of real numbers and a_l, a_u are the lower and upper limits of the interval number respectively.

So, every real number can also be presented by the interval number $[a, a]$, for all $a \in \mathbb{R}$.

Definition 2 : (*Interval-valued function*) Let us consider the interval $[a, b]$ where $a > 0$. The interval $[a, b]$ can be expressed as a function $\phi(p) = a^{(1-p)}b^p$ for $p \in [0, 1]$. This function is called interval-valued function.

Now we define some arithmetic operations on interval-valued functions. Let $A = [a_l, a_u]$ and $B = [b_l, b_u]$ be two interval numbers.

Addition : $A + B = [a_l, a_u] + [b_l, b_u] = [a_l + b_l, a_u + b_u]$. The interval-valued function for the interval number $A + B$ is given by $\phi(p) = a_L^{(1-p)} a_U^p$ where $a_L = a_l + b_l$, and $a_U = a_u + b_u$.

Subtraction : $A - B = [a_l, a_u] - [b_l, b_u] = [a_l - b_u, a_u - b_l]$, provided $a_l - b_u > 0$. The interval-valued function for the interval $A - B$ is given by $\phi(p) = b_L^{(1-p)} b_U^p$ where $b_L = a_l - b_u$, and $b_U = a_u - b_l$.

Scalar multiplication : $\alpha A = \alpha [a_l, a_u] = \begin{cases} [\alpha a_l, \alpha a_u], & \text{if } \alpha \geq 0 \\ [\alpha a_u, \alpha a_l], & \text{if } \alpha < 0 \end{cases}$, provided $a_l > 0$. The interval-valued function for the interval αA is given by $\phi(p) = \begin{cases} c_L^{(1-p)} c_U^p & \text{if } \alpha \geq 0 \\ -d_U^{(1-p)} d_L^p & \text{if } \alpha < 0 \end{cases}$, where $c_L = \alpha a_l$, $c_U = \alpha a_u$, $d_L = |\alpha| a_l$, $d_U = |\alpha| a_u$.

3. Basic Mathematical Model

Maynerd-Smith [26] have considered a two species competing system :

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(t)[K_1 - \alpha_1 N_1(t) - \beta_{12} N_2(t) - \gamma_1 N_1(t) N_2(t)] \\ \frac{dN_2}{dt} &= N_2(t)[K_2 - \alpha_2 N_2(t) - \beta_{21} N_1(t) - \gamma_2 N_1(t) N_2(t)] \end{aligned} \tag{1}$$

with initial data

$$N_1(0) > 0, N_2(0) > 0. \tag{2}$$

Where $N_1(t)$, $N_2(t)$ denote the population biomass of two competing species at time t , having common food resources ; $K_1, K_2, \alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \gamma_1, \gamma_2$ are positive constants. Here K_i is the intrinsic growth rate of species i , α_i represents the intra-species competition coefficient of species i , β_{ij} denote the inter-species competition rate of j th species upon the i th species and γ_i denote the toxic inhibition rate of the i th species ($i \neq j$, $i, j = 1, 2$). Samanta [40] also analyzed dynamical behaviors of such model where a toxic substance is produced at a constant rate.

When the toxic coefficients $\gamma_i > 0$ ($i = 1, 2$), the model (1) represents toxic inhibited species system. If we assume $\gamma_i = -\gamma'_i$ ($i = 1, 2$), where $\gamma'_i > 0$, then the model (1) represents toxic stimulated system.

We assume that each species produces a substance toxic to the other, but only when the other is present. Further, it is also reasonable to assume such model where a toxic substance is produced to the competing species will not be instantaneous, and takes discrete time lag which is regarded as maturity period of the species. Here we introduce the time lag τ for the maturity of the second species. Then the system (1) reduces to

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(t)[K_1 - \alpha_1 N_1(t) - \beta_{12} N_2(t - \tau) - \gamma_1 N_1(t) N_2(t - \tau)] \\ \frac{dN_2}{dt} &= N_2(t)[K_2 - \alpha_2 N_2(t) - \beta_{21} N_1(t) - \gamma_2 N_1(t) N_2(t)] \end{aligned} \tag{3}$$

The initial conditions for the model (3) take the form $N_1(\theta) = \psi_1(\theta) \geq 0$, $N_2(\theta) = \psi_2(\theta) \geq 0$, $\theta \in [-\tau, 0]$, $\psi_1(0) > 0$, $\psi_2(0) > 0$, $(\psi_1(\theta), \psi_2(\theta)) \in (C[-\tau, 0], \mathbb{R}_+^2)$.

3.1. Positivity and Boundedness of the delayed system

In theoretical ecology, positivity and boundedness of a system implies that the system is biologically well posed. The following proposition ensures the positivity and boundedness of the delayed system (3).

Proposition 1 : Each component of the solution of the delayed system (3) is positive and bounded for all $t \geq 0$.

Proof : Since the right hand side of the system (3) is completely continuous and locally Lipschitzian on C , the solution $(N_1(t), N_2(t))$ of (3) exists and is unique on $[0, \xi)$, where $0 < \xi \leq \infty$ [15]. From system (3), we have

$$N_1(t) = N_1(0) \exp \left[\int_0^t (K_1 - \alpha_1 N_1(s) - \beta_{12} N_2(s - \tau) - \gamma_1 N_1(s) N_2(s - \tau)) ds \right] > 0,$$

$$N_2(t) = N_2(0) \exp \left[\int_0^t (K_2 - \alpha_2 N_2(s) - \beta_{21} N_1(s) - \gamma_2 N_1(s) N_2(s)) ds \right] > 0$$

Therefore, $N_1(t) > 0, N_2(t) > 0, \forall t \geq 0$.

Now we assume,

$$W = N_1(t) + N_2(t).$$

Then,

$$\begin{aligned} \frac{dW}{dt} &\leq -\alpha_1 \left(N_1 - \frac{K_1}{\alpha_1} \right)^2 - \alpha_2 \left(N_2 - \frac{K_2}{\alpha_2} \right)^2 + \frac{K_1^2}{\alpha_1} + \frac{K_2^2}{\alpha_2} - K_1 N_1 - K_2 N_2 \\ &\leq \mu - \nu W, \text{ where } \mu = \frac{K_1^2}{\alpha_1} + \frac{K_2^2}{\alpha_2}, \text{ and } \nu = \min \{K_1, K_2\}. \end{aligned}$$

Therefore,

$$\frac{dW}{dt} + \nu W \leq \mu.$$

Applying a theorem on differential inequalities [4], we obtain

$$0 < W(N_1, N_2) \leq \frac{\mu}{\nu} + \frac{W(N_1(0), N_2(0))}{e^{\nu t}}.$$

and for $t \rightarrow \infty$,

$$0 < W \leq \frac{\mu}{\nu}.$$

Thus, all solutions of the system (3) enter into the region

$$B = \left\{ (N_1, N_2) : 0 < W < \frac{\mu}{\nu} + \varepsilon \text{ for any } \varepsilon > 0 \right\}.$$

This proves the theorem.

3.2. Uniform Persistence of the system

To prove the uniform persistence (permanent) of the system (1), we shall use the ‘‘Average Lyapunov’’ function [13, 43]

Proposition 2: System (1) is uniform persistence if $\frac{\beta_{12}}{\alpha_2} < \frac{K_1}{K_2} < \frac{\alpha_1}{\beta_{21}}$

Proof: We consider the average Lyapunov function of the form $V(N_1, N_2) = N_1^{\beta_{21}} N_2^{\beta_{12}}$, where β_{12}, β_{21} are positive constants. In the interior of \mathbb{R}_+^2 , we have

$$\begin{aligned} \frac{\dot{V}}{V} &= \psi(N_1, N_2) \\ &= \beta_{21} K_1 + \beta_{12} K_2 - \beta_{21}(\alpha_1 + \beta_{12}) N_1 - \beta_{12}(\beta_{21} + \alpha_2) N_2 - (\beta_{21} \gamma_1 + \beta_{12} \gamma_2) N_1 N_2 \end{aligned}$$

To prove the uniform persistence of the system, we shall have to show that $\psi(N_1, N_2) > 0$ for all boundary equilibria of the system. The following conditions should be satisfied for equilibrium:

$$E_1\left(\frac{k_1}{\alpha_1}, 0\right); \quad K_2\alpha_1 - K_1\beta_{21} > 0 \tag{C_1}$$

$$E_2\left(0, \frac{k_2}{\alpha_2}\right); \quad K_1\alpha_2 - K_2\beta_{12} > 0 \tag{C_2}$$

It can be easily shown that conditions (C₁) and (C₂) are satisfied if the conditions stated in the proposition holds.

4. Imprecise Competition Model

According to formation of the competition model (3), all the parameters are positive and precise. But due to lack of proper information of the data, the parameters are not always precise. Now if any of the parameters $K_i, \alpha_i, \beta_{ij}, \gamma_i$ ($i \neq j, i, j = 1, 2$) are imprecise, i.e., if any parameter is interval number rather than a single value, then it become difficult to convert the equation to the standard form and analyse the dynamical behaviour of the system. For imprecise parameters, we present the system (3) with interval parameters as described below :

4.1. Competition model with interval parameters

Case I : Toxic inhibition

Let $\hat{K}_i, \hat{\alpha}_i, \hat{\beta}_{ij}, \hat{\gamma}_i$ ($i \neq j, i, j = 1, 2$) be the interval counterparts of $K_i, \alpha_i, \beta_{ij}, \gamma_i$ respectively. Then the imprecise competition delay model (3) becomes :

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(t)[\hat{K}_1 - \hat{\alpha}_1N_1(t) - \hat{\beta}_{12}N_2(t - \tau) - \hat{\gamma}_1N_1(t)N_2(t - \tau)] \\ \frac{dN_2}{dt} &= N_2(t)[\hat{K}_2 - \hat{\alpha}_2N_2(t) - \hat{\beta}_{21}N_1(t) - \hat{\gamma}_2N_1(t)N_2(t)] \end{aligned} \tag{4}$$

where $\hat{K}_i = [K_{il}, K_{iu}]$, $\hat{\alpha}_i = [\alpha_{il}, \alpha_{iu}]$, $\hat{\beta}_{ij} = [\beta_{ijl}, \beta_{iju}]$, $\hat{\gamma}_i = [\gamma_{il}, \gamma_{iu}]$, and $K_{il} > 0, \alpha_{il} > 0, \beta_{ijl} > 0, \gamma_{il} > 0$ ($i \neq j, i, j = 1, 2$).

Case II : Toxic stimulation

In this case, $\hat{\gamma}_1 = -\hat{\gamma}'_1$ and $\hat{\gamma}_2 = -\hat{\gamma}'_2$ where $\hat{\gamma}'_1 = [\gamma'_{1l}, \gamma'_{1u}]$ and $\hat{\gamma}'_2 = [\gamma'_{2l}, \gamma'_{2u}]$ and $\hat{\gamma}'_{1l} > 0, \hat{\gamma}'_{2l} > 0$, i.e., the last term in each of the equations of system (4) gives a positive effect instead of negative effect as in case I. Then the imprecise competition delay model (3) becomes (replacing $\hat{\gamma}'_1$ by $\hat{\gamma}_1$ and $\hat{\gamma}'_2$ by $\hat{\gamma}_2$):

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(t)[\hat{K}_1 - \hat{\alpha}_1N_1(t) - \hat{\beta}_{12}N_2(t - \tau) + \hat{\gamma}_1N_1(t)N_2(t - \tau)] \\ \frac{dN_2}{dt} &= N_2(t)[\hat{K}_2 - \hat{\alpha}_2N_2(t) - \hat{\beta}_{21}N_1(t) + \hat{\gamma}_2N_1(t)N_2(t)] \end{aligned} \tag{5}$$

4.2. Competition model with parametric interval parameters

For fixed m , let us consider the interval-valued function $\phi_m(p) = a_m^{(1-p)} b_m^p$, $p \in [0, 1]$ for an interval $[a_m, b_m]$. Since $\phi_m(p)$ is a strictly increasing and continuous function, the system (4) and (5) can be written in the parametric form as follows :

$$\begin{aligned} \frac{dN_1(t;p)}{dt} &= N_1[(K_{1l})^{(1-p)}(K_{1u})^p - (\alpha_{1l})^{(1-p)}(\alpha_{1u})^p N_1 - (\beta_{12l})^{(1-p)}(\beta_{12u})^p N_2(t - \tau) - (\gamma_{1l})^{(1-p)}(\gamma_{1u})^p N_1 N_2(t - \tau)] \\ \frac{dN_2(t;p)}{dt} &= N_2[(K_{2l})^{(1-p)}(K_{2u})^p - (\alpha_{2l})^{(1-p)}(\alpha_{2u})^p N_2 - (\beta_{21l})^{(1-p)}(\beta_{21u})^p N_1 - (\gamma_{2l})^{(1-p)}(\gamma_{2u})^p N_1 N_2] \end{aligned} \tag{6}$$

where $p \in [0, 1]$,

and

$$\begin{aligned} \frac{dN_1(t;p)}{dt} &= N_1[(K_{1l})^{(1-p)}(K_{1u})^p - (\alpha_{1l})^{(1-p)}(\alpha_{1u})^p N_1 - (\beta_{12l})^{(1-p)}(\beta_{12u})^p N_2(t - \tau) + (\gamma_{1l})^{(1-p)}(\gamma_{1u})^p N_1 N_2(t - \tau)] \\ \frac{dN_2(t;p)}{dt} &= N_2[(K_{2l})^{(1-p)}(K_{2u})^p - (\alpha_{2l})^{(1-p)}(\alpha_{2u})^p N_2 - (\beta_{21l})^{(1-p)}(\beta_{21u})^p N_1 + (\gamma_{2l})^{(1-p)}(\gamma_{2u})^p N_1 N_2] \end{aligned} \tag{7}$$

where $p \in [0, 1]$.

5. Equilibria and Local Stability of the Delayed System

Case I : Toxic inhibition

The system (6) has four positive steady states, namely (i) $E_0(0, 0)$, the trivial equilibrium, (ii) $E_1(\bar{N}_1, 0)$ and $E_2(0, \bar{N}_2)$, the axial equilibrium, where

$$\bar{N}_1 = \frac{(K_{1l})^{(1-p)}(K_{1u})^p}{(\alpha_{1l})^{(1-p)}(\alpha_{1u})^p}, \bar{N}_2 = \frac{(K_{2l})^{(1-p)}(K_{2u})^p}{(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p}, \text{ for all } p \in [0, 1] \tag{8}$$

and (iii) $E^* = (N_1^*, N_2^*)$, the interior equilibrium, where N_1^*, N_2^* can be determined by

$$a_{ij}N_i^{*2} + b_{ij}N_i^* + c_{ij} = 0, i \neq j, i, j = 1, 2 \tag{9}$$

where

$$\begin{aligned} a_{ij} &= (\beta_{ijl})^{(1-p)}(\beta_{iju})^p(\gamma_{il})^{(1-p)}(\gamma_{iu})^p - (\alpha_{il})^{(1-p)}(\alpha_{iu})^p(\gamma_{jl})^{(1-p)}(\gamma_{ju})^p, \\ b_{ij} &= (K_{il})^{(1-p)}(K_{iu})^p(\gamma_{jl})^{(1-p)}(\gamma_{ju})^p - (K_{jl})^{(1-p)}(K_{ju})^p(\gamma_{il})^{(1-p)}(\gamma_{iu})^p \\ &\quad - (\alpha_{il})^{(1-p)}(\alpha_{iu})^p(\alpha_{jl})^{(1-p)}(\alpha_{ju})^p + (\beta_{ijl})^{(1-p)}(\beta_{iju})^p(\beta_{jil})^{(1-p)}(\beta_{jiu})^p, \\ c_{ij} &= (K_{il})^{(1-p)}(K_{iu})^p(\alpha_{jl})^{(1-p)}(\alpha_{ju})^p - (K_{jl})^{(1-p)}(K_{ju})^p(\beta_{jl})^{(1-p)}(\beta_{ju})^p, \end{aligned} \tag{10}$$

for all $p \in [0, 1]$.

Then

$$N_i^* = \frac{1}{2a_{ij}}(-b_{ij} \pm \sqrt{b_{ij}^2 - 4a_{ij}c_{ij}}), i, j = 1, 2 \tag{11}$$

exists with the conditions

$$a_{ij} \neq 0, b_{ij}^2 - 4a_{ij}c_{ij} \geq 0. \tag{12}$$

The variational matrix of the system (6) at $E_0(0, 0)$ is given by

$$V(E_0) = \begin{bmatrix} (K_{1l})^{(1-p)}(K_{1u})^p & 0 \\ 0 & (K_{2l})^{(1-p)}(K_{2u})^p \end{bmatrix}.$$

Clearly, $E_0(0, 0)$ is a saddle node and hence unstable.

The variational matrix of the system (6) at $E_1(\bar{N}_1, 0)$ is given by

$$V(E_1) = \begin{bmatrix} -(\alpha_{1l})^{(1-p)}(\alpha_{1u})^p \bar{N}_1 & -(\beta_{12l})^{(1-p)}(\beta_{12u})^p \bar{N}_1 - (\gamma_{1l})^{(1-p)}(\gamma_{1u})^p \bar{N}_1^2 e^{-\lambda \tau} \\ 0 & (K_{2l})^{(1-p)}(K_{2u})^p - (\beta_{21l})^{(1-p)}(\beta_{21u})^p \bar{N}_1 \end{bmatrix},$$

then the eigenvalues are $\lambda_1 = -(\alpha_{1l})^{(1-p)}(\alpha_{1u})^p \bar{N}_1$ and $\lambda_2 = (K_{2l})^{(1-p)}(K_{2u})^p - (\beta_{21l})^{(1-p)}(\beta_{21u})^p \bar{N}_1$. Therefore the equilibrium point $E_1(\bar{N}_1, 0)$ is asymptotically stable if $\frac{(K_{1l})^{(1-p)}(K_{1u})^p}{(K_{2l})^{(1-p)}(K_{2u})^p} > \frac{(\alpha_{1l})^{(1-p)}(\alpha_{1u})^p}{(\beta_{21l})^{(1-p)}(\beta_{21u})^p}$ and unstable if

$$\frac{(K_{1l})^{(1-p)}(K_{1u})^p}{(K_{2l})^{(1-p)}(K_{2u})^p} < \frac{(\alpha_{1l})^{(1-p)}(\alpha_{1u})^p}{(\beta_{21l})^{(1-p)}(\beta_{21u})^p}. \tag{13}$$

Again, the variational matrix of the system (6) at $E_2(0, \bar{N}_2)$ is given by

$$V(E_2) = \begin{bmatrix} (K_{1l})^{(1-p)}(K_{1u})^p - (\beta_{12l})^{(1-p)}(\beta_{12u})^p \bar{N}_2 e^{-\lambda \tau} & 0 \\ -\bar{N}_2 \{(\beta_{21l})^{(1-p)}(\beta_{21u})^p + (\gamma_{2l})^{(1-p)}(\gamma_{2u})^p \bar{N}_2\} & -(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p \bar{N}_2 \end{bmatrix},$$

then the characteristic equation of the system at E_2 is of the form:

$$\{\lambda + (\alpha_{2l})^{(1-p)}(\alpha_{2u})^p \bar{N}_2\} \{\lambda - (K_{1l})^{(1-p)}(K_{1u})^p + (\beta_{12l})^{(1-p)}(\beta_{12u})^p \bar{N}_2 e^{-\lambda \tau}\} = 0.$$

Here $\lambda = -(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p \bar{N}_2$ is an eigenvalue. Now we consider the equation

$$\lambda = (K_{1l})^{(1-p)}(K_{1u})^p - (\beta_{12l})^{(1-p)}(\beta_{12u})^p \bar{N}_2 e^{-\lambda \tau}. \tag{14}$$

If $\tau = 0$, then the equilibrium point $E_2(0, \bar{N}_2)$ is locally asymptotically stable if

$\frac{(K_{2l})^{(1-p)}(K_{2u})^p}{(K_{1l})^{(1-p)}(K_{1u})^p} > \frac{(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p}{(\beta_{12l})^{(1-p)}(\beta_{12u})^p}$ and unstable if

$$\frac{(K_{2l})^{(1-p)}(K_{2u})^p}{(K_{1l})^{(1-p)}(K_{1u})^p} < \frac{(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p}{(\beta_{12l})^{(1-p)}(\beta_{12u})^p}. \tag{15}$$

Again by substituting $\lambda = i\eta$ in equation (14) and equating real and imaginary parts, we obtain

$$(K_{1l})^{(1-p)}(K_{1u})^p = (\beta_{12l})^{(1-p)}(\beta_{12u})^p \bar{N}_2 \cos \eta \tau$$

$$\eta = (\beta_{12l})^{(1-p)}(\beta_{12u})^p \bar{N}_2 \sin \eta \tau,$$

eliminating τ , we have

$$\eta^2 = \{(\beta_{12l})^{(1-p)}(\beta_{12u})^p\}^2 \bar{N}_2^2. \tag{16}$$

We know that the equation (16) has a positive root η_+ if $\frac{(K_{2l})^{(1-p)}(K_{2u})^p}{(K_{1l})^{(1-p)}(K_{1u})^p} > \frac{(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p}{(\beta_{12l})^{(1-p)}(\beta_{12u})^p}$.

Therefore, there is a positive constant τ_+ such that for $\tau > \tau_+$, $E_2(0, \bar{N}_2)$ is unstable.

Hence combining (13) and (15), the condition required for the persistence of both the species is

$$\frac{(K_{il})^{(1-p)}(K_{iu})^p}{(K_{jl})^{(1-p)}(K_{ju})^p} < \frac{(\alpha_{il})^{(1-p)}(\alpha_{iu})^p}{(\beta_{jil})^{(1-p)}(\beta_{jiu})^p} \quad (i \neq j; i, j = 1, 2) \text{ for all } p \in [0, 1]. \tag{17}$$

Condition (17) gives $c_{ij} > 0$ in equation (9). Then the system (6) has unique positive equilibrium if

$$a_{ij} < 0 \text{ i.e., if } \frac{(\alpha_{il})^{(1-p)}(\alpha_{iu})^p}{(\beta_{jil})^{(1-p)}(\beta_{jiu})^p} > \frac{(\gamma_{il})^{(1-p)}(\gamma_{iu})^p}{(\gamma_{jl})^{(1-p)}(\gamma_{ju})^p} \quad (i \neq j; i, j = 1, 2) \text{ for all } p \in [0, 1]. \quad (18)$$

So, combining equation (17) and (18), we have

$$\frac{(\alpha_{il})^{(1-p)}(\alpha_{iu})^p}{(\beta_{jil})^{(1-p)}(\beta_{jiu})^p} > \max\left\{\frac{(\gamma_{il})^{(1-p)}(\gamma_{iu})^p}{(\gamma_{jl})^{(1-p)}(\gamma_{ju})^p}, \frac{(K_{il})^{(1-p)}(K_{iu})^p}{(K_{jl})^{(1-p)}(K_{ju})^p}\right\} \quad (i \neq j; i, j = 1, 2) \text{ for all } p \in [0, 1], \quad (19)$$

as the condition of existence of unique positive interior equilibrium of the system (6).

Now to investigate the local stability of the interior equilibrium $E^*(N_1^*, N_2^*)$, we linearize the system (6) by using the following transformations:

$$N_1 = N_1^* + n_1, \quad N_2 = N_2^* + n_2.$$

Then the linear system is given by

$$\frac{dU}{dt} = MU(t) + NU(t - \tau), \quad (20)$$

where

$$U(t) = [n_1 \ n_2]^T, \quad M = (m_{ij})_{2 \times 2}, \quad N = (n_{ij})_{2 \times 2},$$

and

$$\begin{aligned} m_{11} &= -N_1^*[(\alpha_{1l})^{(1-p)}(\alpha_{1u})^p + (\gamma_{1l})^{(1-p)}(\gamma_{1u})^p N_2^*], \quad m_{12} = 0, \\ m_{21} &= -N_2^*[(\beta_{21l})^{(1-p)}(\beta_{21u})^p + (\gamma_{2l})^{(1-p)}(\gamma_{2u})^p N_2^*], \quad m_{22} = -N_2^*[(\alpha_{2l})^{(1-p)}(\alpha_{2u})^p + (\gamma_{2l})^{(1-p)}(\gamma_{2u})^p N_1^*], \\ n_{12} &= -N_1^*[(\beta_{12l})^{(1-p)}(\beta_{12u})^p + (\gamma_{1l})^{(1-p)}(\gamma_{1u})^p N_1^*], \text{ and all other } n_{ij} = 0. \end{aligned} \quad (21)$$

We now use the following theorem [12] to find the necessary and sufficient conditions for stability in the absence or presence of delay.

Theorem 1 : A set of necessary and sufficient conditions for E^* to be asymptotically stable for $\tau \geq 0$ is the following:

a) The real parts of all the roots of $\Delta(\lambda, 0) = 0$ are negative.

b) For all real k and $\tau \geq 0$, $\Delta(ik, \tau) \neq 0$, where $i = \sqrt{-1}$.

Theorem 2 : The unique interior equilibrium E^* of the system (6) with toxic inhibition is locally asymptotically stable for all $\tau \geq 0$.

To prove this theorem, we assume a solution of the model (6) of the form $U(t) = \rho e^{\lambda t}$, $0 \neq \rho \in \mathfrak{R}$. This leads to the following characteristic equation :

$$\Delta(\lambda, \tau) = \lambda^2 + a_1\lambda + a_2 + a_3e^{-\lambda\tau} = 0, \quad (22)$$

where $a_1 = -(m_{11} + m_{22})$; $a_2 = m_{11}m_{22}$; $a_3 = -m_{21}n_{12}$.

When $\tau = 0$, then equation (22) becomes $\Delta(\lambda, 0) = \lambda^2 + a_1\lambda + a_2 + a_3 = 0$.

$$\therefore \lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4(a_2 + a_3)}}{2}.$$

Since, $a_1 > 0$ and $a_2 + a_3 > 0$ [from equations (17) and (18)], then both the roots of $\Delta(\lambda, 0) = 0$ are negative. Hence the interior equilibrium E^* is asymptotically stable for $\tau = 0$.

The stability criteria of the system (6) for $\tau = 0$ will not necessarily ensure the stability of the same system with positive delay $\tau > 0$. Now we assume $\tau \neq 0$.

It is well known that the signs of the real parts of the solution (22) characterize the stability behavior of E^* . Therefore, by substituting $\lambda = \xi + i\eta$ in (22), we obtain real and imaginary parts, respectively as

$$\xi^2 - \eta^2 + a_1\xi + a_2 + a_3 \cos \eta\tau e^{-\xi\tau} = 0 \tag{23}$$

$$2\xi\eta + a_1\eta - a_3 \sin \eta\tau e^{-\xi\tau} = 0 \tag{24}$$

A necessary condition for the change of stability of the equilibrium E^* is that the characteristic equation (22) should have imaginary solutions. Hence to obtain the stability criterion, we take $\xi = 0$ in (23) and (24). Then we have

$$-\eta^2 + a_2 + a_3 \cos \eta\tau = 0 \tag{25}$$

$$a_1\eta - a_3 \sin \eta\tau = 0 \tag{26}$$

eliminating τ by squaring and adding (25) and (26), we get the equation for determining η as

$$\eta^4 + d_1\eta^2 + d_2 = 0 \tag{27}$$

where $d_1 = a_1^2 - 2a_2$, and $d_2 = a_2^2 - a_3^2$

As $d_1 > 0$ and $a_2 - a_3 > 0$, then we must have $a_2 + a_3 > 0$. Thus the equation (27) has no positive root η_+^2 . Hence the characteristic equation (22) has no purely imaginary roots $\pm i\eta_+$ for all values of the parameter $p \in [0, 1]$.

Therefore the interior equilibrium E^* of the toxic inhibited system is always asymptotically stable for all $\tau \geq 0$ and hence the delay has no effect on the system.

Case II : Toxic stimulation

In this case, we study the system (7), which is same as the system (6) unless in the last term, where the toxic inhibition rate γ_i ($i = 1, 2$) are negative i.e. either species produces auxin which stimulate the growth of the other species.

Here the local stability analysis of the system (7) remains same as in case I for the trivial and axial equilibrium points and so as the criteria for persistence of both the species. Now we study the local stability of the system (7) for the interior equilibrium point $E^*(N_1^*, N_2^*)$ which is determined by the equation

$$a'_{ij}N_i^{*2} + b'_{ij}N_i^* + c'_{ij} = 0, \quad i \neq j, \quad i, j = 1, 2 \tag{28}$$

where

$$\begin{aligned} a'_{ij} &= (\alpha_{il})^{(1-p)}(\alpha_{iu})^p(\gamma_{jl})^{(1-p)}(\gamma_{ju})^p - (\beta_{ijl})^{(1-p)}(\beta_{iju})^p(\gamma_{il})^{(1-p)}(\gamma_{iu})^p, \\ b'_{ij} &= (K_{jl})^{(1-p)}(K_{ju})^p(\gamma_{il})^{(1-p)}(\gamma_{iu})^p - (K_{il})^{(1-p)}(K_{iu})^p(\gamma_{jl})^{(1-p)}(\gamma_{ju})^p \\ &\quad - (\alpha_{il})^{(1-p)}(\alpha_{iu})^p(\alpha_{jl})^{(1-p)}(\alpha_{ju})^p + (\beta_{ijl})^{(1-p)}(\beta_{iju})^p(\beta_{jil})^{(1-p)}(\beta_{jiu})^p, \\ c'_{ij} &= (K_{il})^{(1-p)}(K_{iu})^p(\alpha_{jl})^{(1-p)}(\alpha_{ju})^p - (K_{jl})^{(1-p)}(K_{ju})^p(\beta_{il})^{(1-p)}(\beta_{iu})^p, \end{aligned} \tag{29}$$

for all $p \in [0, 1]$.

Since, $c'_{ij} > 0$ [from (15)], then the system (7) has unique positive interior equilibrium $E^*(N_1^*, N_2^*)$ exists if

$$a'_{12} > 0, \quad a'_{21} < 0, \quad \text{and} \quad b'_{21} < 0 \tag{30}$$

according to the persistence condition of both the species (17) together with the conditions (18) and (19). To study the local stability properties of $E^*(N_1^*, N_2^*)$, we linearize the system (7), then the relation (21) becomes

$$\begin{aligned} m'_{11} &= -N_1^*[(\alpha_{11})^{(1-p)}(\alpha_{1u})^p - (\gamma_{11})^{(1-p)}(\gamma_{1u})^p N_2^*], \quad m'_{12} = 0, \\ m'_{21} &= -N_2^*[(\beta_{211})^{(1-p)}(\beta_{21u})^p - (\gamma_{21})^{(1-p)}(\gamma_{2u})^p N_2^*], \quad m'_{22} = -N_2^*[(\alpha_{21})^{(1-p)}(\alpha_{2u})^p - (\gamma_{21})^{(1-p)}(\gamma_{2u})^p N_1^*], \end{aligned}$$

$$n'_{12} = -N_1^*[(\beta_{12l})^{(1-p)}(\beta_{12u})^p - (\gamma_{1l})^{(1-p)}(\gamma_{1u})^p N_1^*], \text{ and all other } n'_{ij} = 0. \tag{31}$$

and the characteristic equation becomes

$$\Delta'(\lambda, \tau) = \lambda^2 + a'_1\lambda + a'_2 + a'_3e^{-\lambda\tau} = 0 \tag{32}$$

where $a'_1 = -(m'_{11} + m'_{22})$; $a'_2 = m'_{11}m'_{22}$; $a'_3 = -m'_{21}n'_{12}$.

From equation (32), for $\tau = 0$, the interior equilibrium E^* is locally asymptotically stable if

$$a'_2 + a'_3 > 0 \text{ i.e. } m'_{11}m'_{22} - m'_{21}n'_{12} > 0 \tag{33}$$

Now to study the stability behaviour for $\tau \neq 0$, we assume, $\lambda = i\eta$, ($\eta > 0$) is a root of the equation (32), then we have,

$$-\eta^2 + ia'_1\eta + a'_2 + a'_3e^{-i\eta\tau} = 0 \tag{34}$$

equating real and imaginary parts of equation (34), we have,

$$-\eta^2 + a'_2 - a'_3 \cos \eta\tau = 0 \tag{35}$$

and

$$a'_1\eta - a'_3 \sin \eta\tau = 0 \tag{36}$$

Eliminating τ by squaring and adding (35) and (36), we get the following fourth degree equation for determining η as

$$\eta^4 + (a_1'^2 - 2a_2')\eta^2 + (a_2'^2 - a_3'^2) = 0 \tag{37}$$

solving the above quadratic equation in η^2 , we get,

$$\eta^2 = -\frac{1}{2}(a_1'^2 - 2a_2') \pm \frac{1}{2}\sqrt{(a_1'^2 - 2a_2')^2 - 4(a_2'^2 - a_3'^2)}$$

Here we assume, $a_2'^2 - a_3'^2 < 0$ i.e. $-a_3' < a_2' < a_3'$, so that the equation (37) has only one positive solution η^{*2} . Therefore the characteristic equation (32) has a purely imaginary roots $\pm i\eta^*$ for all values of the parameter $p \in [0, 1]$. Putting the value of η^{*2} in the equation (36) and solving for τ , we get,

$$\tau_j^* = \frac{1}{\eta^*} \arcsin\left(\frac{a_1'\eta^*}{a_3'}\right) + \frac{2j\pi}{\eta^*}, \quad j = 0, 1, 2, \dots \tag{38}$$

From the above discussion we have the following lemma.

Lemma 1 : for $\tau < \tau_0^*$, the equation (32) has a pair of imaginary roots $\pm i\eta^*$ for all p .

Theorem 3 : Let τ_j^* be defined by (38) and the condition (33) is satisfied, then the equilibrium point $E^*(N_1^*, N_2^*)$ of the system (7) is asymptotically stable for $\tau < \tau_0^*$ and unstable for $\tau > \tau_0^*$. Further as τ increases through τ_0^* , E^* bifurcates into small amplitude periodic solutions, where $\tau_0^* = \tau_j^*$ for $j = 0$ and for all $p \in [0, 1]$.

Proof: For $\tau = 0$, the equilibrium point $E^*(N_1^*, N_2^*)$ is asymptotically stable if the condition (33) is satisfied. Hence by Butler's lemma [9], $E^*(N_1^*, N_2^*)$ remains stable for $\tau < \tau_0^*$ for all $p \in [0, 1]$. Now we have to show that $\frac{d(Re\lambda)}{d\tau}|_{\tau=\tau_0^*, \eta=\eta^*} > 0$. This indicate that there exists at least one eigenvalue with positive real part for $\tau > \tau_0^*$ for all $p \in [0, 1]$. Also the condition of Hopf-bifurcation are then satisfied yielding the required periodic solution for all $p \in [0, 1]$. Now differentiating (32) with respect to τ we get

$$[2\lambda + a'_1 - \tau a'_3 e^{-\lambda\tau}] \frac{d\lambda}{d\tau} = \lambda a'_3 e^{-\lambda\tau} \Rightarrow \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + a'_1}{\lambda a'_3 e^{-\lambda\tau}} - \frac{\tau}{\lambda} \tag{39}$$

Since, $e^{-\lambda\tau} = -\frac{\lambda^2+a'_1\lambda+a'_2}{a'_3}$, the equation (39) becomes $\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda+a'_1}{\lambda(\lambda^2+a'_1\lambda+a'_2)} - \frac{\tau}{\lambda}$. Thus

$$\begin{aligned} \operatorname{sign}\left[\frac{d(\operatorname{Re}\lambda)}{d\tau}\right]_{\lambda=i\eta^*} &= \operatorname{sign}\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\eta^*} = \operatorname{sign}\left[\operatorname{Re}\left(-\frac{\lambda(2\lambda+a'_1)}{\lambda^2(\lambda^2+a'_1\lambda+a'_2)} - \frac{\tau}{\lambda}\right)\right]_{\lambda=i\eta^*} \\ &= \frac{1}{\eta^{*2}} \operatorname{sign}\left[\frac{a'_1{}^2\eta^{*2} - 2\eta^{*2}(a'_2 - \eta^{*2})}{a'_1{}^2\eta^{*2} + (a'_2 - \eta^{*2})^2}\right] = \frac{1}{\eta^{*2}} \operatorname{sign}\left[\frac{2\eta^{*4} + (a'_1{}^2 - 2a'_2)\eta^{*2}}{a'_1{}^2\eta^{*2} + (a'_2 - \eta^{*2})^2}\right] \end{aligned}$$

As $a'_1{}^2 - 2a'_2$ is always positive, we have, $\left[\frac{d(\operatorname{Re}\lambda)}{d\tau}\right]_{\eta=\eta^*, \tau=\tau_0^*} > 0$. Therefore, the transversality condition holds and hence Hopf-bifurcation occurs at $\eta = \eta^*$, $\tau = \tau_0^*$ for all $p \in [0, 1]$. Hence the theorem.

6. Direction and Stability of Hopf Bifurcation

In this section our attention is focussed an investigation of the direction, stability and period of the periodic solution bifurcating from a stable equilibrium $E^* = (N_1^*, N_2^*)$. Following the ideas of Hassard et. al [19], we derive the explicit formula for determining the Hopf bifurcation at the critical value of τ_j by using normal form and centre manifold theory. Without loss of generality, we denote any one of the critical values τ_j (where $j = 0, 1, 2, \dots$) by τ^* at which equation (37) has a pair of purely imaginary roots $\pm i\eta$ and the system (7) undergoes a Hopf bifurcation at E^* . Let $u_1(t) = N_1 - N_1^*$, $u_2(t) = N_2 - N_2^*$, $\tau = \mu + \tau^*$ and $t \rightarrow \frac{t}{\tau}$ where $\mu \in \mathbb{R}$, then at $\mu = 0$ the system (7) undergoes Hopf bifurcation. Linearizing system (7), we have

$$\begin{aligned} \frac{du_1}{dt} &= m'_{11}u_1 + n'_{12}u_2(t - \tau) \\ \frac{du_2}{dt} &= m'_{21}u_1 + m'_{22}u_2 \end{aligned} \tag{40}$$

In space $\mathbb{C}([-1, 0], \mathbb{R}^2)$, the system (40) is transformed into a functional differential equation as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu u_t) \tag{41}$$

where $u_t = (u_1(t), u_2(t))^T \in \mathbb{R}^2$ and $L_\mu : \mathbb{C} \rightarrow \mathbb{R}$, $f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}$ are represented by

$$L_\mu(\phi) = (\mu + \tau^*) \begin{pmatrix} m'_{11} & 0 \\ m'_{21} & m'_{22} \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\mu + \tau^*) \begin{pmatrix} 0 & n'_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}, \tag{42}$$

$$f(\mu, \phi) = (\mu + \tau^*) \begin{pmatrix} a_{11}\phi_1^2(0) + a_{12}\phi_1(0)\phi_2(-1) + a_{22}\phi_1^2(0)\phi_2(-1) \\ b_{11}\phi_1(0)\phi_2(0) + b_{12}\phi_2^2(0) + b_{22}\phi_1(0)\phi_2^2(0) \end{pmatrix}, \tag{43}$$

$\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in \mathbb{C}^2$; the entries a_{ij} and b_{ij} are given as

$$\begin{aligned} a_{11} &= -\hat{\alpha}_1 + \hat{\gamma}_1 N_2^*, a_{12} = -\hat{\beta}_{12} + 2\hat{\gamma}_1 N_1^*, a_{22} = \hat{\gamma}_1, \\ b_{11} &= -\hat{\beta}_{21} + 2\hat{\gamma}_2 N_2^*, b_{12} = -\hat{\alpha}_2 + \hat{\gamma}_2 N_1^*, a_{22} = \hat{\gamma}_2 \end{aligned}$$

By Riesz representation theorem, there exist a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, 0)\phi(\theta), \text{ for } \phi \in \mathbb{C} \tag{44}$$

In fact we can choose

$$\eta(\theta, \mu) = (\mu + \tau^*) \begin{pmatrix} m'_{11} & 0 \\ m'_{21} & m'_{22} \end{pmatrix} \delta(\theta) - (\mu + \tau^*) \begin{pmatrix} 0 & n'_{12} \\ 0 & 0 \end{pmatrix} \delta(\theta + 1), \tag{45}$$

where δ is a dirac delta function. For $\phi = \mathbb{C}'([-1, 0], \mathbb{R}^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \text{for } \theta \in [-1, 0); \\ \int_{-1}^0 d\eta(\mu, s)\phi(s), & \text{for } \theta = 0. \end{cases} \tag{46}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \text{for } \theta \in [-1, 0); \\ f(\mu, \phi), & \text{for } \theta = 0. \end{cases} \tag{47}$$

The system (41) is equivalent to

$$\dot{u}_t = A_\mu(u_t) + R_\mu(u_t), \tag{48}$$

where $u_\theta = u_{t+\theta}$ for $\theta \in [-1, 0)$.

For $\psi = \mathbb{C}'([0, 1], (\mathbb{R}^2)^*)$, where $(\mathbb{R}^2)^*$ is the 2-dimensional space of row vectors defined by

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & \text{for } s \in (0, 1]; \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & \text{for } s = 0. \end{cases} \tag{49}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) = \int_{-1}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \quad \eta(\theta) = \eta(\theta, 0) \tag{50}$$

Then $A(0)$ and A^* are adjoint operators. Suppose that $q(\theta)$ and $q^*(\theta)$ are eigen vectors of A and A^* corresponding to $i\eta^*\tau^*$ and $-i\eta^*\tau^*$ respectively. By direct computation, we have

$$q(\theta) = \left(1, \frac{m'_{11} + i\eta^*}{n'_{12}e^{-i\eta^*\tau^*}} \right) e^{i\eta^*\tau^*\theta}$$

$$q^*(s) = \bar{D} \left(\frac{m'_{22} - i\eta^*}{m'_{21}}, 1 \right) e^{i\eta^*\tau^*\theta}$$

where

$$\bar{D} = \frac{1}{\frac{m'_{11} + i\eta^*}{n'_{12}e^{-i\eta^*\tau^*}} + \frac{m'_{22} - i\eta^*}{m'_{21}} - \tau^*(m'_{21} + m'_{11} - i\eta^*)e^{-i\eta^*\tau^*}} \tag{51}$$

and $\langle q^*(s), q(\theta) \rangle = 1, \langle q^*(s), \bar{q}(\theta) \rangle = 0$.

Let u_t be the solution of (48) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t \rangle, w(t, \theta) = u_t(\theta) - 2\text{Re}(z(t)q(\theta)) \tag{52},$$

on the centre manifold \mathbb{C}_0 , we have $w(t, \theta) = w(z(t), \bar{z}(t), \theta)$, where

$$w(z, \bar{z}, \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + w_{30}(\theta)\frac{z^3}{6} + \dots \tag{53}$$

z and \bar{z} are local coordinates for the centre manifold \mathbb{C}_0 in the direction of q^* and \bar{q}^* . Note that w is real if u_t is real and we consider only real solutions. For solution $u_t \in \mathbb{C}_0$ of (48) since $\mu = 0$, we have

$$\dot{z}(t) = i\eta^*\tau^*z + \bar{q}^*(0)f(0, w(z, \bar{z}, 0) + 2\text{Re}(zq(\theta))) \stackrel{\text{def}}{=} i\eta^*\tau^*z + \bar{q}^*(0)f_0(z, \bar{z})$$

We rewrite this equation as

$$\dot{z}(t) = i\eta^*\tau^*z(t) + g(z, \bar{z}), \tag{54}$$

where $g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots$
 It follows form (52) and (53)

$$u_t(\theta) = w(t, \theta) + 2\text{Re}\{z(t)q(t)\} = w_{20}(\theta) \frac{z^2}{2} + w_{11}(\theta) z\bar{z} + w_{02}(\theta) \frac{\bar{z}^2}{2} + (1, q_1)^T e^{i\eta^* \tau^* \theta} z + (1, \bar{q}_1)^T e^{-i\eta^* \tau^* \theta} \bar{z} + \dots, \tag{55}$$

where $q_1 = -\frac{m'_{11} - i\eta^*}{m'_{11} e^{-i\eta^* \tau^* \theta}}$. It follows together with (43) that

$$g(z, \bar{z}) = \bar{q}^* f_0(z, \bar{z}) = \bar{q}^* f(0, u_t) = \tau^* \bar{D} \{ [\bar{q}_2 (a_{11} + a_{12} q_1 e^{-i\eta^* \tau^*} + a_{22} q_1 e^{-i\eta^* \tau^*}) + (b_{11} q_1 + b_{12} q_1^2 + b_{22} q_1^2)] z^2 + \{ \bar{q}_2 (2a_{11} + 2a_{12} \text{Re}(q_1) + 2a_{22} \text{Re}(q_1)) + (2b_{11} \text{Re}(q_1) + 2b_{12} |q_1|^2 + 2b_{22} |q_1|^2) \} z\bar{z} + \{ \bar{q}_2 (a_{11} + a_{12} \bar{q}_1 e^{i\eta^* \tau^*} + a_{22} \bar{q}_1 e^{i\eta^* \tau^*}) + (b_{11} \bar{q}_1 + b_{12} \bar{q}_1^2 + b_{22} \bar{q}_1^2) \} \bar{z}^2 + \{ \bar{q}_2 (a_{11} (2w_{11}^{(1)}(0) + w_{20}^{(1)}(0))) + a_{12} (\frac{1}{2} (\bar{q}_1 w_{20}^{(1)}(-1) + w_{20}^{(2)}(-1)) e^{i\eta^* \tau^*} + (q_1 w_{11}^{(1)}(-1) + w_{11}^{(2)}(-1)) e^{-i\eta^* \tau^*}) + \bar{q}_2 a_{22} (2q_1 w_{11}^{(2)}(-1) e^{-i\eta^* \tau^*} + w_{20}^{(2)}(-1) \bar{q}_1 e^{i\eta^* \tau^*}) + b_{11} (2w_{11}^{(1)}(0) + w_{20}^{(1)}(0)) + b_{12} (\frac{1}{2} (\bar{q}_1 w_{11}^{(1)}(0) + w_{20}^{(1)}(0)) + q_1 (w_{11}^{(1)}(0) + w_{20}^{(2)}(0))) + b_{22} (2w_{11}^{(2)}(0) + w_{20}^{(2)}(0) \bar{q}_1) \} z^2 \bar{z}] \tag{56}$$

where $g_{20} = \frac{m'_{22} - i\eta^*}{m'_{21}}$. Comparing the coefficients of z^2 , \bar{z}^2 , $z\bar{z}$ and $z^2 \bar{z}$, we have

$$g_{02} = 2\tau^* \bar{D} (\bar{q}_2 (a_{11} + a_{12} q_2 e^{-i\eta^* \tau^*} + a_{22} q_1 e^{-i\eta^* \tau^*})) + (b_{11} q_1 + b_{12} q_1^2);$$

$$g_{11} = 2\tau^* \bar{D} (\bar{q}_2 (a_{11} + a_{12} \bar{q}_1 e^{i\eta^* \tau^*} + a_{22} q_1 e^{i\eta^* \tau^*} + (b_{11} \bar{q}_1 + b_{12} \bar{q}_1^2 + b_{22} \bar{q}_1^2)));$$

$$g_{02} = 2\tau^* \bar{D} (\bar{q}_2 (2a_{11} + 2a_{12} \text{Re}(q_1) + 2a_{22} \text{Re}(q_1)) + (2b_{11} \text{Re}(q_1) + 2b_{12} |q_1|^2 + 2b_{22} |q_1|^2));$$

$$g_{21} = \tau^* \bar{D} (\bar{q}_2 (a_{11} (2w_{11}^{(1)}(0) + w_{20}^{(1)}(0)) + a_{12} ((\bar{q}_1 w_{20}^{(1)}(-1) + w_{20}^{(2)}(-1)) e^{-i\eta^* \tau^*}) + 2\bar{q}_2 a_{22} (2q_1 w_{11}^{(2)}(-1) e^{-i\eta^* \tau^*} + w_{20}^{(2)}(-1) \bar{q}_1 e^{i\eta^* \tau^*}) + 2b_{11} (2w_{11}^{(1)}(0) + w_{20}^{(1)}(0)) + b_{12} ((\bar{q}_1 w_{11}^{(1)}(0) + w_{20}^{(1)}(0)) + 2(q_1 (w_{11}^{(1)}(0) + w_{20}^{(2)}(0))) + 2b_{22} (2w_{11}^{(2)}(0) + w_{20}^{(2)}(0) \bar{q}_1)) \tag{57}$$

Since there are $w_{20}(\theta)$ and $w_{11}(\theta)$ in q_{21} , we need to find out their values at $\theta = 0$ and $\theta = -1$. From the definition given by equation (48) and (52), we have

$$\dot{w} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} Aw - 2\text{Re}\{\bar{q}^*(0) f_0 q(\theta)\}, & \text{for } \theta \in [-1, 0); \\ Aw - 2\text{Re}\{\bar{q}^*(0) f_0 q(0)\} + f_0, & \text{for } \theta = 0. \end{cases} = Aw + H(z, \bar{z}, \theta) \tag{58}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \tag{59}$$

Substituting the above series and computing the corresponding coefficients, we have

$$(A - i\eta^* \tau^*) w_{20} = -H_{20}(\theta), \quad Aw_{11}(\theta) = -Hw_{11}(\theta), \dots \tag{60}$$

For $\theta \in [-1, 0)$, we know that

$$H(z, \bar{z}, \theta) = \bar{q}^*(0) f_0 q(\theta) - q^*(0) \bar{f}_0 \bar{q}(\theta) = -g((z, \bar{z}, \theta)) q(\theta) - \bar{g}(z, \bar{z}, \theta) \bar{q}(\theta) \tag{61}$$

Comparing the coefficients, we have

$$H_{20}\theta = -g_{20}(\theta)\bar{q}(\theta) - \bar{g}_{02}(\theta)\bar{q}(\theta) \text{ and } H_{11}(\theta) = -g_{11}(\theta)q(\theta) - \bar{g}_{11}(\theta)\bar{q}(\theta) \tag{62}$$

From equation (60), (62) and the definition of A, it follows that

$$\dot{w}_{20} = 2i\eta^* \tau^* w_{20}(\theta) + q_{20}(\theta)\bar{q}(\theta) + \bar{q}_{02}(\theta)\bar{q}(\theta).$$

We know that $q(\theta) = (1, q_1)^T e^{i\eta^* \tau^* \theta}$. Hence

$$w_{20}\theta = \frac{i\bar{g}_{20}}{\eta^* \tau^*} q(0)e^{i\eta^* \tau^* \theta} + \frac{i\bar{g}_{02}}{3\eta^* \tau^*} \bar{q}(0) + E_1 e^{2i\eta^* \tau^* \theta}, \tag{63}$$

where $E_1 = (E_1^1, E_1^2) \in \mathbb{R}^*$ is a constant vector. Similarly we can obtain

$$w_{11}\theta = \frac{-i\bar{g}_{11}}{\eta^* \tau^*} q(0)e^{i\eta^* \tau^* \theta} + \frac{i\bar{g}_{11}}{\eta^* \tau^*} \bar{q}(0)e^{-i\eta^* \tau^* \theta} + E_2, \tag{64}$$

where $E_2 = (E_2^1, E_2^2) \in \mathbb{R}^2$ is a constant vector.

Now we seek the appropriate value of E_1, E_2 . From the definition of A and (60), we obtain

$$\int_{-1}^0 d\eta(\theta)w_{20}(\theta) = 2\eta^* \tau^* w_{20}(0) - H_{20}(0) \tag{65}$$

$$\text{and } \int_{-1}^0 d\eta(\theta)w_{11}(\theta) = -H_{11}(0), \tag{66}$$

where $\eta(\theta) = \eta(0, \theta)$. From equation (58), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau^* \begin{pmatrix} a_{11} + a_{12}q_1 e^{-i\eta^* \tau^*} + a_{22}q_1 e^{-i\eta^* \tau^*} \\ b_{11} + b_{12}q_1 + b_{22}q_1^2 \end{pmatrix} \tag{67}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau^* \begin{pmatrix} a_{11} + a_{12}Re(q_1) + a_{22}Re(q_1) \\ b_{11}Re(q_1) + b_{12}|q_1|^2 + b_{22}|q_1|^2 \end{pmatrix} \tag{68}$$

Substituting (63) and (67) into (65) and noticing that

$$\left(i\eta^* \tau^* I - \int_{-1}^0 e^{i\eta^* \tau^* \theta} d\eta(\theta) \right) q(0) = 0 \text{ and } \left(-i\eta^* \tau^* I - \int_{-1}^0 e^{-i\eta^* \tau^* \theta} d\eta(\theta) \right) \bar{q}(0) = 0,$$

$$\text{we obtain } \left(2i\eta^* \tau^* I - \int_{-1}^0 e^{2i\eta^* \tau^* \theta} d\eta(\theta) \right) E_1 = 2\tau^* \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{69}$$

$$\text{where } B_1 = a_{11} + a_{12}q_1 e^{-i\eta^* \tau^*} + a_{22}q_1 e^{-i\eta^* \tau^*}, \quad B_2 = b_{11} + b_{12}q_1 + b_{22}q_1^2$$

Hence, equation (69) reduces to

$$\begin{pmatrix} 2i\eta^* - m'_{11} & -n'_{12} e^{-2i\eta^* \tau^*} \\ -m'_{21} & 2i\eta^* + m'_{22} \end{pmatrix} E_1 = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

By using Cramer’s rule, we have,

$$E_1^{(1)} = \frac{2}{\Delta} \begin{vmatrix} B_1 & -n'_{12} e^{-2i\eta^* \tau^*} \\ B_2 & 2i\eta^* + m'_{22} \end{vmatrix}, \quad E_1^{(2)} = \frac{2}{\Delta} \begin{vmatrix} 2i\eta^* - m'_{11} & B_1 \\ -m'_{21} & B_2 \end{vmatrix},$$

where $A = \begin{vmatrix} 2i\eta^* - m'_{11} & -n'_{12}e^{-2i\eta^*\tau^*} \\ -m'_{21} & 2i\eta^* + m'_{22} \end{vmatrix}$

Similarly, substituting (64) and (68) into (66), we have

$$\begin{pmatrix} m'_{11} & n'_{12} \\ m'_{21} & m'_{22} \end{pmatrix} E_2 = -2 \begin{pmatrix} a_{11} + a_{12}Re(q_1) + a_{22}Re(q_1) & \\ b_{11}Re(q_1) + b_{12}|q_1|^2 + b_{22}|q_1|^2 & \end{pmatrix}$$

Hence we have

$$E_1^{(1)} = -\frac{2}{A^*} \begin{vmatrix} a_{11} + a_{12}Re(q_1) + a_{22}Re(q_1) & n'_{12} \\ b_{11}Re(q_1) + b_{12}|q_1|^2 + b_{22}|q_1|^2 & m'_{22} \end{vmatrix},$$

$$E_1^{(2)} = -\frac{2}{A^*} \begin{vmatrix} m'_{11} & a_{11} + a_{12}Re(q_1) + a_{22}Re(q_1) \\ m'_{21} & b_{11}Re(q_1) + b_{12}|q_1|^2 + b_{22}|q_1|^2 \end{vmatrix},$$

where $A = -A|_{\theta=0, \eta^*=0}$. Then, we can find out $w_{20}(0)$ and $w_{11}(0)$ from relation (63) and (64). Furthermore, we can determine g_{12} by the system parameters and delay in (57). Thus we can compute the following results:

$$C_1 = \frac{i}{2\eta^*\tau^*} \left(g_{11}g_{20} - 2|g_{11}|^2 + \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \tag{70a}$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau^*)\}}, \tag{70b}$$

$$\beta_2 = 2Re\{C_1(0)\}, \tag{70c}$$

$$\tau_2 = Im \frac{\{C_1(0)/0\} + \mu_2 Im\{\lambda'(\tau^*)\}}{\tau^*\eta^*}, \tag{70d}$$

which determine the bifurcating periodic solution in the center manifold at the critical value τ^* , i.e., μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau^*$ ($\tau < \tau^*$). Here β_2 determines the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$) and τ_2 the period of periodic solution: the period increase (decrease) if $\tau > 0$ ($\tau < 0$).

6.1. Existence of switching stability

The characteristic equation corresponding to the Jacobian matrix J^* at the interior equilibrium $E^*(N_1^*, N_1^*)$ of the system with positive delay is

$$\lambda^2 + (-m'_{11} - m'_{22})\lambda + m'_{11}m'_{22} = m'_{21}n'_{12}e^{-\lambda\tau}$$

This equation can be written as $P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0$,

where $P(\lambda) = \lambda^2 + (-m'_{11} - m'_{22})\lambda$ and $Q(\lambda) = -m'_{21}n'_{12}$

Clearly, $P(\lambda)$ and $Q(\lambda)$ are both analytic function in $Re(\lambda) > 0$ how we have the following results:

(i) $P(0) + Q(0) = m'_{11}m'_{22} - m'_{21}n'_{12}$

$$\begin{aligned}
 &= N_1^* N_2^* [(\hat{\alpha}_1 \hat{\alpha}_2 + \hat{\beta}_{21} \hat{\gamma}_1 N_1^* + \hat{\beta}_{21} \hat{\gamma}_2 N_2^*) - (\hat{\beta}_{12} \hat{\beta}_{21} + \hat{\alpha}_1 \hat{\gamma}_2 N_1^* + \hat{\alpha}_2 \hat{\gamma}_1 N_2^*)] \\
 &\neq 0 \text{ if } \hat{\alpha}_1 \hat{\alpha}_2 + \hat{\beta}_{21} \hat{\gamma}_1 N_1^* + \hat{\beta}_{21} \hat{\gamma}_2 N_2^* > \hat{\beta}_{12} \hat{\beta}_{21} + \hat{\alpha}_1 \hat{\gamma}_2 N_1^* + \hat{\alpha}_2 \hat{\gamma}_1 N_2^*
 \end{aligned}$$

(ii) $P(-iN_2) = P(i\bar{N}_2)$, $Q(-iN_2) = Q(i\bar{N}_2)$

$$\begin{aligned}
 \text{Now } F(N_2) &= |P(iN_2)|^2 - |Q(iN_2)|^2 \\
 &= N_2^4 + (m_{11}'^2 + m_{22}'^2)N_2^2 + m_{11}'^2 m_{22}'^2 - m_{21}'^2 n_{12}'^2
 \end{aligned}$$

which is a quadratic in N_2^2 .

Therefore $F(N_2) = 0$ has at least one positive root if $|m_{11}' m_{22}'| < |m_{21}' n_{12}'|$. Hence by applying theorem 4.1 in Kuang [22], we see that the system (7) possesses at most finite number of stability switches.

7. Numerical Illustration

Analytical studies can never be completed without numerical verification of the results. In this section we present computer simulation of some solutions of the system (6) and (7). Beside verification of our analytical findings, these numerical solutions are very important from practical point of view. We consider all fictitious set of data through analysis to establish our theoretical findings.

Let us consider a set of imprecise biological values of parameters as follows in appropriate units :

$$\hat{K}_1 = [1.9, 2.2], \hat{K}_2 = [0.9, 1.1], \hat{\alpha}_1 = [0.05, 0.08], \hat{\alpha}_2 = [0.06, 0.09], \hat{\beta}_{12} = [0.04, 0.07], \hat{\beta}_{21} = [0.01, 0.02], \hat{\gamma}_1 = [0.0008, 0.001], \hat{\gamma}_2 = [0.0025, 0.005] \text{ and } p \in [0, 1].$$

Case I : Toxic Inhibition

Using the parametric form of interval numbers and assuming the initial condition $(N_1(0), N_2(0)) = (1.0, 1.0)$, we find the conditions given in (19) are satisfied, which imply that the unique interior equilibrium point $E^*(N_1^*, N_2^*)$ exist of the system (6) for all values of $p \in [0, 1]$. From Table 1 we observe that for different values of the parameter p , the system (6) corresponds a unique positive equilibrium point which are locally asymptotically stable. We present the dynamics of the model for different values of p ($p = 0.0, 0.3, 0.5, 0.8, 1.0$) in fig.1a-e. These figures show that the interior equilibrium E^* exist and asymptotically stable for all values of $p \in [0, 1]$. But the values are different for different values of p . Fig.1f shows that both the species population decrease with increasing p , but N_1 species is decreasing rapidly where as N_2 species decreasing slowly. However, in this case, analytically we have already shown that the system is always stable for all $\tau \geq 0$.

Table 1 : Stable equilibrium points of toxic inhibited system for different p

p	Equilibrium Points
0.0	(32.6443,4.0502)
0.3	(29.8189,3.6902)
0.5	(28.0503,3.4648)
0.8	(25.5764,3.1465)
1.0	(24.0367,2.9463)

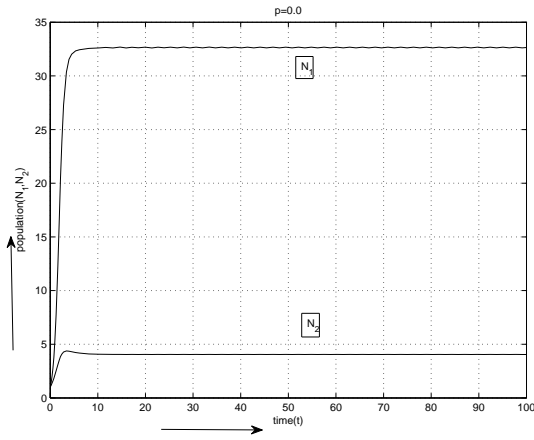


Fig.1a

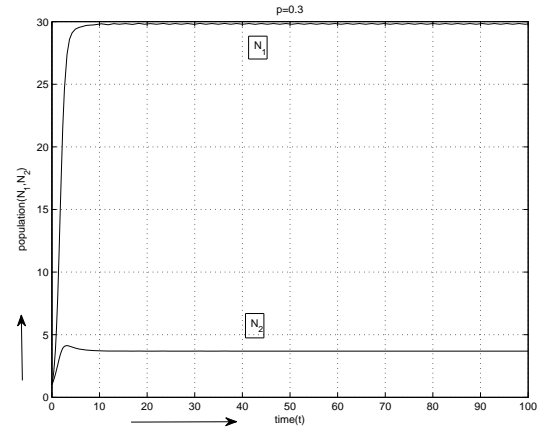


Fig.1b

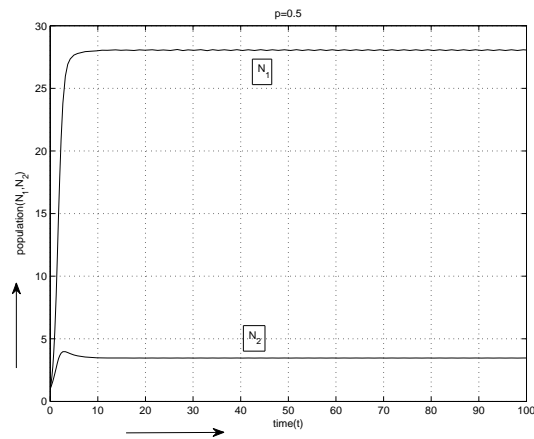


Fig.1c

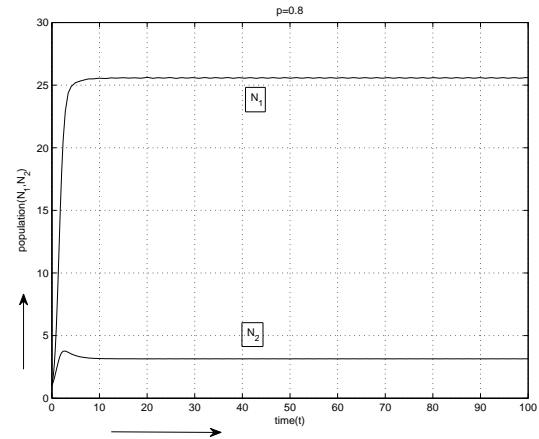


Fig.1d

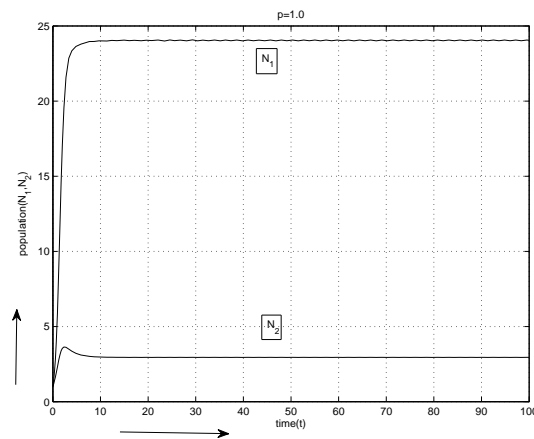


Fig.1e

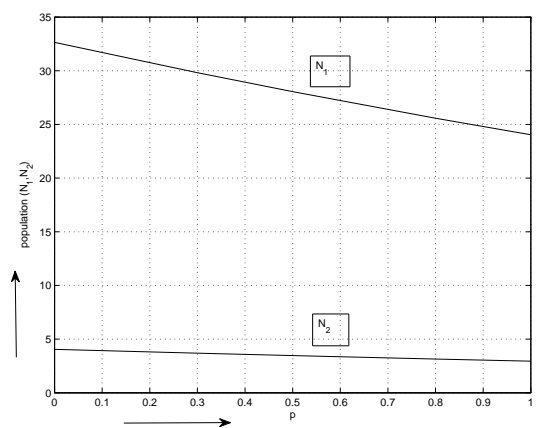


Fig.1f

Fig.1. Behavior of the toxic inhibited species with time by using the imprecise parameters values and initial conditions $(N'_1(0), N'_2(0)) = (1.0, 1.0)$ for $p = 0.0, 0.3, 0.5, 0.8, 1.0$ and Fig.1f shows the dynamical behavior of the two species population (N'_1, N'_2) with respect to p when the values of the other parameters are same

Case II : Toxic Stimulation

Here also we consider the same parametric form of interval numbers and assuming the initial condition $(N_1'(0), N_2'(0)) = (1.0, 1.0)$, we find the conditions given in (30) are satisfied which imply that the unique interior equilibrium point $E^*(N_1^*, N_2^*)$ exist of the system (5) for all values of $p \in [0, 1]$. Also when the delay value $\tau = 0$, the condition (33) is satisfied and the interior equilibrium point E^* is locally asymptotically stable. From Table 2, we observe that for different values of the parameter p , the system (7) corresponds a unique positive equilibrium point which are locally asymptotically stable. we present the dynamics of the model for different values of p ($p = 0.0, 0.3, 0.5, 0.8, 1.0$) in fig.2a-e. These figures show that the interior equilibrium E^* exist and asymptotically stable for all values of $p \in [0, 1]$. But values are different for different values of p . Fig.2f shows that both the species population decrease with increasing p , but N_2 species is decreasing rapidly where as N_1 species decreasing slowly.

Table 2 : Stable equilibrium points of toxic stimulated system for different p

p	Equilibrium Points
0.0	(16.6667,40.0000)
0.3	(14.3634,33.0759)
0.5	(13.0319,29.5765)
0.8	(11.2801,25.3857)
1.0	(10.2498,23.0964)

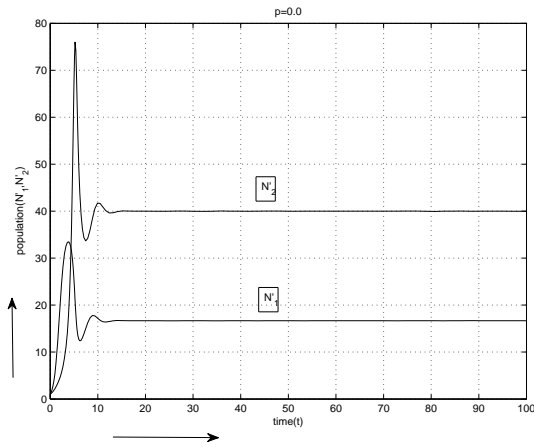


Fig.2a

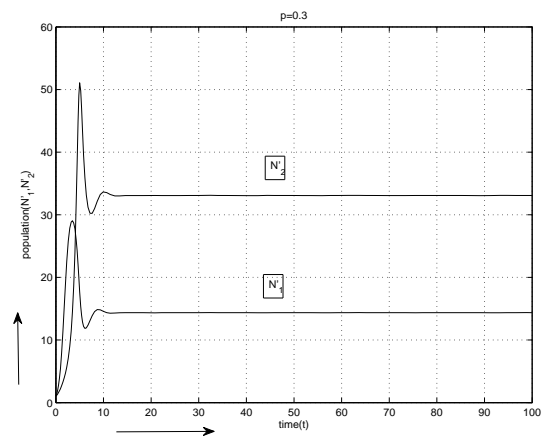


Fig.2b

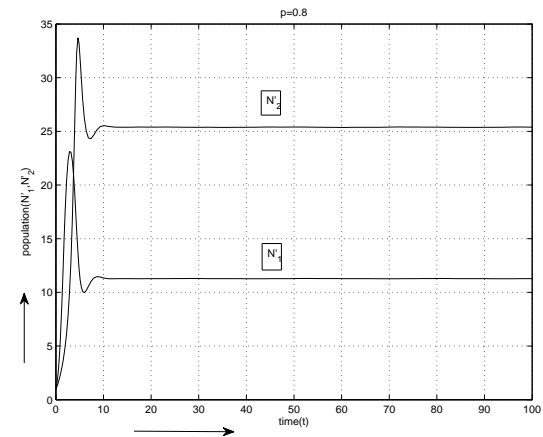
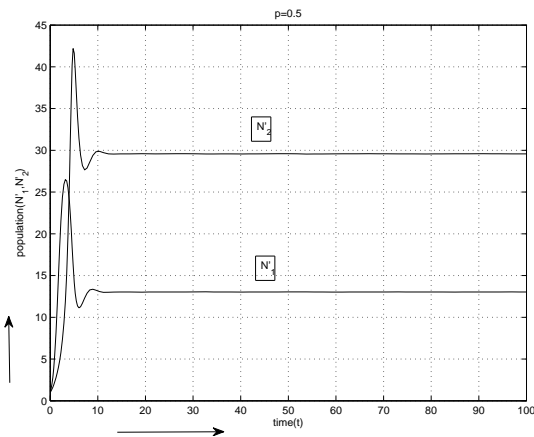


Fig.2c

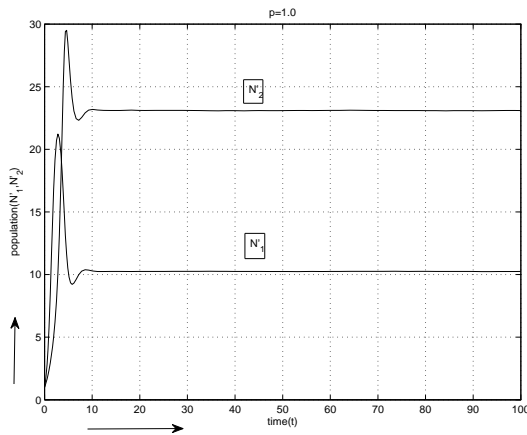


Fig.2e

Fig.2d

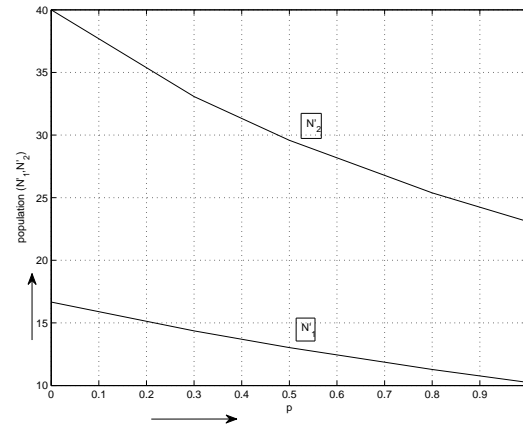


Fig.2f

Fig.2. Behavior of the toxic stimulated species with time by using the imprecise parameters values and initial conditions $(N'_1(0), N'_2(0)) = (1.0, 1.0)$ for $p = 0.0, 0.3, 0.5, 0.8, 1.0$ and Fig.2f shows the dynamical behavior of the two species population (N'_1, N'_2) with respect to p when the values of the other parameters are same

Finally we investigate the dynamical behavior of the system (7) numerically in the presence of time delay using the same data. For the given values of the interval parameters the equation (37) has unique positive solution namely η^* which takes different value depending on the values of the parameter p given in Table 3. Now depending on these values of η^* for $j = 0$, from equation (38) we obtain the initial values of the bifurcation parameter (time delay) τ say τ^* given in Table 3.

For a particular value of p we get a definite critical value of the delay τ^* as shown in Table 3. Now for a particular value of p , if value of τ is below the critical value τ^* then fig.3a-e shows that the interior equilibrium point $E^*(N'_1, N'_2)$ is asymptotically stable and both the species converge to their steady states in finite time. Now if we gradually increase only the value of delay, the stability of the equilibrium point $E^*(N'_1, N'_2)$ may switch over. By theorem 2 as τ passes through the critical value τ^* as per Table 3, E^* losses its stability as shown in fig.3a-e, which is the case of Hopf-bifurcation. Also from fig.4a-e, the positive equilibrium point $E^*(N'_1, N'_2)$ is unstable with a periodic orbit near $E^*(N'_1, N'_2)$ as τ passes through τ^* .

Table 3 : Values of η^* and critical values of time delay (τ^*) of toxic stimulated system for different p

p	0.0	0.3	0.5	0.8	1.0
η^*	1.1409	1.0548	1.0051	0.9371	0.8941
τ^*	0.7261	0.9627	1.1181	1.3568	1.5254

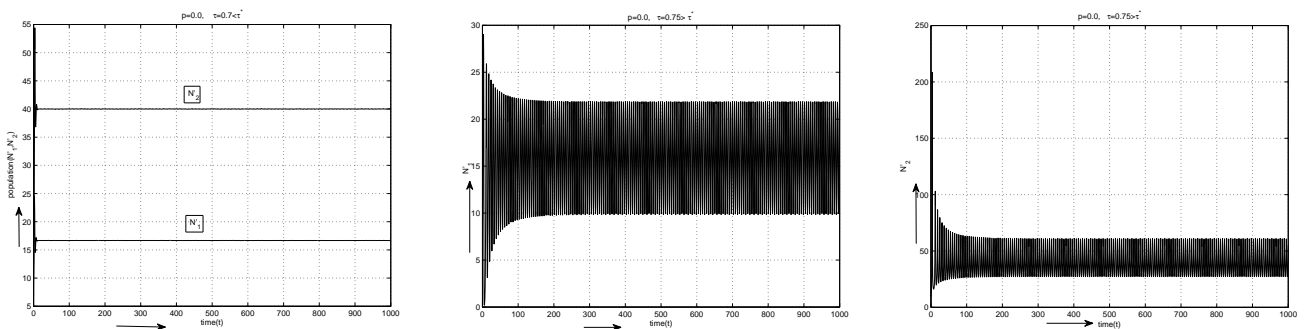


Fig.3a

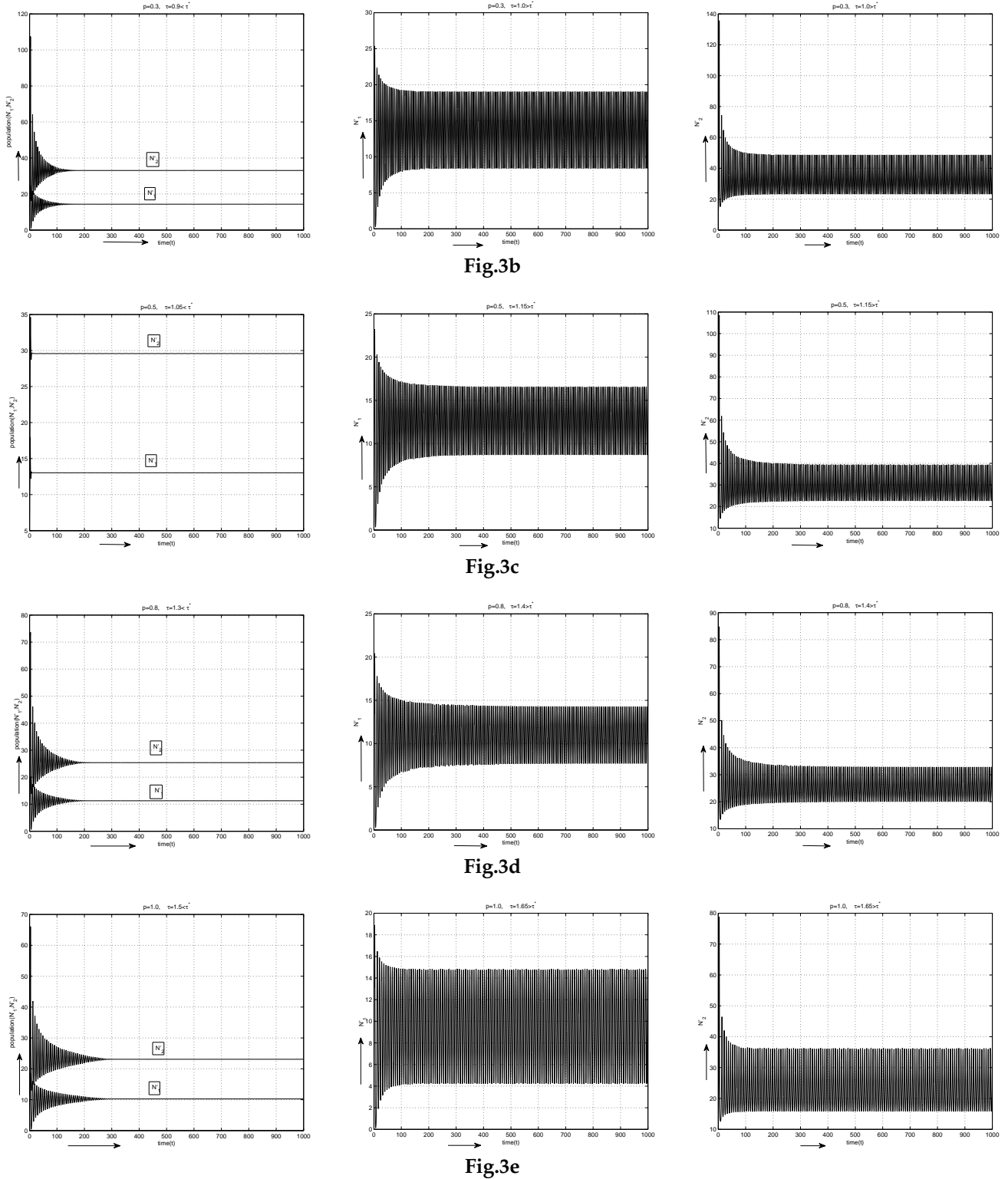


Fig.3. Time course of two competitive species variation when $\tau < \tau^*$ and $\tau > \tau^*$ with same imprecise parameter values for $p = 0.0, 0.3, 0.5, 0.8, 1.0$

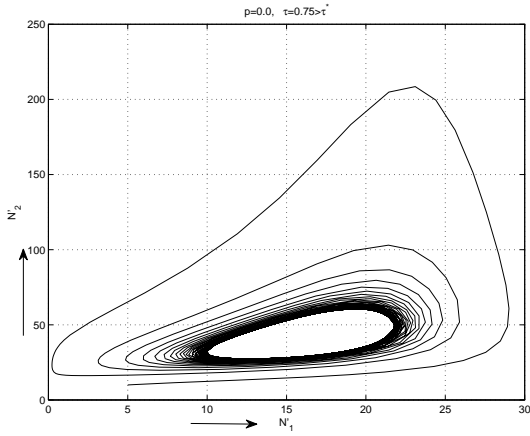


Fig.4a

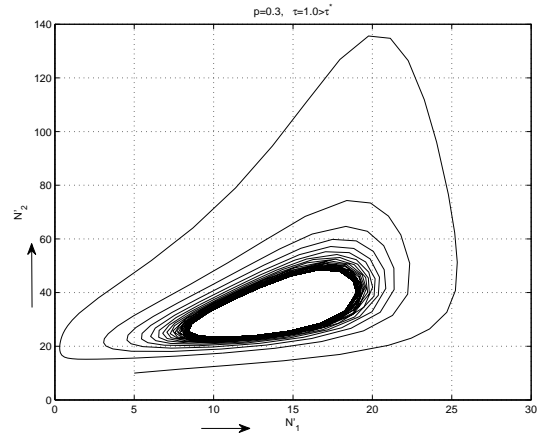


Fig.4b

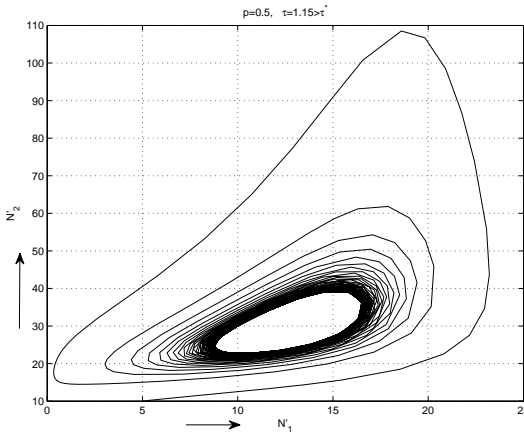


Fig.4c

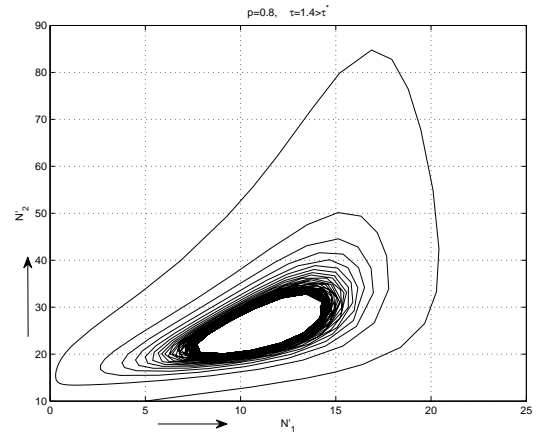


Fig.4d

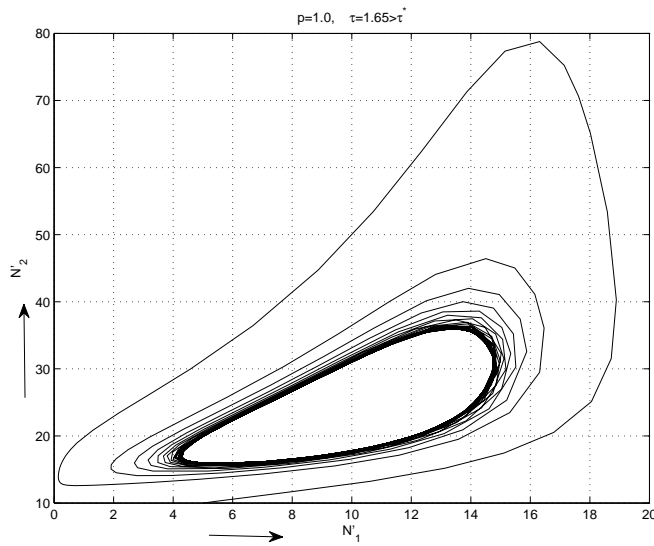


Fig.4e

Fig.4. Phase portrait of species when $\tau > \tau^*$ with same imprecise parameter values for $p = 0.0, 0.3, 0.5, 0.8, 1.0$

8. Discussion

In this paper, we have studied a delayed two species competition model with the effect of toxic substances. The equations of the model are obtained from the modified Lotka-Volterra type competition of two species which are affected by toxicant. These toxic substances may be toxic inhibitory or stimulatory to the other species, which are studied extensively in our paper. Most of the competition models are generally based on the assumption that the biological parameters are precisely known. But in real life situation, it is not possible to know the values of all biological parameters precisely. In this paper, we consider a two species competition model with delay in the second species using some imprecise parameters. Here we introduce the concept of interval numbers to the model by considering the biological parameters $\hat{K}_1, \hat{K}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\gamma}_1, \hat{\gamma}_2$ which are imprecise in nature. We have used parametric functional form of the interval number to convert the imprecise competition model to the corresponding parametric competition delay model.

We consider all fictitious set of data through analysis to establish our theoretical findings. We examine the dynamical behaviour of both the cases, toxic inhibitory and stimulatory model system in absence as well as presence of time delay for different values of the parameter $p \in [0, 1]$. We have discussed the existence and stability of various equilibrium points of both the systems. We get different equilibrium level of the species and the critical value of the time delay depending upon values of the parameter p .

Analytically it is obtained that the time delay does not affect the stability on the toxic inhibited system. But in case of toxic stimulatory system, it is shown that the system becomes unstable at different time delays for different values of the parameter p and leads to stable limit cycle periodic solutions through Hopf-bifurcation.

All our important mathematical findings for the dynamical behaviour of the toxicant affected two species competition model are also numerically verified and graphical representation of a variety of solutions of system (6) and (7) are depicted by using MATLAB with some imprecise parameter values. These numerical results are very important to understand the system in both mathematical and ecological points of view.

Finally, we conclude that impreciseness of biological parameters have great impact on the behaviour of the delay model. For the first time we use the concept of interval number to present imprecise delay competition model, which makes the situation more realistic as always it is not possible to know the parameter values precisely. Here we consider all the biological parameters are imprecise, except the delay parameter τ . The delay model can be made more realistic when incorporated with impreciseness in the delay term, it makes the model more interesting and is left for future work consideration.

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