Filomat 31:16 (2017), 5305–5314
https://doi.org/10.2298/FIL1716305B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Improvement in Companion of Ostrowski Type Inequalities for Mappings Whose First Derivatives are of Bounded Variation and Applications

Hüseyin Budak[a], Mehmet Zeki Sarikaya[a], Ather Qayyum[b]

[a]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey*
[b]*Department of Mathematics, University of Ha'il, 2440, Saudi Arabia.*

Abstract. The main aim of this paper is to obtain a improved and generalized version of companion of Ostrowski type integral inequalities for mappings whose first derivatives are of bounded variation. Some previous results are also recaptured as special cases. New quadrature formulae are also provided.

1. Introduction

In 1938, Ostrowski [15] established the following interesting integral inequality associated with the differentiable mappings.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\left\Vert f'\right\Vert_{\infty}:=\sup\limits_{t\in(a,b)}\left\vert f'(t)\right\vert<\infty$. Then, we have the inequality*

$$\left\vert f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right\vert\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\Vert f'\right\Vert_{\infty}, \tag{1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. During the past few years, many authors have studied on Ostrowski type inequalities for function of bounded variation, see for example ([1]-[3], [5]-[13]). Uptil now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications.

The following definitions will be frequently used to prove our results.

2010 *Mathematics Subject Classification*. Primary 26D15, 26A45, 26D10, 41A55
Keywords. Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integral
Received: 14 February 2016; Accepted: 08 February 2017
Communicated by Dragan S. Djordjević
Email addresses: hsyn.budak@gmail.com (Hüseyin Budak), sarikayamz@gmail.com (Mehmet Zeki Sarikaya), atherqayyum@gmail.com (Ather Qayyum)

Definition 1.2. *Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} \left|\Delta f(x_i)\right|$$

is bounded for all such partitions.

Definition 1.3. *Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^{n} \left|\Delta f(x_i)\right|$ corresponding to the partition P of $[a, b]$. The number*

$$\bigvee_a^b (f) := \sup\left\{\sum \Delta f(P) : P \in P([a, b])\right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [10], Dragomir proved the following Ostrowski type inequalities related to functions of bounded variation:

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a) f(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

In [14], Liu proved the following Ostrowski type inequalities for functions with first derivatives of bounded variation:

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then for any $x \in [a, b]$ and $\theta \in [0, 1]$ we have*

$$\begin{aligned} &\left|\int_a^b f(t)dt - (b-a)\left[(1-\theta) f(x) + \theta\frac{f(a)+f(b)}{2} - (1-\theta)\left(x - \frac{a+b}{2}\right) f'(x)\right]\right| \\ \le\ & \frac{1}{16}\left[4(x-b)^2 - 4\theta(b-a)(b-x) + \theta^2(b-a)^2 \left|4(x-b)^2 - 4\theta(b-a)(b-x) - \theta^2(b-a)^2\right|\right] \bigvee_a^b (f') \end{aligned}$$

for $a \le x \le \frac{a+b}{2}$ and

$$\begin{aligned} &\left|\int_a^b f(t)dt - (b-a)\left[(1-\theta) f(x) + \theta\frac{f(a)+f(b)}{2} - (1-\theta)\left(x - \frac{a+b}{2}\right) f'(x)\right]\right| \\ \le\ & \frac{1}{16}\left[4(x-a)^2 - 4\theta(b-a)(x-a) + \theta^2(b-a)^2 \left|4(x-a)^2 - 4\theta(b-a)(x-a) - \theta^2(b-a)^2\right|\right] \bigvee_a^b (f') \end{aligned}$$

for $\frac{a+b}{2} \le x \le b$.

In [7], authors gave the following Ostrowski type inequality:

Theorem 1.6. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)-f'(a+b-x)\right]\right|$$
$$\leq \frac{1}{16}\left[\frac{5(x-a)^2-2(x-a)(b-x)+(b-x)^2}{b-a}+4\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f')$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

In [4], Budak and Sarikaya obtained the following Ostrowski type inequality in weighted form for the mappings whose first derivatives are of bounded variation:

Theorem 1.7. *Let $w:[a,b]\to\mathbb{R}$ be nonnegative and continous and let $f:[a,b]\to\mathbb{R}$ be differentiable mapping on $[a,b]$. If f' is of bounded variation on $[a,b]$, then we have the weighted inequality*

$$\left|\left(\int_a^b (u-x)\,w(u)du\right)f'(x)+\left(\int_a^b w(u)du\right)f(x)-\int_a^b w(t)f(t)dt\right|$$
$$\leq \left(\int_a^x (u-x)\,w(u)du\right)\bigvee_a^x(f')+\left(\int_x^b (u-x)\,w(u)du\right)\bigvee_x^b(f')$$

for any $x\in[a,b]$.

In [17], authors established a new version of Ostrowski's type integral inequality by using a new type of kernel with five sections. Then, Budak and Sarikaya obtained a companion of Ostrowski type inequalities for mappings of bounded variation with the help of this 5-step kernel [8]. Recently, Qayyum et. al [16], proved Ostrowski inequality using a 5-step quadratic kernel. In this paper, using this five step quadratic kernel, we establish a new companion of Ostrowski type integral inequalities for functions whose first derivatives are of bounded variation by the similar way that used in [8]. At the end, we apply our results for new efficient quadrature rules. The results presented here would provide extensions of those given in [7].

2. Derivation of companion of Ostrowski type integral inequalities

Before we prove our results for the 5-step quadratic kernel, we give the following lemma.

Lemma 2.1. *Consider the kernel $P(x,t)$ defined by Qayyum et al. in [16]*

$$P(x,t)=\begin{cases}\frac{1}{2}(t-a)^2, & t\in\left(a,\frac{a+x}{2}\right]\\ \frac{1}{2}\left(t-\frac{3a+b}{4}\right)^2, & t\in\left(\frac{a+x}{2},x\right]\\ \frac{1}{2}\left(t-\frac{a+b}{2}\right)^2, & t\in(x,a+b-x]\\ \frac{1}{2}\left(t-\frac{a+3b}{4}\right)^2, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right]\\ \frac{1}{2}(t-b)^2, & t\in\left[\frac{a+2b-x}{2},b\right)\end{cases} \tag{2}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$*, then the following identity*

$$\int_a^b P(x,t)df'(t) \tag{3}$$
$$= \int_a^b f(t)dt - \frac{b-a}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right.$$
$$\left. + \left(x - \frac{5a+3b}{8}\right)\{f'(a+b-x) - f'(x)\} + \frac{1}{2}\left(x - \frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right) - f'\left(\frac{a+x}{2}\right)\right\}\right].$$

holds.

Proof. By using (2), we have

$$\int_a^b P(x,t)df'(t) \tag{4}$$
$$= \frac{1}{2}\left[\int_a^{\frac{a+x}{2}} (t-a)^2 df'(t) + \int_{\frac{a+x}{2}}^{x} \left(t - \frac{3a+b}{4}\right)^2 df'(t) + \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right)^2 df'(t)\right.$$
$$\left. + \int_{a+b-x}^{\frac{a+2b-x}{2}} \left(t - \frac{a+3b}{4}\right)^2 df'(t) + \int_{\frac{a+2b-x}{2}}^{b} (t-b)^2 df'(t)\right]$$
$$= \frac{1}{2}\left[I_1 + I_2 + I_3 + I_4 + I_5\right].$$

Integrating by parts, we obtain

$$I_1 = \int_a^{\frac{a+x}{2}} (t-a)^2 df'(t) \tag{5}$$
$$= (t-a)^2 f'(t)\Big|_a^{\frac{a+x}{2}} - 2\int_a^{\frac{a+x}{2}} (t-a) f'(t)dt$$
$$= \frac{(x-a)^2}{4} f'\left(\frac{a+x}{2}\right) - (x-a) f\left(\frac{a+x}{2}\right) + 2\int_a^{\frac{a+x}{2}} f(t)dt.$$

Similarly, using integration by part, we have

$$I_2 = \int_{\frac{a+x}{2}}^{x} \left(t - \frac{3a+b}{4}\right)^2 df'(t) \tag{6}$$
$$= \left(x - \frac{3a+b}{4}\right)^2 f'(x) - \frac{1}{4}\left(x - \frac{a+b}{2}\right)^2 f'\left(\frac{a+x}{2}\right)$$
$$-2\left(x - \frac{3a+b}{4}\right) f(x) + \left(x - \frac{a+b}{2}\right) f\left(\frac{a+x}{2}\right) + 2\int_{\frac{a+x}{2}}^{x} f(t)dt,$$

$$\begin{aligned} I_3 &= \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 df'(t) \qquad (7)\\ &= \left(\frac{a+b}{2}-x\right)^2 f'(a+b-x)-\left(\frac{a+b}{2}-x\right)^2 f'(x)\\ &\quad -2\left(\frac{a+b}{2}-x\right)f(a+b-x)-2\left(\frac{a+b}{2}-x\right)f(x)+2\int_x^{a+b-x} f(t)dt, \end{aligned}$$

$$\begin{aligned} I_4 &= \int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2 df'(t) \qquad (8)\\ &= \frac{1}{4}\left(\frac{a+b}{2}-x\right)^2 f'\left(\frac{a+2b-x}{2}\right)-\left(\frac{3a+b}{4}-x\right)^2 f'(a+b-x)\\ &\quad -\left(\frac{a+b}{2}-x\right)f\left(\frac{a+2b-x}{2}\right)+2\left(\frac{3a+b}{4}-x\right)f(a+b-x)+2\int_{a+b-x}^{\frac{a+2b-x}{2}} f(t)dt \end{aligned}$$

and

$$\begin{aligned} I_5 &= \int_{\frac{a+2b-x}{2}}^{b}(t-b)^2 df'(t) \qquad (9)\\ &= -\frac{(x-a)^2}{4}f'\left(\frac{a+2b-x}{2}\right)-(x-a)f\left(\frac{a+2b-x}{2}\right)+2\int_{\frac{a+2b-x}{2}}^{b} f(t)dt. \end{aligned}$$

If we substitute the equalities (5)-(9) in (4), we get the required identity (3). □

Now using above identity, we state and prove the following theorem.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt-\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\right. \qquad (10)\\ &\left.\left.+\left(x-\frac{5a+3b}{8}\right)\{f'(a+b-x)-f'(x)\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\right]\right|\\ \leq\ & \frac{1}{2}\max\left\{\left(x-\frac{3a+b}{4}\right)^2,\frac{1}{4}\left(\frac{a+b}{2}-x\right)^2,\frac{(x-a)^2}{4}\right\}\bigvee_a^b(f'), \end{aligned}$$

where $x\in\left[a,\frac{a+b}{2}\right]$ and $\bigvee_a^b(f')$ denotes the total variation of f' on $[a,b]$.

Proof. From Lemma 2.1, we have

$$\left|\int_a^b P(x,t)df'(t)\right| \tag{11}$$
$$\leq \frac{1}{2}\left[\left|\int_a^{\frac{a+x}{2}}(t-a)^2\,df'(t)\right| + \left|\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2 df'(t)\right| + \left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 df'(t)\right|\right.$$
$$\left.+\left|\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2 df'(t)\right| + \left|\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\,df'(t)\right|\right].$$

It is well known that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right| \leq \sup_{t\in[a,b]}\left|g(t)\right|\bigvee_a^b(f). \tag{12}$$

By using (12) for each term in (11), we get

$$\left|\int_a^{\frac{a+x}{2}}(t-a)^2\,df'(t)\right| \leq \sup_{t\in\left[a,\frac{a+x}{2}\right]}(t-a)^2\bigvee_a^{\frac{a+x}{2}}(f') = \frac{(x-a)^2}{4}\bigvee_a^{\frac{a+x}{2}}(f'), \tag{13}$$

$$\left|\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2 df'(t)\right| \leq sup_{t\in\left[\frac{a+x}{2},x\right]}\left(t-\frac{3a+b}{4}\right)^2\bigvee_{\frac{a+x}{2}}^{x}(f') \tag{14}$$
$$= \max\left\{\left(x-\frac{3a+b}{4}\right)^2, \frac{1}{4}\left(\frac{a+b}{2}-x\right)^2\right\}\bigvee_{\frac{a+x}{2}}^{x}(f'),$$

$$\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 df'(t)\right| \leq \sup_{t\in[x,a+b-x]}\left(t-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f') = \left(\frac{a+b}{2}-x\right)^2\bigvee_x^{a+b-x}(f') \tag{15}$$

$$\left|\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2 df'(t)\right| \leq \sup_{t\in\left[a+b-x,\frac{a+2b-x}{2}\right]}\left(t-\frac{a+3b}{4}\right)^2\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f') \tag{16}$$
$$= \max\left\{\left(x-\frac{3a+b}{4}\right)^2, \frac{1}{4}\left(\frac{a+b}{2}-x\right)^2\right\}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f'),$$

and

$$\left| \int_{\frac{a+2b-x}{2}}^{b} (t-b)^2 \, df'(t) \right| \leq \sup_{t \in \left[\frac{a+2b-x}{2}, b\right]} (t-b)^2 \bigvee_{\frac{a+2b-x}{2}}^{b} (f') = \frac{(x-a)^2}{4} \bigvee_{\frac{a+2b-x}{2}}^{b} (f'), \tag{17}$$

respectively. Using (13)-(17) in (11), we have the inequality

$$\begin{aligned}
&\left| \int_a^b P(x,t) df'(t) \right| \\
\leq\ & \frac{1}{2} \left[\frac{(x-a)^2}{4} \bigvee_a^{\frac{a+x}{2}} (f') + \max\left\{ \left(x - \frac{3a+b}{4}\right)^2, \frac{1}{4}\left(\frac{a+b}{2} - x\right)^2 \right\} \bigvee_{\frac{a+x}{2}}^{x} (f') + \left(\frac{a+b}{2} - x\right)^2 \bigvee_{x}^{a+b-x} (f') \right. \\
& \left. + \max\left\{ \left(x - \frac{3a+b}{4}\right)^2, \frac{1}{4}\left(\frac{a+b}{2} - x\right)^2 \right\} \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f') + \frac{(x-a)^2}{4} \bigvee_{\frac{a+2b-x}{2}}^{b} (f') \right] \\
\leq\ & \frac{1}{2} \max\left\{ \left(x - \frac{3a+b}{4}\right)^2, \left(\frac{a+b}{2} - x\right)^2, \frac{(x-a)^2}{4} \right\} \bigvee_a^b (f').
\end{aligned}$$

Thus, the proof is completed. □

Remark 2.3. *If we choose $x = a$ in Theorem 2.2, we get the result proved by Budak et al. [7].*

Corollary 2.4. *Under the assumption of Theorem 2.2 with $x = \frac{a+b}{2}$, then we have the following inequality*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \right. \\
&\left. - \frac{b-a}{32} \left\{ f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right\} \right| \\
\leq\ & \frac{b-a}{32} \bigvee_a^b (f').
\end{aligned} \tag{18}$$

Corollary 2.5. *Under the assumption of Theorem 2.2 with $x = \frac{3a+b}{4}$, then we get the inequality*

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \frac{b-a}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] \right. \\
&\left. + \frac{b-a}{32} \left\{ f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right\} \right| \\
\leq\ & \frac{b-a}{32} \bigvee_a^b (f').
\end{aligned} \tag{19}$$

Under assumption of of Theorem 2.2, we also have the following corollaries:

Corollary 2.6. *Let $f \in C^2[a,b]$. Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt - \frac{b-a}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right.\right. \\ &\left.\left. + \left(x - \frac{5a+3b}{8}\right)\{f'(a+b-x) - f'(x)\} + \frac{1}{2}\left(x - \frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right) - f'\left(\frac{a+x}{2}\right)\right\}\right]\right| \\ \leq\ & \frac{1}{2}\max\left\{\left(x-\frac{3a+b}{4}\right)^2, \frac{1}{4}\left(\frac{a+b}{2}-x\right)^2, \frac{(x-a)^2}{4}\right\}\left\|f''\right\|_{[a,b],1} \end{aligned} \tag{20}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, where $\|.\|_{[a,b],1}$ is the L_1–norm, namely

$$\left\|f''\right\|_{[a,b],1} = \int_a^b \left|f''(t)\right| dt.$$

Corollary 2.7. *Let $f' : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constants $L > 0$. Then, we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt - \frac{b-a}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right.\right. \\ &\left.\left. + \left(x - \frac{5a+3b}{8}\right)\{f'(a+b-x) - f'(x)\} + \frac{1}{2}\left(x - \frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right) - f'\left(\frac{a+x}{2}\right)\right\}\right]\right| \\ \leq\ & \frac{L(b-a)}{2}\max\left\{\left(x-\frac{3a+b}{4}\right)^2, \frac{1}{4}\left(\frac{a+b}{2}-x\right)^2, \frac{(x-a)^2}{4}\right\} \end{aligned} \tag{21}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. As f' is L-Lipschitzian on $[a,b]$, it is also of bounded variation. If $P([a,b])$ denotes the family of divisions on $[a,b]$, then

$$\bigvee_a^b (f') = \sup_{P \in P([a,b])} \sum_{i=0}^{n-1} \left|f'(x_{i+1}) - f'(x_i)\right| \leq L \sup_{P \in P([a,b])} \sum_{i=0}^{n-1} |x_{i+1} - x_i| = L(b-a)$$

and the required result (21) is proved. □

3. Derivation of New Quadrature Rule

Our obtained inequalities have many applications but in this paper, we apply our result only for efficient quadrature rule.

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$ with $h_i := x_{i+1} - x_i$ and $\upsilon(h) := \max\{h_i | \ i = 0, ..., n-1\}$. Then the following Theorem holds:

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$ and $\xi_i \in \left[x_i, \frac{x_i+x_{i+1}}{2}\right]$ $(i = 0, ..., n-1)$. Then we have the quadrature formula:*

$$\begin{aligned}
&\int_a^b f(t)dt \\
=\ & \frac{1}{4}\sum_{i=0}^{n-1}\left[f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f\left(\frac{x_i+\xi_i}{2}\right) + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right)\right]h_i \\
&+\frac{1}{4}\sum_{i=0}^{n-1}\left(\xi_i - \frac{5x_i + 3x_{i+1}}{8}\right)\left\{f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{a+\xi_i}{2}\right)\right\}h_i \\
&+\frac{1}{8}\sum_{i=0}^{n-1}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)\left\{f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{x_i+\xi_i}{2}\right)\right\}h_i + R(I_n, f, \xi).
\end{aligned}$$

The remainder term $R(I_n, f, \xi)$ satisfies

$$\begin{aligned}
&\left|R(I_n, f, \xi)\right| \qquad (22)\\
\le\ & \frac{1}{2}\max_{i\in\{0,...,n-1\}}\left\{\max\left\{\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^2, \frac{1}{4}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right)^2, \frac{(\xi_i - x_i)^2}{4}\right\}\right\}\bigvee_a^b(f').
\end{aligned}$$

Proof. Applying Theorem 2.2 to interval $[x_i, x_{i+1}]$, we have

$$\begin{aligned}
&\Bigg|\int_{x_i}^{x_{i+1}} f(t)dt - \frac{h_i}{4}\Bigg[f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f\left(\frac{x_i+\xi_i}{2}\right) + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) \qquad (23)\\
&+\left(\xi_i - \frac{5x_i + 3x_{i+1}}{8}\right)\left\{f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{a+\xi_i}{2}\right)\right\} \\
&+\frac{1}{2}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)\left\{f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{x_i+\xi_i}{2}\right)\right\}\Bigg]\Bigg| \\
\le\ & \frac{1}{2}\max\left\{\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^2, \frac{1}{4}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right)^2, \frac{(\xi_i - x_i)^2}{4}\right\}\bigvee_{x_i}^{x_{i+1}}(f').
\end{aligned}$$

Summing the inequality (23) over i from 0 to $n-1$, then we have

$$\begin{aligned}
&\left|R(I_n, f, \xi)\right| \\
\le\ & \frac{1}{2}\sum_{i=0}^{n-1}\max\left\{\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^2, \frac{1}{4}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right)^2, \frac{(\xi_i - x_i)^2}{4}\right\}\bigvee_{x_i}^{x_{i+1}}(f') \\
\le\ & \frac{1}{2}\max_{i\in\{0,...,n-1\}}\left\{\max\left\{\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^2, \frac{1}{4}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right)^2, \frac{(\xi_i - x_i)^2}{4}\right\}\right\}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f') \\
\le\ & \frac{1}{2}\max_{i\in\{0,...,n-1\}}\left\{\max\left\{\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^2, \frac{1}{4}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right)^2, \frac{(\xi_i - x_i)^2}{4}\right\}\right\}\bigvee_a^b(f').
\end{aligned}$$

This completes the proof of the Theorem. □

Corollary 3.2. *Under the assumption of Theorem 3.1 with $\xi_i = x_i$, we have*

$$\int_a^b f(t)dt = \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1}) - \frac{f'(x_{i+1}) - f'(x_i)}{4}h_i\right]h_i + R(I_n, f)$$

where remainder term $R(I_n, f)$ satisfies

$$\left|R(I_n, f)\right| \leq \frac{(\nu(h))^2}{8}\bigvee_a^b (f').$$

4. Concluding Remark

In this paper, we presented an improved version of companion of Ostrowski type inequalities for function whose first derivatives are of bounded variation. A further study could be assess similar inequalities by using different types of quadratic kernels.

References

[1] M. W. Alomari, *A Generalization of Weighted Companion of Ostrowski Integral Inequality for Mappings of Bounded Variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.

[3] H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, Turkish Journal of Analysis and Number Theory, 2017, Vol. 5, No. 5, 191–196.

[4] H. Budak and M.Z. Sarıkaya, *New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation* TJMM, 8 (2016), No. 1, 21-27.

[5] H. Budak and M.Z. Sarikaya, *A new generalization of Ostrowski type inequality for mappings of bounded variation*, Lobachevskii Journal of Mathematics, in press.

[6] H. Budak and M.Z. Sarikaya, *On generalization of weighted Ostrowski type inequalities for functions of bounded variation,* Asian-European Journal of Mathematics, in press.

[7] H. Budak and M. Z. Sarikaya, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan Journal of Pure and Applied Analysis, Volume 2(1), 2016, Pages 1–11.

[8] H. Budak and M. Z. Sarikaya, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, Transactions of A. Razmadze Mathematical Institute, 171(2), 2017, 136-143.

[9] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1) (1999), 495–508.

[10] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4 (2001), no. 1, 59–66.

[11] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5 (2014) No. 1, 89–97 pp.

[12] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Research Report Collection, 16 (2013), Art. 93.

[13] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, (2012).

[14] Z. Liu, *Some Ostrowski type inequalities*, Mathematical and Computer Modelling 48 (2008) 949–960.

[15] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226–227.

[16] A. Qayyum, M. Shoaib and I. Faye, *Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications*, Journal of Nonlinear Science and Applications, 9 (2016), 537–552.

[17] A. Qayyum, M. Shoaib and I. Faye, *A companion of Ostrowski type integral inequality using a 5-step kernel with some applications*, Filomat, 30:13 (2016), 36013614.