Filomat 31:16 (2017), 5057–5064
https://doi.org/10.2298/FIL1716057T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Symmetric Difference Between Pseudo B-Fredholm Spectrum and Spectra Originated from Fredholm Theory

Abdelaziz Tajmouati[a], Mohamed Amouch[b], Mohammed Karmouni[a]

[a] *Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, Laboratory of Mathematical Analysis and Applications Fez, Morocco*
[b] *Department of Mathematics University Chouaib Doukkali, Faculty of Sciences, Eljadida. 24000, Eljadida, Morocco*

Abstract. In this paper, we continue the study of the pseudo B-Fredholm operators of Boasso, and the pseudo B-Weyl spectrum of Zariouh and Zguitti; in particular we find that the pseudo B-Weyl spectrum is empty whenever the pseudo B-Fredholm spectrum is, and look at the symmetric differences between the pseudo B-Weyl and other spectra.

1. Introduction and Preliminaries

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. Berkani [6] has defined $T \in \mathcal{B}(X)$ to be a "B-Fredholm operator" if there is an integer $n \geq 0$ for which the range $R(T^n) = T^n(X)$ is closed, while the restriction T_n to $R(T^n)$ is Fredholm in the usual sense, and then T to be "B-Weyl" if also the Fredholm operator T_n has index zero. The "B-Fredholm" and "B-Weyl" spectrum of $T \in \mathcal{B}(X)$ are now defined in the obvious way, as the Fredholm and Weyl spectrum of T_n. Berkani [4] also showed that T is B-Fredholm iff it has a direct sum decomposition $T = T_1 \oplus T_0$ with T_1 Fredholm and T_0 nilpotent; further [5] this decomposition respects the index: T is B-Weyl iff T_1 is Weyl. Boasso [7] has used the decomposition to extend the Berkani concept to "pseudo B" Fredholm and Weyl operators, $T = T_1 \oplus T_0 \in \mathcal{B}(X)$ for which T_1 is Fredholm, respectively Weyl, while T_0 is only quasinilpotent, see also [28]. The pseudo B-Fredholm and pseudo B-Weyl spectrum are defined by

$$\sigma_{pBF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Fredholm}\}.$$

$$\sigma_{pBW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Weyl}\}.$$

For $T \in \mathcal{B}(X)$ we denote by T^*, $R(T)$, $N(T)$, $\sigma(T)$, respectively the adjoint, the range, the null space and the spectrum of T. Recall that $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP for short) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. An operator T is said to

2010 *Mathematics Subject Classification*. Primary 47A53; Secondary 47B10
Keywords. Fredholm operators, pseudo Fredholm operators, pseudo B-Fredholm operator, pseudo B-Weyl spectrum.
Received: 13 September 2016; Accepted: 13 February 2017
Communicated by Dragan S. Djordjević
Email addresses: abdelaziz.tajmouati@usmba.ac.ma (Abdelaziz Tajmouati), mohamed.amouch@gmail.com (Mohamed Amouch), mohammed.karmouni@usmba.ac.ma (Mohammed Karmouni)

have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, hence T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum.

An operator $T \in \mathcal{B}(X)$ is said to be semi-regular, if $R(T)$ is closed and $N(T) \subseteq R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$. The corresponding spectrum is the semi-regular spectrum $\sigma_{se}(T)$ defined by $\sigma_{se}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not semi-regular $\}$, see [1].

In the other hand, recall that an operator $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, (GKD for short), if there exists two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T_1 = T\ |_{X_1}$ is semi-regular and $T_0 = T\ |_{X_2}$ is quasi-nilpotent, in this case T is said a pseudo Fredholm operator. If we assume in the definition above that $T_0 = T\ |_{X_2}$ is nilpotent, then T is said to be of Kato type. Clearly, every semi-regular operator is of Kato type and a quasi-nilpotent operator has a GKD, see [18, 21] for more information about generalized Kato decomposition.

Recall that $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that

1. $R(T^n) \cap N(T) = R(T^d) \cap N(T)$ for all $n \geq d$;
2. $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are closed in X.

An operator is quasi-Fredholm if it is quasi-Fredholm of some degree d. Note that semi-regular operators are quasi-Fredholm of degree 0 and by results of Labrousse [18], in the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators. For every bounded operator $T \in \mathcal{B}(X)$, let us define the generalized Kato spectrum as follows :

$$\sigma_{GK}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato decomposition}\}.$$

It is know that $\sigma_{GK}(T)$ is always a compact subsets of the complex plane contained in the spectrum $\sigma(T)$ of T [13, Corollary 2.3]. Note that $\sigma_{GK}(T)$ is not necessarily non-empty, see [13, 14] for more information about $\sigma_{GK}(T)$.

In [4], Berkani proved that every B-Fredholm operator in Hilbert space is a quasi-Fredholm operator. The proof is based on the decomposition of quasi-Fredholm operators of Labrousse [18] which was proved only for Hilbert-spaces operators. This gap was subsequently filled by Müller in [26, Theorem 7] and the result holds in more general setting of Banach space.

As a continuation of [7] and [28], in section two, we prove that every pseudo B-Fredholm operator is a pseudo-Fredholm operator. Also, we study the relationships between the class of pseudo B-Fredholm and other class of operator. we characterize when the pseudo B-Fredholm spectrum is empty or most countable. In section tree, we study the components of the complement of the pseudo B-Fredholm spectrum $\sigma_{pBF}(T)$, to obtain a classification of the components by using the constancy of the subspaces quasi-nilpotent part and analytic core. In the last section, we show that the symmetric difference $\sigma_{se}(T)\Delta\sigma_{pBF}(T)$ is at most countable.

2. The Class of Pseudo B-Fredholm Operators

In the following theorem we prove that every pseudo B-Fredholm operator is pseudo Fredholm.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$. If T is pseudo B-Fredholm, then T is pseudo Fredholm.*

Proof. Let $T \in \mathcal{B}(X)$. If T is pseudo B-Fredholm operator, then there exists two closed subsets M and N of X such that $X = M \oplus N$ and $T = T_1 \oplus T_2$ with $T_1 = T_{|M}$ is a Fredholm operator and $T_2 = T_{|N}$ is quasi-nilpotent. Since T_1 is Fredholm then T admits a Kato decomposition, hence there exists M', M'' two closed subsets of M such that $M = M' \oplus M''$, $T_1 = T_1' \oplus T_1''$ with $T_1' = T_{1|M'}$ is a semi-regular operator and $T_1'' = T'_{1|M''}$ is nilpotent. Then $X = M' \oplus M'' \oplus N$, and $T = S \oplus R$ where $S = T_1'$ is a semi-regular operator and $R = T_1'' \oplus T_2$ is a quasi-nilpotent operator, hence T is a pseudo Fredholm operator. $\square$

The following example (Müller [25]) shows that the pseudo B-Fredholm operators form a proper subclass of the pseudo Fredholm operators.

Example 2.2. *Let H be the Hilbert space with an orthonormal basis* $(e_{i,j})$, *where i and j are integers such that* $ij \leq 0$. *Define operator* $T \in \mathcal{B}(H)$ *by :*

$$Te_{i,j} = \begin{cases} 0 & \text{if } i = 0, j > 0 \\ e_{i+1,j} & \text{Otherewise} \end{cases}$$

We have $N(T) = \bigvee_{j>0}\{e_{0,j}\} \subset R^{\infty}(T)$ *and* $R(T)$ *is closed, then T is a semi-regular operator but T is not a Fredholm operator, since* $dimN(T) = \infty$.
Let Q a quasinilpotent operator in H which is not nilpotent and no commute with T, then $S = T \oplus Q$ *is a pseudo Fredholm operator but is not pseudo B-Fredholm operator, hence the class of pseudo B-Fredholm operator is a proper subclass of pseudo Fredholm operator.*

Remark 2.3. *In [28, Remark 2.5] and [8, Proposition 1.2], If T is a bilateral shift on* $l^2(\mathbb{N})$, *we have :*

1. *T is pseudo B-Weyl if and only if T is Weyl or T is quasi-nilpotent operator.*
2. *T is pseudo Fredholm if and only if T is semi-regular or T is quasi-nilpotent operator.*

By the same argument we can prove :

1. *T is pseudo B-Fredholm if and only if T is Fredholm or T is quasi-nilpotent operator.*
2. *T is generalized Drazin if and only if T is invertible or T is quasi-nilpotent operator.*

Corollary 2.4. *Let* $T \in \mathcal{B}(X)$. *Then*

$$\sigma_{GK}(T) \subset \sigma_{pBF}(T) \subset \sigma_{pBW}(T)$$

Lemma 2.5. *[22] Let* $T \in \mathcal{B}(X)$ *and let G a connected component of* $\rho_{se}(T) = \mathbb{C}\backslash\sigma_{se}(T)$. *Then*

$$G \setminus \sigma(T) \neq \emptyset \Longrightarrow G \cap \sigma(T) = \emptyset$$

Lemma 2.6. *[8] Let* $T \in \mathcal{B}(X)$.
$\sigma_{se}(T) \setminus \sigma_{GK}(T)$ *is at most countable*

Since $\sigma_{se}(T) \setminus \sigma_{pBF}(T) \subset \sigma_{se}(T)\backslash\sigma_{GK}(T)$, we can easily obtain that:

Corollary 2.7. *Let* $T \in \mathcal{B}(X)$.
$\sigma_{se}(T) \setminus \sigma_{pBF}(T)$ *is at most countable.*

Proposition 2.8. *Let* $T \in \mathcal{B}(X)$. *Then the following statements are equivalent :*

1. $\sigma_{pBF}(T)$ *is at most countable*
2. $\sigma_{pBW}(T)$ *is at most countable*
3. $\sigma(T)$ *is at most countable*

Proof. 1) $\Longrightarrow$ 3) Suppose that $\sigma_{pBF}(T)$ is at most countable then $\rho_{pBF}(T)$ is connexe, by corollary 2.7 $\rho_{pBF}(T) \setminus \rho_{se}(T)$ is at most countable. Hence $\rho_{se}(T) \cap \rho_{pBF}(T) = \rho_{pBF}(T) \setminus (\rho_{pBF}(T) \setminus \rho_{se}(T))$ is connexe. By lemma 2.5 $\sigma(T) = \sigma_{se}(T) \cup \sigma_{pBF}(T)$. Therefore $\sigma(T) = \sigma_{pBF}(T) \cup (\rho_{pBF}(T) \setminus \rho_{se}(T))$ is at most countable.
3) $\Longrightarrow$ 1) Obvious.
2) $\Longrightarrow$ 3) If $\sigma_{pBW}(T)$ is at most countable then $\rho_{pBW}(T)$ is connexe, since every pseudo B-Weyl operator is a pseudo B-Fredholm operator by corollary 2.7 $\rho_{pBW}(T) \setminus \rho_{se}(T)$ is at most countable. Hence $\rho_{se}(T) \cap \rho_{pBW}(T) = \rho_{pBW}(T) \setminus (\rho_{pBW}(T) \setminus \rho_{se}(T))$ is connexe. By lemma 2.5 $\sigma(T) = \sigma_{se}(T) \cup \sigma_{pBW}(T)$. Therefore $\sigma(T) = \sigma_{pBW}(T) \cup (\rho_{pBW}(T) \setminus \rho_{se}(T))$ is at most countable.
3) $\Longrightarrow$ 2) Obvious. □

Corollary 2.9. *Let* $T \in \mathcal{B}(X)$, *if* $\sigma_{GK}(T)$ *is at most countable. Then:*
T is a spectral operator if and only if T is similar to a paranormal operator.

Proof. See [23, Theoerem 2.4 and Corollary 2.5] $\square$

Let $T \in \mathcal{B}(X)$. The operator range topology on $R(T)$ is the topology induced by the norm $\|.\|_T$ defined by $\|y\|_T := \inf_{x \in X}\{\|x\| : y = Tx\}$. For a detailed discussion of operator ranges and their topology we refer the reader to [11].
T is said to have uniform descent for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for $n \geq d$. If in addition, $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then T is said to have topological uniform descent (TUD for brevity) for $n \geq d$. The topological uniform descent spectrum :

$$\sigma_{ud}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \text{ does not have TUD}\}$$

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exists we let $a(T) = \infty$. Analogously the descent of T is $d(T) = min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exists we let $d(T) = \infty$ [20]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$. The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by :

$$\sigma_{des}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ has not finite descent}\}$$

$$\sigma_{ac}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ has not finite ascent }\}.$$

On the other hand, a bounded operator $T \in \mathcal{B}(X)$ is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS,\ \ T^{k+1}S = T^k \ \ and\ S^2T = S.$$

This is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is nilpotent. Recall that an operator T is Drazin invertible if it has a finite ascent and descent. The concept of Drazin invertible operators has been generalized by Koliha [17]. In fact $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc\sigma(T)$ the set of all points of accumulation of $\sigma(T)$, which is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasinilpotent. The Drazin and generalized Drazin spectra of $T \in \mathcal{B}(X)$ are defined by :

$$\sigma_D(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not Drazin invertible}\}$$

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin}\}$$

We denote by, $\sigma^e_{des}(T)$, $\sigma_{rGD}(T)$ and $\sigma_{lGD}(T)$ respectively the essential descent, right generalized Drazin and left generalized Drazin spectra of T. According to corollary 2.8, [8, Theorem 3.3] and [16, corollary 3.4], we have the following:

Corollary 2.10. *Let $T \in \mathcal{B}(X)$. Then the following statements are equivalent*

1. $\sigma(T)$ *is at most countable;*
2. $\sigma_{pBF}(T)$ *is at most countable;*
3. $\sigma_{pBW}(T)$ *is at most countable;*
4. $\sigma_{ud}(T)$ *is at most countable;*
5. $\sigma_{GK}(T)$ *is at most countable;*
6. $\sigma_{GD}(T)$ *is at most countable;*
7. $\sigma_{lGD}(T)$ *is at most countable;*
8. $\sigma_{rGD}(T)$ *is at most countable;*
9. $\sigma_D(T)$ *is at most countable;*
10. $\sigma_{se}(T)$ *is at most countable;*
11. $\sigma_{BF}(T)$ *is at most countable;*
12. $\sigma_{BW}(T)$ *is at most countable;*
13. $\sigma_{des}(T)$ *is at most countable;*
14. $\sigma^e_{des}(T)$ *is at most countable;*

In [15], the showed that an operator with TUD for $n \geq d$, $K(T) = R^\infty(T)$ and $\overline{H_0(V)} = \overline{N^\infty(V)}$, where $K(T)$ and $H_0(T)$ are the analytic core and the quasinilpotent part of T. For a pseudo B-Fredholm operator, these properties do not necessarily hold. Indeed: let X be the Banach space of continuous functions on $[0, 1]$, denoted by $C([0, 1])$, provided with the infinity norm. We define by V, the Volterra operator, X by :

$$Vf(x) := \int_0^x f(x)\,\mathrm{d}x.$$

V is injective and quasi-nilpotent. In addition, $N^\infty(V) = \{0\}$, $K(V) = \{0\}$ and we have $R^\infty(V) = \{f \in C^\infty[0, 1] : f^{(n)}(0) = 0,\ n \in \mathbb{N}\}$, thus $R^\infty(V)$ is not closed. Hence:

1. $K(V) \neq R^\infty(V)$
2. $\overline{H_0(V)} \neq \overline{N^\infty(V)}$

Note that V is a compact operator, then $R(V)$ is not closed.

Theorem 2.11. *There exists a pseudo B-Fredholm operator T such that :*

1. $K(T) \neq R^\infty(T)$,
2. $\overline{H_0(T)} \neq \overline{N^\infty(T)}$,
3. $R(T)$ *is not closed.*

Proposition 2.12. *Let $T \in \mathcal{B}(X)$. Then the following statements are equivalent*

1. $\sigma_{pBF}(T)$ *is empty*
2. $\sigma_{pBW}(T)$ *is empty*
3. $\sigma_{GK}(T)$ *is empty*
4. $\sigma(T)$ *is finite*

Proof. 3) $\Longleftrightarrow$ 4) see [8, Theorem 3.3].
1) $\Longrightarrow$ 4) If $\sigma_{pBF}(T)$ is empty then $\sigma(T) = \rho_{pBF}(T) \setminus \rho_{se}(T)$. By corollary 2.7 $\rho_{pBF}(T) \setminus \rho_{se}(T)$ is at most countable and this set is bounded, hence it is finite.
4) $\Longrightarrow$ 1) Suppose that $\sigma(T)$ is finite then every $\lambda_0 \in \sigma(T)$ is isolated, then $X = H_0(T - \lambda_0) \oplus K(T - \lambda_0)$, [27, Theorem 4] $(T - \lambda_0)_{|H_0(T-\lambda_0)}$ is quasi-nilpotent and $(T - \lambda_0)_{|K(T-\lambda_0)}$ is surjective, hence $(T - \lambda_0)_{|K(T-\lambda_0)}$ is Fredholm. Indeed, λ_0 is an isolated point, then T has the SVEP at λ_0, hence $(T - \lambda_0)_{|K(T-\lambda_0)}$ has the SVEP at 0 and it is surjective by [1, corollary 2.24] $(T - \lambda_0)_{|K(T-\lambda_0)}$ is bijective. Thus every $\lambda_0 \in \sigma(T)$, $T - \lambda_0 I$ is pseudo B-Fredholm, so $\sigma_{pBF}(T)$ is empty.
2) $\Longleftrightarrow$ 4) similar to 1) $\Longleftrightarrow$ 4). $\square$

A bounded operator $T \in \mathcal{B}(X)$ is said to be a Riesz operator if $T - \lambda I$ is a Fredholm operator for every $\lambda \in \mathbb{C}\setminus\{0\}$.

Corollary 2.13. *Let $T \in \mathcal{B}(X)$ a Riesz operator, then the following statements are equivalent*

1. $\sigma_{pBF}(T)$ *is empty,*
2. $\sigma_{pBW}(T)$ *is empty,*
3. $\sigma_{GK}(T)$ *is empty,*
4. $\sigma(T)$ *is finite,*
5. $K(T)$ *is closed,*
6. $K(T^*)$ *is closed,*
7. $K(T)$ *is finite-dimensional,*
8. $K(T - \lambda)$ *is closed for all* $\lambda \in \mathbb{C}$,
9. $codim H_0(T) < \infty$,
10. $codim H_0(T^*) < \infty$,
11. $T = Q + F$, *with* $Q, F \in \mathcal{B}(X)$, $QF = FQ = 0$, $\sigma(Q) = \{0\}$ *and F is a finite rank operator.*

Proof. Direct consequence of Proposition 2.12 and [9, Theorem 2.3] and [24, Corollary 9] □

In the following, we will prove that if T is with finite descent, then T is pseudo B-Fredholm if and only if T is a B-Fredholm operator.

Proposition 2.14. *Let $T \in \mathcal{B}(X)$ with finite descent. Then T is a pseudo B-Fredholm if and only if T is a B-Fredholm.*

Proof. Obviously if T is B-Fredholm then T is pseudo B-Fredholm.
If T is a pseudo B-Fredholm then $T = T_1 \oplus T_2$ with T_1 is Fredholm operator and T_2 is quasinilpotent. Since T has finite descent then T_1 and T_2 have finite descent. We have T_2 is quasinilpotent with finite descent implies that is a nilpotent operator. Thus T is a B-Fredholm operator. □

3. Classification of the Components of Pseudo B-Fredholm Resolvent

We begin this section by the following lemmas which will be needed in the sequel.

Lemma 3.1. *Let $T \in \mathcal{B}(X)$ a pseudo B-Fredholm, then there exists $\varepsilon > 0$ such that for all $|\lambda| < \varepsilon$, we have:*

1. $K(T-\lambda) + H_0(T-\lambda) = K(T) + H_0(T)$.
2. $K(T-\lambda) \cap \overline{H_0(T-\lambda)} = K(T) \cap \overline{H_0(T)}$.

Proof. By Theorem 2.1, T is a pseudo Fredholm operator, hence we conclude by [8, Theorem 4.2] the result. □

The pseudo B-Fredholm resolvent set is defined as $\rho_{pBF}(T) = \mathbb{C}\backslash\sigma_{pBF}(T)$.

Corollary 3.2. *Let $T \in \mathcal{B}(X)$ a pseudo B-Fredholm operator, then the mappings*

$\lambda \longrightarrow K(T-\lambda) + H_0(T-\lambda)$, $\lambda \longrightarrow K(T-\lambda) \cap \overline{H_0(T-\lambda)}$ *are constant on the components of* $\rho_{pBF}(T)$.

We denote by $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the approximate point spectrum and the surjectivity spectrum of T.

Lemma 3.3. *Let $T \in \mathcal{B}(X)$ a pseudo B-Fredholm operator. Then the following statements are equivalent:*

1. *T has the SVEP at 0,*
2. *$\sigma_{ap}(T)$ does not cluster at 0.*

Proof. Without loss of generality, we can assume that $\lambda_0 = 0$.
2) ⇒ 1) See [1].
1) ⇒ 2) Suppose that T is a pseudo B-Fredholm operator, then there exists two closed T-invariant subspaces X_1 , $X_2 \subset X$ such that $X = X_1 \oplus X_2$, $T_{|X_1}$ is Fredholm, $T_{|X_2}$ is quasi-nilpotent and $T = T_{|X_1} \oplus T_{|X_2}$. Since $T_{|X_1}$ is Fredholm, then $T_{|X_1}$ is of Kato type by [2, Theorem 2.2] there exists a constant $\varepsilon > 0$ such that for all $\lambda \in D^*(0,\varepsilon)$, $\lambda I - T$ is bounded below. Since $T_{|X_2}$ is quasi-nilpotent, $\lambda I - T$ is bounded below for all $\lambda \neq 0$. Hence $\lambda I - T$ is bounded below for all $\lambda \in D^*(0,\varepsilon)$. Therefore $\sigma_{ap}(T)$ does not cluster at λ_0. □

By duality we have :

Lemma 3.4. *Let $T \in \mathcal{B}(X)$ a pseudo B-Fredholm operator. Then the following statements are equivalent:*

1. *T^* has the SVEP at 0,*
2. *$\sigma_{su}(T)$ does not cluster at 0.*

Theorem 3.5. *Let $T \in \mathcal{B}(X)$ and Ω a component of $\rho_{pBF}(T)$. Then the following alternative holds:*

1. *T has the SVEP for every point of Ω. In this case, $\sigma_{ap}(T)$ does not have limit points in Ω, every point of Ω is not an eigenvalue of T execpt a subset of Ω which consists of at most countably many isolated points.*
2. *T has the SVEP at no point of Ω. In this case, every point of Ω is an eigenvalue of T.*

Proof. 1) Assume that T has the SVEP at $\lambda_0 \in \Omega$. By [1, Theorem 3.14] we have $K(T-\lambda_0) \cap H_0(T-\lambda_0) = K(T-\lambda_0) \cap \overline{H_0(T-\lambda_0)} = \{0\}$. According to corollary 3.2, we have $K(T-\lambda_0) \cap \overline{H_0(T-\lambda_0)} = K(T-\lambda) \cap \overline{H_0(T-\lambda)} = \{0\}$ for all $\lambda \in \Omega$. Hence $K(T-\lambda) \cap \overline{H_0(T-\lambda)} = \{0\}$ and therefore T has the SVEP at every $\lambda \in \Omega$ [1, Theorem 3.14]. By Lemma 3.3, $\sigma_{ap}(T)$ does not cluster at any $\lambda \in \Omega$. Consequently every point of Ω is not an eigenvalue of T execpt a subset of Ω which consists of at most countably many isolated points.
2) Suppose that T has the SVEP at not point of Ω. From [1, Theorem 2.22], we have $N(T-\lambda) \neq \{0\}$, for all $\lambda \in \Omega$, hence every point of Ω is an eigenvalue of T. □

Theorem 3.6. *Let $T \in \mathcal{B}(X)$ and Ω a component of $\rho_{pBF}(T)$. Then the following alternative holds:*

1. *T^* has the SVEP for every point of Ω. In this case, $\sigma_{su}(T)$ does not have limit points in Ω, every point of Ω is not a deficiency value of T execpt a subset of Ω which consists of at most countably many isolated points.*
2. *T^* has the SVEP at no point of Ω. In this case, every point of Ω is a deficiency value of T.*

Proof. 1) Assume that T^* has the SVEP at $\lambda_0 \in \Omega$, by [1, Theorem 3.15] we have $K(T-\lambda_0) + H_0(T-\lambda_0) = X$. According to corollary 3.2, we have $K(T-\lambda_0) + H_0(T-\lambda_0) = K(T-\lambda) + H_0(T-\lambda) = X$ for all $\lambda \in \Omega$. Hence $K(T-\lambda) + H_0(T-\lambda) = X$ and therefore T has the SVEP at every $\lambda \in \Omega$ [1, Theorem 3.15]. By lemma 3.4, $\sigma_{su}(T)$ does not cluster at any $\lambda \in \Omega$. Consequently every point of Ω is not a deficiency value of T execpt a subset of Ω which consists of at most countably many isolated points.
2) Suppose that T^* has the SVEP at no point of Ω. Assume that there exists a $\lambda_0 \in \Omega$ such that $T-\lambda_0$ is surjective, then $T^*-\lambda_0$ is injective this implies that T^* has the SVEP at λ_0. Contraduction and hence every point of Ω is a deficiency value of T. □

Remark 3.7. *We have $\sigma_{pBF}(.) \subset \sigma_{gD}(.)$, this inclusion is proper. Indeed: Consider the operator T defined in $l^2(\mathbb{N})$ by*

$$T(x_1, x_2, ...) = (0, x_1, x_2, ...), \quad T^*(x_1, x_2, ...) = (x_2, x_3, ...).$$

Let $S = T \oplus T^*$. Then $\sigma_{gD}(S) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$ and we have $0 \notin \sigma_{pBF}(S)$. This shows that the inclusion $\sigma_{pBF}(S) \subset \sigma_{gD}(S)$ is proper.

Next we obtain a condition on an operator such that its pseudo B-Fredholm spectrum coincide with the generalized Drazin spectrum.

Theorem 3.8. *Suppose that $T \in \mathcal{B}(X)$ and $\rho_{pBF}(T)$ has only one component. Then*

$$\sigma_{pBF}(T) = \sigma_{gD}(T)$$

Proof. $\rho_{pBF}(T)$ has only one component, then $\rho_{pBF}(T)$ is the unique component. Since T has the SVEP on $\rho(T) \subset \rho_{pBF}(T)$. By Theorem 3.5, T has the SVEP on $\rho_{pBF}(T)$. Similar T^* also has the SVEP on $\rho_{pBF}(T)$ by Theorem 3.6. (This since $\rho(T^*) = \rho(T) \subset \rho_{pBF}(T)$). From Lemma 3.3 and Lemma 3.4, $\sigma(T)$ does not cluster at any $\lambda \in \rho_{pBF}(T)$. Therefore $\rho_{pBF}(T) \subset iso\sigma(T) \cup \rho(T) = \rho_{gD}(T)$, hence $\rho_{pBF}(T) = \rho_{gD}(T)$. □

4. Symmetric Difference for Pseudo B-Fredholm Spectrum

Let in the following we give symmetric difference between $\sigma_{pBF}(T)$ and other parts of the spectrum. Denoted by $\rho_{fK}(T) = \{\lambda \in \mathbb{C}, K(T-\lambda)$ is not closed $\}$, $\sigma_{fK}(T) = \mathbb{C} \setminus \rho_{fK}(T)$ and $\rho_{cr}(T) = \{\lambda \in \mathbb{C}, R(T-\lambda)$ is closed $\}$, $\sigma_{cr}(T) = \mathbb{C} \setminus \rho_{cr}(T)$ the Goldberg spectrum. Most of the classes of operators, for example, in Fredholm theory, require that the operators have closed ranges. Thus, it is natural to consider the closed-range spectrum or Goldberg spectrum of an operator.

Proposition 4.1. *If $\lambda \in \sigma_*(T)$ is non-isolated point then $\lambda \in \sigma_{pBF}(T)$, where $* \in \{fK, cr\}$.*

Proof. Let $\lambda \in \sigma_*(T)$ an isolated point. Suppose that $T-\lambda$ is a pseudo B-Fredholm, by Lemma 2.6 there exists a constant $\varepsilon > 0$ such that for all $\mu \in D^*(\lambda, \varepsilon)$, $\mu - T$ is semi-regular. Then $R(T-\mu)$ and $K(T-\mu)$ are closed for all $\mu \in D^*(\lambda, \varepsilon)$, then λ is an isolated point of $\sigma_*(T)$, contradiction. □

Corollary 4.2. *$\sigma_*(T)\backslash\sigma_{pBF}(T)$ is at most countable, where $* \in \{fK, cr\}$.*

Proposition 4.3. *Let $T \in \mathcal{B}(X)$ such that $\sigma_{cr}(T) = \sigma(T)$ and every λ is non-isolated in $\sigma(T)$. Then*

$$\sigma(T) = \sigma_{cr}(T) = \sigma_{pBF}(T) = \sigma_{pBW}(T) = \sigma_e(T) = \sigma_{se}(T) = \sigma_{ap}(T)$$

Proof. Since every $\lambda \in \sigma(T) = \sigma_{cr}(T)$ is non-isolated then by Proposition 4.1, we have $\sigma(T) = \sigma_{cr}(T) \subseteq \sigma_{pBF}(T) \subseteq \sigma_{pBW}(T) \subseteq \sigma_e(T) \subseteq \sigma(T)$ and since $\sigma(T) = \sigma_{cr}(T) \subseteq \sigma_{se}(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$, we deduce the statement of the theorem. □

Proposition 4.4. *Let $T \in \mathcal{B}(X)$. The symmetric difference $\sigma_{se}(T)\Delta\sigma_{pBF}(T)$ is at most countable.*

Proof. By corollary 2.7, $\sigma_{se}(T)\backslash\sigma_{pBF}(T)$ is at most countable. We have $\sigma_e(T)\backslash\sigma_{se}(T)$ consists of at most countably many isolated points (see [1, Theorem 1.65] and $\sigma_{pBF}(T)\backslash\sigma_{se}(T) \subseteq \sigma_e(T)\backslash\sigma_{se}(T)$, hence $\sigma_{pBF}(T)\backslash\sigma_{se}(T)$ is at most countable. Since

$$\sigma_{se}(T)\Delta\sigma_{pBF}(T) = (\sigma_{se}(T)\backslash\sigma_{pBF}(T)) \bigcup (\sigma_{pBF}(T)\backslash\sigma_{se}(T))$$

Therefore $\sigma_{se}(T)\Delta\sigma_{pBF}(T)$ is at most countable. □

Acknowledgement The authors thank the referees for his suggestions, remarks and comments thorough reading of the manuscript.

References

[1] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers,* Kluwer. Acad. Press, 2004.
[2] P. Aiena, E. Rosas, *Single-valued extension property at the points of the approximate point spectrum,* J. Math. Anal. Appl. (2003) 279 (1) 180-188.
[3] E. Albrecht, *A characterization of spectral operators on Hilbert Spaces,* Glasgow Math. J. 23 (1982), 91-95.
[4] M. Berkani, *On a class of quasi-Fredholm operators,* Integral Equations Operator Theory 34 (1999) 244-249.
[5] M. Berkani, *B-Weyl spectrum and poles of the resolvent,* J. Math. Anal. Appl. (2002) 272 (2), 596-603.
[6] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem,* Proc. Amer. Math. Soc. 130 (2002), 1717-1723.
[7] E. Boasso, *Isolated spectral points and Koliha-Drazin invertible elements in quotient Banach algebras and homomorphism ranges,* Math. Proc. Royal Irish Acad. Vol. 115A, No. 2 (2015), pp. 1-15.
[8] W. Bouamama, *Opérateurs Pseudo Fredholm dans les espaces de Banach,* Rend. Circ. Mat. Parelmo, (2004) 313-324.
[9] W. Bouamama, *Opérateurs de Riesz dont le coeur analytique est fermé,* Studia Mathematica 162 (1) (2004) 15-23.
[10] S. V. Djordjević, G. Kantún-Montiel, *On generalised T-Fredholm elements in Banach algebras,* Math. Proc. Royal Irish Acad. 109A (2009) 61-66.
[11] S. Grabiner, *Ranges of products of operators,* Canad. J. Math. 26, (1974), 1430-1441.
[12] S. Grabiner, *Uniform ascent and descent of bounded operators,* J. Math. Soc. Japan 34, 2, (1982), 317-337.
[13] Q. Jiang, H. Zhong, *Generalized Kato decomposition, single-valued extension property and approximate point spectrum*, J. Math. Anal. Appl. 356 (2009), 322-327.
[14] Q. Jiang and H. Zhong, *Components of generalized Kato resolvent set and single-valued extension property,* Frontiers of Mathematics in China August (2012), Vol 7, Issue 4, pp 695-702.
[15] Q. Jiang and H. Zhong, *Topological uniform descent and localized SVEP,* J. Math. Anal. Appl. 390 (2012) 355-361.
[16] Q. Jiang, H. Zhong, S. Zhang, *Components topological uniform descent resolvent set and local spectral theory,* linear algebra and it applications 438(2013), 1149-1158.
[17] J. J. Koliha, *A generalized Drazin inverse,* Glasgow Math. J. (1996), 38:367-81.
[18] J. P. Labrousse, *Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm,* Rend. Circ. Math. Palermo (2), XXIX (1980) 161-258.
[19] K. B. Laursen, M. M. Neumann, *An introduction to Local Spectral Theory,* London Mathematical Society Monograph, New series, Vol. 20, Clarendon Press, Oxford, 2000.
[20] D. Lay, A. Taylor, *Introduction to functional analysis,* J. Wiley and Sons, N. York. 1980.
[21] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux,* Glasgow Math J, 29 (1987) 159-175.
[22] M. Mbekhta, A. Ouahab, *Opérateurs s-régulier dans un espace de Banach et Théorie spectrale,* Acta Sci, Math (Szeged). 59, (1994) 525-543.
[23] M. Mbekhta, *Ascente, descente et spéctre essentiel quasi-Fredholm,* Rend. Circ. Mat. Palermo 46 (1997), 175-196.
[24] T. L. Miller, V. G. Miller, M. Neumann, *On Operators with Closed Analytic core,* Ren del Circ. Math. Di Palermo. S2. (2002), 495-502.
[25] V. Müller, *On the regular spectrum,* J. Oper. Theory, 31(1994) 363-380.
[26] V. Müller, *On the Kato-decomposition of quasi-Fredholm and B-Fredholm operators,* Vienna, Preprint ESI 1013 (2001).
[27] C. Schmoeger, *On isolated points of the spectrum of a bounded linear operator,* Proc. Amer. Math. Soc. 117 (1993), 715-719.
[28] H. Zariouh, H. Zguitti, *On pseudo B-Weyl operators and generalized drazin invertible for operator matrices,* Linear and Multilinear Algebra, Vol 64, Issue 7, (2016), 1245-1257.