Filomat 31:16 (2017), 5065–5071
https://doi.org/10.2298/FIL1716065A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Warped Product Bi-Slant Submanifolds of Cosymplectic Manifolds

Lamia Saeed Alqahtani[a], Mića S. Stanković[b], Siraj Uddin[a]

[a]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[b]*Department of Mathematics, Faculty of Sciences and Mathematics, University of Nis, Serbia*

Abstract. In this paper, we study warped product bi-slant submanifolds of cosymplectic manifolds. It is shown that there is no proper warped product bi-slant submanifold other than pseudo-slant warped product. Finally, we give an example of warped product pseudo-slant submanifolds.

1. Introduction

In [6], Cabrerizo et al. introduced the notion of bi-slant submanifolds of almost contact metric manifolds as a generalization of contact CR-submanifolds, slant and semi-slant submanifolds. They have obtained non-trivial examples of such submanifolds. One of the class of such submanifolds is that of pseudo-slant submanifolds [8]. We note that the pseudo-slant submanifolds are also studied under the name of hemi-slant submanifolds [19].

Warped product submanifolds have been studied rapidly and actively, since Chen introduced the notion of CR-warped products of Kaehler manifolds [10, 11]. Different types of warped product submanifolds have been studied in several kinds of structures for last fifteen years (see [2, 15, 18, 20, 22]). The related studies on this topic can be found in Chen's book and a survey article [12, 13].

Recently, warped product submanifolds of cosymplectic manifolds were studied in ([1],[15], [20–22]). In this paper, we study warped product bi-slant submanifolds of cosymplectic manifolds. We prove the non-existence of proper warped product bi-slant submanifolds of a cosymplectic manifold. Finally, we give an example of special class of warped product bi-slant submanifolds known as warped product pseudo-slant submanifolds studied in [23].

2. Preliminaries

Let $(\widetilde{M}, g)$ be an odd dimensional Riemannian manifold with a tensor field φ of type $(1,1)$, a global vector field ξ (*structure vector field* and a dual 1-form η of ξ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1}$$

2010 *Mathematics Subject Classification*. 53C15; 53C40; 53C42; 53B25

Keywords. Warped products, bi-slant submanifolds, warped product bi-slant submanifolds, pseudo-slant warped product, cosymplectic manifolds

Received: 23 September 2016; Accepted: 09 December 2016

Communicated by Dragan S. Djordjević

The research was supported by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. G-688-247-37. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Email addresses: lalqahtani@kau.edu.sa (Lamia Saeed Alqahtani), stmica@ptt.rs (Mića S. Stanković), siraj.ch@gmail.com (Siraj Uddin)

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{M}$ is called an *almost contact metric manifold* [4], where $\Gamma(T\widetilde{M})$ denotes the set all vector fields of $\widetilde{M}$ and I being the identity transformation on $T\widetilde{M}$. As a consequence, the dimension of $\widetilde{M}$ is odd $(= 2m + 1)$, $\varphi(\xi) = 0 = \eta \circ \varphi$ and $\eta(X) = g(X, \xi)$. The fundamental 2-form Φ of $\widetilde{M}$ is defined $\Phi(X, Y) = g(X, \varphi Y)$. An almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ is said to be *cosymplectic* if $[\varphi, \varphi] = 0$ and $d\eta = 0$, $d\Phi = 0$, where $[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$ and d is an exterior differential operator.

Let $\widetilde{\nabla}$ denotes the Levi-Civita connedtion on $\widetilde{M}$ with respect to the Riemannian metric g. Then in terms of the covariant derivative of φ, the cosymplectic structure is characterized by the relation $(\widetilde{\nabla}_X \varphi)Y = 0$, for any $X, Y \in \Gamma(T\widetilde{M})$ [16]. From the formula $(\widetilde{\nabla}_X \varphi)Y = 0$, it follows that $\widetilde{\nabla}_X \xi = 0$.

Let M be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M. Let ∇ be the Levi-Civita connection on M, then the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{3}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_N is the shape operator of M with respect to N. Moreover, $h : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\widetilde{M}$. Furthermore, A_N and h are related by

$$g(h(X, Y), N) = g(A_N X, Y) \tag{4}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

For any X tanget to M, we write

$$\varphi X = TX + FX, \tag{5}$$

where TX and FX are the tangential and normal components of φX, respectively. Then T is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$\varphi N = BN + CN, \tag{6}$$

where BN and CN are the tangential and normal components of φN, respectively. Moreover, from (1) and (5), we have

$$g(TX, Y) = -g(X, TY), \tag{7}$$

for any $X, Y \in \Gamma(TM)$.

A sumanifold M is said to be *φ-invariant* if F is identically zero, i.e., $\varphi X \in \Gamma(TM)$, for any $X \in \Gamma(TM)$. On the other hand, M is said to be *φ-anti-invariant* if T is identically zero i.e., $\varphi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in an almost contact metric manifold were considered. Throughout the paper we consider the structure vector field ξ is tangent to the submanifold otherwise it is a C-totally real submanifold.

(1) A submanifold M of an almost contact metric manifold $\widetilde{M}$ is called a *contact CR-submanifold* [1] of $\widetilde{M}$ if there exist a differentiable distribution $\mathcal{D} : p \to \mathcal{D}_p \subset T_pM$ such that $\mathcal{D}$ is invariant with respect to φ, i.e., $\varphi(\mathcal{D}) = \mathcal{D}$ and the complementary distribution $\mathcal{D}^\perp$ is anti-invariant with respect to φ, i.e., $\varphi(\mathcal{D}^\perp) \subset T^\perp M$ and TM has the orthogonal decomposition $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is a 1-dimensional distribution which is spanned by ξ.

(2) A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be slant [7], if for each non-zero vector X tangent to M such that X is not proportional to $\langle\xi\rangle$, the angle $\theta(X)$ between φX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle\xi_p\rangle$. A slant submanifold is said to be *proper slant* if $\theta \neq 0$ and $\neq \frac{\pi}{2}$.

(3) A submanifold M of an almost contact metric manifold $\widetilde{M}$ is called semi-slant [6], if it is endowed with two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$, such that $TM = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$ where $\mathcal{D}$ is invariant with respect to φ and $\mathcal{D}^\theta$ is proper slant, i.e., $\theta(X)$ is the angle between φX and $\mathcal{D}^\theta_p$ is constant for any $X \in \mathcal{D}^\theta_p$ and $p \in M$.

(4) A submanifold M of a an almost contact metric manifold $\widetilde{M}$ is said be pseudo-slant (or hemi–slant) [8], if it is endowed with two orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$, where $\mathcal{D}^\perp$ is anti-invariant with respect to φ and $\mathcal{D}^\theta$ is proper slant.

We note that on a slant submanifold if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

It is known that [7] if M is a submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$, then M is a slant submanifold with slant angle θ if and only if

$$T^2 = \cos^2\theta\,(-I + \eta\otimes\xi) \tag{8}$$

The following relations are the consequences of (8) as

$$g(TX, TY) = \cos^2\theta\,(g(X,Y) - \eta(X)\eta(Y))\,, \tag{9}$$

$$g(FX, FY) = \sin^2\theta\,(g(X,Y) - \eta(X)\eta(Y)) \tag{10}$$

for any $X, Y \in \Gamma(TM)$. Another characterization of a slant submanifold of an almost contact metric manifold is obtained by using (5), (6) and (8) as

$$BFX = \sin^2\theta\,(-X + \eta(X)\xi)\,, \quad CFX = -FTX \tag{11}$$

for any $X \in \Gamma(TM)$.

In [6], Cabrerizo et al defined and studied bi-slant submanifolds of almost contact metric manifolds as follows:

Definition 2.1. *Let $\widetilde{M}$ be an almost contact metric manifold and M a real submanifold of $\widetilde{M}$. Then, we say that M is a bi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that*

(i) The tangent space TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \langle\xi\rangle$.
(ii) $T\mathcal{D}_1 \perp \mathcal{D}_2$ and $T\mathcal{D}_2 \perp \mathcal{D}_1$
(iii) For any $i = 1, 2$, $\mathcal{D}_i$ is a slant distribution with slant angle θ_i.

Let d_1 and d_2 denote the dimensions of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Then from the above definition, it is clear that

(i) If $d_1 = 0$ or $d_2 = 0$, then M is a slant submanifold.
(ii) If $d_1 = 0$ and $\theta_2 = 0$, then M is invariant.
(iii) If $d_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is an invariant submanifold.
(iv) If neither $d_1 = 0$ nor $d_2 = 0$ and $\theta_1 = 0$, then M is a semi-slant submanifold with slant angle θ_2.
(v) If neither $d_1 = 0$ nor $d_2 = 0$ and $\theta_1 = \frac{\pi}{2}$, then M is a pseudo-slant submanifold with slant angle θ_2.

A bi-slant submanifold of an almost contact metric manifold $\widetilde{M}$ is called *proper* if the slant distributions $\mathcal{D}_1, \mathcal{D}_2$ are of slant angles $\theta_1,\ \theta_2 \neq 0, \frac{\pi}{2}$.

We refer to [6] and [14] for non-trivial examples of bi-slant submanifolds.

3. Warped product bi-slant submanifolds

In [3], Bishop and O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y)$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant. Let X be an unit vector field tangent to M_1 and Z be an another unit vector field on M_2, then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z \tag{12}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ be a warped product manifold then M_1 is a totally geodesic submanifold of M and M_2 is a totally umbilical submanifold of M [3, 10].

Definition 3.1. *A warped product $M_1 \times_f M_2$ of two slant submanifolds M_1 and M_2 with slant angles θ_1 and θ_2 of a cosymplectic manifold $\widetilde{M}$ is called a warped product bi-slant submanifold.*

A warped product bi-slant submanifold $M_1 \times_f M_2$ is called *proper* if both M_1 and M_2 are proper slant submanifolds with slant angle $\theta_1, \ \theta_2 \neq 0, \frac{\pi}{2}$ of $\widetilde{M}$. A warped product $M_1 \times_f M_2$ is contact CR-warped product if $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ discussed in [22]. Also, a warped product bi-slant submanifold $M = M_1 \times_f M_2$ is pseudo-slant warped product if $\theta_2 = \frac{\pi}{2}$ [23].

In this section, we investigate the geometry of warped product bi-slant submanifolds of the form $M_1 \times_f M_2$ of a cosymplectic manifold $\widetilde{M}$, where M_1 and M_2 are slant submanifolds of $\widetilde{M}$. It is noted that on a warped product submanifold $M = M_1 \times_f M_2$ of a cosymplectic manifold $\widetilde{M}$ if the structure vector field ξ is tangent to M_2, then warped product is simply a Riemannian product (trivial) [15]. Now, throughout we consider the structure vector field ξ is tangent to the base manifold M_1.

First, we give the following lemma for later use.

Lemma 3.2. *Let $M = M_1 \times {}_f M_2$ be a warped product bi-slant submanifold of a cosymplectic manifold $\widetilde{M}$ such that ξ is tangent to M_1, where M_1 and M_2 are slant submanifolds of $\widetilde{M}$. Then*

$$g(h(X, Y), FV) = g(h(X, V), FY) \tag{13}$$

for any $X, Y \in \Gamma(TM_1)$ and $V \in \Gamma(TM_2)$.

Proof. For any $X, Y \in \Gamma(TM_1)$ and $V \in \Gamma(TM_2)$, we have

$$\begin{aligned} g(h(X, Y), FV) &= g(\widetilde{\nabla}_X Y, FV) \\ &= g(\widetilde{\nabla}_X Y, \varphi V) - g(\widetilde{\nabla}_X Y, TV) \\ &= -g(\widetilde{\nabla}_X \varphi Y, V) + g(\widetilde{\nabla}_X TV, Y). \end{aligned}$$

Then from (2), (5) and (12), we obtain

$$g(h(X, Y), FV) = -g(\widetilde{\nabla}_X TY, V) - g(\widetilde{\nabla}_X FY, V) + (X \ln f)\, g(TV, Y).$$

Last term in the right hand side of above relation vanishes identically by the orthogonality of the vector fields, thus we have

$$g(h(X, Y), FV) = g(TY, \widetilde{\nabla}_X V) + g(A_{FY}X, V).$$

Again from (12), (4) and the orthogonality of vector fields, we get the desired result. □

Lemma 3.3. *Let $M = M_1 \times {}_f M_2$ be a warped product bi-slant submanifold of a cosymplectic manifold $\widetilde{M}$ such that ξ is tangent to M_1, where M_1 and M_2 are proper slant submanifolds of $\widetilde{M}$ with slant angles θ_1 and θ_2, respectively. Then*

$$g(h(X,Z),FV) = g(h(X,V),FZ) \tag{14}$$

for any $X \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$.

Proof. For any $X \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have

$$g(\widetilde{\nabla}_X Z, V) = (X \ln f)\, g(Z,V). \tag{15}$$

On the other hand, we also have

$$g(\widetilde{\nabla}_X Z, V) = g(\varphi \widetilde{\nabla}_X Z, \varphi V) = g(\widetilde{\nabla}_X \varphi Z, \varphi V).$$

Using (5), we derive

$$g(\widetilde{\nabla}_X Z, V) = g(\widetilde{\nabla}_X TZ, TV) + g(\widetilde{\nabla}_X TZ, FV) + g(\widetilde{\nabla}_X FZ, \varphi V).$$

Then from (1), (2), (12), (9) and the cosymplectic characteristic, we find that

$$g(\widetilde{\nabla}_X Z, V) = \cos^2\theta_2\,(X \ln f) g(Z,V) + g(h(X,TZ),FV) - g(\widetilde{\nabla}_X \varphi FZ, V).$$

By using (6), we arrive at

$$g(\widetilde{\nabla}_X Z, V) = \cos^2\theta_2\,(X \ln f) g(Z,V) + g(h(X,TZ),FV) - g(\widetilde{\nabla}_X BFZ, V) - g(\widetilde{\nabla}_X CFZ, V).$$

Then from (11), we obtain

$$\begin{aligned} g(\widetilde{\nabla}_X Z, V) &= \cos^2\theta_2\,(X \ln f) g(Z,V) + g(h(X,TZ),FV) + \sin^2\theta_2\, g(\widetilde{\nabla}_X Z, V) + g(\widetilde{\nabla}_X FTZ, V). \\ &= (X \ln f) g(Z,V) + g(h(X,TZ),FV) - g((X,V),FTZ). \end{aligned} \tag{16}$$

Then from (15) and (16), we compute

$$g(h(X,TZ),FV) = g((X,V),FTZ)$$

Interchanging Z by TZ and using (8), we obtain

$$\cos^2\theta_2\, g(h(X,Z),FV) = \cos^2\theta_2\, g((X,V),FZ).$$

Since M is proper, then $cos^2\theta_2 \neq 0$, thus from the above relation we get (14), which proves the lemma completely. □

Theorem 3.4. *There does not exist any proper warped product bi-slant submanifold $M = M_1 \times {}_f M_2$ of a cosymplectic manifold $\widetilde{M}$ such that M_1 and M_2 are proper slant submanifolds of $\widetilde{M}$.*

Proof. When the structure vector field ξ is tangent to M_2, then warped product is trivial. Now, we consider $\xi \in \Gamma(TM_1)$ and for any $X \in \Gamma(TM_1)$ and $Z, V \in \Gamma(TM_2)$, we have

$$g(h(X,Z),FV) = g(\widetilde{\nabla}_Z X, FV) = g(\widetilde{\nabla}_Z X, \varphi V) - g(\widetilde{\nabla}_Z X, TV).$$

Using (1), (2), (5), (12) and the cosymplectic characteristic equation, we derive

$$\begin{aligned} g(h(X,Z),FV) &= -g(\widetilde{\nabla}_Z \varphi X, V) - (X \ln f)\, g(Z,TV) \\ &= -g(\widetilde{\nabla}_Z TX, V) - g(\widetilde{\nabla}_Z FX, V) - (X \ln f)\, g(Z,TV). \end{aligned}$$

Again, from (2), (3), (4) and (12), we find that

$$\begin{aligned} g(h(X,Z),FV) &= -(TX\ln f)\,g(Z,V) + g(A_{FX}Z,V) - (X\ln f)\,g(Z,TV) \\ &= -(TX\ln f)\,g(Z,V) + g(h(Z,V),FX) \\ &\quad - (X\ln f)\,g(Z,TV). \end{aligned} \tag{17}$$

Interchanging Z by V in (17) and using (1), we obtain

$$\begin{aligned} g(h(X,V),FZ) &= -(TX\ln f)\,g(Z,V) + g(h(Z,V),FX) \\ &\quad + (X\ln f)\,g(Z,TV). \end{aligned} \tag{18}$$

Then from (17), (18) and Lemma 3.3, we arrive at

$$(X\ln f)\,g(Z,TV) = 0. \tag{19}$$

Interchanging Z by TZ in (19) and using (9), we get

$$\cos^2\theta_2 (X\ln f)\,g(Z,V) = 0. \tag{20}$$

Since M is proper, then $\cos^2\theta_2 \neq 0$, thus from (20) we conclude that f is constant. Hence, the theorem is proved completely. □

From relation (20) of Theorem 3.4, if M is not proper and $\theta_1 = \frac{\pi}{2}$, then M is a pseudo-slant warped product of the form $M_{\theta_1} \times_f M_\perp$ and this a case which has been discussed in [23] for its characterisation and inequality.

Now, we give an example of such warped products.

Example 3.5. Let $\mathbb{R}^7$ be the Euclidean 7-space endowed with the standard metric and cartesian coordinates $(x_1,\, x_2,\, x_3,\, y_1,\, y_2,\, y_3,\, t)$ and with the canonical structure given by

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \le i, j \le 3.$$

If we assume a vector field $X = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial t}$ of $\mathbb{R}^7$, then $\varphi X = \lambda_i \frac{\partial}{\partial y_i} - \mu_j \frac{\partial}{\partial x_j}$ and $\varphi^2(X) = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -X + \nu \frac{\partial}{\partial t}$. Also, we can see that $g(X,X) = \lambda_i^2 + \mu_j^2 + \nu^2$ and $g(\varphi X, \varphi X) = \lambda_i^2 + \mu_j^2$, where g is the Euclidean metric tensor of $\mathbb{R}^7$. Then, we have $g(\varphi X, \varphi X) = g(X,X) - \eta(X)\xi$, where $\xi = \frac{\partial}{\partial t}$ and hence (φ, ξ, η, g) is an almost contact structure on $\mathbb{R}^7$. Consider a submanifold M of $\mathbb{R}^7$ defined by

$$\phi(u,\, v,\, w,\, t) = (u\cos v,\, u\sin v,\, w,\, w\cos v,\, w\sin v,\, 2u,\, t),\ v \neq 0, \frac{\pi}{2}$$

for non-zero u and w. The tangent bundle TM of M is spanned by

$$Z_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial y_3},$$

$$Z_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} - w\sin v\frac{\partial}{\partial y_1} + w\cos v\frac{\partial}{\partial y_2}$$

$$Z_3 = \frac{\partial}{\partial x_3} + \cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial y_2}, \quad Z_4 = \frac{\partial}{\partial t}.$$

Then, we have

$$\varphi Z_1 = \cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial y_2} - 2\frac{\partial}{\partial x_3},$$

$$\varphi Z_2 = -u\sin v\frac{\partial}{\partial y_1} + u\cos v\frac{\partial}{\partial y_2} + w\sin v\frac{\partial}{\partial x_1} - w\cos v\frac{\partial}{\partial x_2}$$

$$\varphi Z_3 = \frac{\partial}{\partial y_3} - \cos v\frac{\partial}{\partial x_1} - \sin v\frac{\partial}{\partial x_2}, \quad \varphi Z_4 = 0.$$

Clearly, the vector fields φZ_2 is orthogonal to TM. Then the anti-invariant and proper slant distributions of M respectively are $\mathcal{D}_1 = Span\{Z_2\}$ and $\mathcal{D}_2 = Span\{Z_1, Z_3\}$ with slant angle $\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$ such that $\xi = Z_4$ is tangent to M. Hence, M is a proper pseudo-slant submanifold of $\mathbb{R}^7$. Furthermore, it is easy to se that both the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable. We denote the integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$ by $M_\perp$ and M_θ, respectively. Then the metric tensor g of the product manifold M of $M_\perp$ and M_θ is

$$\begin{aligned} g &= 5\,du^2 + 2\,dw^2 + dt^2 + \left(u^2 + w^2\right) dv^2 \\ &= g_{M_\theta} + \left(\sqrt{u^2 + w^2}\right)^2 g_{M_\perp}. \end{aligned}$$

Thus M is a warped product pseudo-slant submanifold of $\mathbb{R}^7$ of the form $M_\theta \times_f M_\perp$ with warping function $f = \sqrt{u^2 + w^2}$ such that ξ is tangent to M_θ.

References

[1] M. Atceken, *Contact CR-warped product submanifolds in cosymplectic space forms*, Collect. Math. **62** (2011), 17-26.
[2] M. Atceken, *Warped product semi-slant submanifolds in locally Riemannian product manifolds*, Bull. Austral. Math. Soc. **77** (2008), 177-186.
[3] R.L. Bishop and B. O'Neill, *Manifolds of Negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1-49.
[4] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, New York, (1976).
[5] D.E. Blair and S.I. Goldberg, *Topology of almost contact manifolds*, J. Diff. Geometry **1** (1967), 347-354.
[6] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata **78** (1999), 183-199.
[7] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J. **42** (2000), 125-138.
[8] A. Carriazo, *New developments in slant submanifolds theory*, Narosa Publishing House, New Delhi, (2002).
[9] B.-Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc. **41** (1990), 135-147.
[10] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. **133** (2001), 177–195.
[11] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II*, Monatsh. Math. **134** (2001), 103–119.
[12] B.-Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific, Hackensack, NJ, (2011).
[13] B.-Y. Chen, *Geometry of warped product submanifolds: a survey*, J. Adv. Math. Stud. **6** (2013), no. 2, 1–43.
[14] R.S. Gupta, B.-Y. Chen's inequalities for bi-slant submanifolds in cosymplectic space forms, Sarajevo J. Math. **9** (21) (2013), 117–128.
[15] K.A. Khan, V.A. Khan and S. Uddin, *Warped product submanifolds of cosymplectic manifolds*, Balkan J. Geom. Its Appl. **13** (2008), 55-65.
[16] G.D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Diff. Geometry, **4** (1970), 237-244.
[17] N. Papaghiuc, *Semi-slant submanifolds of Kaehlerian manifold*, Ann. St. Univ. Iasi **9** (1994), 55-61.
[18] Sahin, B. *Warped product semi-invariant submanifolds of locally product Riemannian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie, Tome **49 (97)** (2006) 383-394.
[19] B. Sahin, *Warped product submanifolds of Kaehler manifolds with a slant factor*, Ann. Pol. Math. **95** (2009), 207-226.
[20] S. Uddin, Warped product CR-submanifolds of LP-cosymplectic manifolds Filomat **24** (1) (2010), 87–95.
[21] S. Uddin, V.A. Khan and K.A. Khan, *A note on warped product submanifolds of cosymplectic manifolds*, Filomat **24** (3) (2010), 95–102.
[22] S. Uddin and K.A. Khan, *Warped product CR-submanifolds of cosymplectic manifolds*, Ricerche di Matematica **60** (2011), 143–149.
[23] S. Uddin and F.R. Al-Solamy, *Warped product pseudo-slant submanifolds of cosymplectic manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat (N.S.) Tome LXIII (2016), f_2 vol. 3, 901-913.
[24] S. Uddin, B.-Y. Chen and F.R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math., 14 (2017), 1-10.
[25] Yano, K. and Kon, M. *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.