Filomat 31:17 (2017), 5489–5495
https://doi.org/10.2298/FIL1717489B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ulam Stability of Some Functional Inclusions for Multi-valued Mappings

Janusz Brzdęk[a], **Magdalena Piszczek**[a]

[a]*Department of Mathematics, Pedagogical University, Podchorążych 2, 30-084 Kraków, Poland*

Abstract. We show that some multifunctions $F : K \to n(Y)$, satisfying functional inclusions of the form

$$\Psi\Big(x, F(\xi_1(x)), \dots, F(\xi_n(x))\Big) \subset F(x)G(x),$$

admit near-selections $f : K \to Y$, fulfilling the functional equation

$$\Psi\Big(x, f(\xi_1(x)), \dots, f(\xi_n(x))\Big) = f(x),$$

where functions $G : K \to n(Y)$, $\Psi\colon K \times Y^n \to Y$ and $\xi_1, \dots, \xi_n \in K^K$ are given, n is a fixed positive integer, K is a nonempty set, $(Y, \cdot)$ is a group and $n(Y)$ denotes the family of all nonempty subsets of Y.

Our results have been motivated by the notion of Ulam stability and some earlier outcomes. The main tool in the proofs is a very recent fixed point theorem for nonlinear operators, acting on some spaces of multifunctions.

1. Introduction

The question *under what conditions an approximate solution to an equation can be replaced by an exact solution to it (or conversely) and what error we thus commit* seems to be very natural. The theory of Ulam (often also called the Hyers-Ulam) type stability provides some convenient tools to investigate such issues. Let us only mention that the study of such stability has been motivated by a problem raised by S. Ulam in 1940 and a solution to it given by Hyers in [3]. For some updated information and further references concerning that type of stability we refer to [1, 4, 5, 7]. We continue those investigations for some classes of inclusions for multifunctions and our main results correspond to and/or generalize the earlier outcomes in [8–12].

In this paper K is a nonempty set, $(Y, \cdot)$ is a group with the neutral element e, d is a complete metric in Y, $n(Y)$ is the family of all nonempty subsets of Y, $bd(Y)$ is the family of all nonempty and bounded subsets of Y, and $bcl(Y)$ is the family of all closed sets from $bd(Y)$. Moreover, as usual, B^A denotes the family of all functions mapping a set $A \neq \emptyset$ into a set $B \neq \emptyset$.

Let $n \in \mathbb{N}$ (positive integers), $\xi_1, \dots, \xi_n \in K^K$ and $\Psi\colon K \times Y^n \to Y$. We mainly investigate the Ulam stability of the functional equation

$$\phi(x) = \Psi(x, \phi(\xi_1(x)), \dots, \phi(\xi_n(x))), \qquad x \in K, \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 39B62, 39B82, 54C60, 54C65.
Keywords. Fixed point; Functional inclusion; Near-selection; Nonlinear operator; Set-valued map; Ulam stability.
Received: 30 July 2017; Accepted: 22 September 2017
Communicated by Erdal Karapinar
Email addresses: jbrzdek@up.krakow.pl (Janusz Brzdęk), magdap@up.krakow.pl (Magdalena Piszczek)

in the class of functions $\phi : K \to n(Y)$, but actually we study even more general issue of multifunctions fulfilling several particular cases of the inclusion of the form

$$\Psi\Big(x, \phi(\xi_1(x)), \ldots, \phi(\xi_n(x))\Big) \subset \phi(x)G(x), \qquad x \in K, \tag{2}$$

with a given $G : K \to n(Y)$, where

$$\Psi\Big(x, \phi(\xi_1(x)), \ldots, \phi(\xi_n(x))\Big) := \Psi\Big(\{x\} \times \phi(\xi_1(x)) \times \ldots \times \phi(\xi_n(x))\Big).$$

2. The main results

The following two theorems are the main results of this paper. The proofs of them are provided in the last section.

In what follows, $\mathbb{R}_+$ denotes the set of nonnegative reals, the number (possibly also ∞)

$$\delta(A) = \sup\{d(x, y) : \ x, y \in A\}$$

is said to be the diameter of $A \in n(Y)$ and h stands for the Hausdorff distance, induced by the metric d in Y, and given by

$$h(A, B) := \max\Big\{\sup_{x\in A} \inf_{y\in B} d(x, y), \sup_{y\in B} \inf_{x\in A} d(x, y)\Big\}, \qquad A, B \in n(Y).$$

It is well known that h is a metric if restricted to $bcl(Y)$.

For $g \colon K \to Y$ we denote by $\widehat{g}$ the multifunction defined by

$$\widehat{g}(x) := \{g(x)\}, \qquad x \in K.$$

Next, we write

$$AB := \{xy : x \in A, y \in B\}, \qquad A, B \in n(Y),$$

$$\prod_{i=1}^{1} x_i = x_1, \qquad \prod_{i=1}^{n+1} x_i = x_{n+1} \prod_{i=1}^{n} x_i, \qquad x_1, \ldots, x_{n+1} \in Y, n \in \mathbb{N},$$

and

$$\prod_{i=1}^{n} A_i = \left\{\prod_{i=1}^{n} x_i : x_1 \in A_1, \ldots, x_n \in A_n\right\}, \qquad A_1, \ldots, A_n \in n(Y), n \in \mathbb{N}.$$

In what follows we always assume that $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in K^K$ and $G \colon K \to bd(Y)$ are fixed and

$$e \in G(x), \qquad x \in K. \tag{3}$$

Our first main result reads as follows.

Theorem 2.1. *Let $M \colon K \to \mathbb{R}_+$, $c_1, \ldots, c_n \colon K \to [0, 1)$ and $\Psi \colon K \times Y^n \to Y$ be such that*

$$\lambda(x) := c_1(x) + \ldots + c_n(x) < 1, \qquad x \in K,$$

$$\max_{i=1,\ldots,n} \lambda(\xi_i(x)) \le \lambda(x), \qquad x \in K, \tag{4}$$

$$\max_{i=1,\ldots,n} M(\xi_i(x)) \le M(x), \qquad x \in K, \tag{5}$$

$$d(\Psi(x, z_1, \ldots, z_n), \Psi(x, w_1, \ldots, w_n)) \le \sum_{i=1}^{n} c_i(x) d(z_i, w_i), \tag{6}$$

$$x \in K, z_1, \ldots, z_n, w_1, \ldots, w_n \in Y.$$

Assume that $F\colon K \to bd(Y)$ *fulfils the functional inclusion*

$$\Psi\Big(x, F(\xi_1(x)), \ldots, F(\xi_n(x))\Big) \subset F(x)G(x), \qquad x \in K, \tag{7}$$

and

$$\delta(F(x)G(x)) \le M(x), \qquad x \in K. \tag{8}$$

Then there exists a unique function $f\colon K \to Y$ *such that*

$$\Psi(x, f(\xi_1(x)), \ldots, f(\xi_n(x))) = f(x), \qquad x \in K, \tag{9}$$

$$h(\widehat{f}(x), F(x)) \le \frac{M(x)}{1-\lambda(x)}, \qquad x \in K. \tag{10}$$

If d is invariant (i.e., $d(xz, yz) = d(x, y) = d(zx, zy)$ for $x, y, z \in Y$), then a very simple example of $\Psi\colon K \times Y^n \to Y$ satisfying (6) is given by

$$\Psi(x, z_1, \ldots, z_n) = \prod_{i=1}^{n} \Psi_i(x, z_i), \qquad x \in K,\ z_1, \ldots, z_n \in Y, \tag{11}$$

with some $\Psi_1, \ldots, \Psi_n\colon K \times Y \to Y$ and $c_1, \ldots, c_n\colon K \to [0, 1)$ such that

$$d(\Psi_i(x, z), \Psi_i(x, w)) \le c_i(x) d(z, w), \qquad x \in K, z, w \in Y, i = 1, \ldots, n. \tag{12}$$

The next theorem deals with such situation in the particular case where d is non-Archimedean (an ultrametric).

Let us remind that a metric ρ in a set Z is non-Archimedean (or an ultrametric) provided

$$\rho(z, w) \le \max\{\rho(z, y), \rho(y, w)\}, \qquad y, z, w \in Z;$$

then we say that (Z, ρ) is an ultrametric space (for some information on non-Archimedean analysis see, e.g., [6]).

Theorem 2.2. *Let* d *be invariant and non-Archimedean,* $c_1, \ldots, c_n\colon K \to [0, 1)$, $\Psi_1, \ldots, \Psi_n\colon K \times Y \to Y$, $M\colon K \to \mathbb{R}_+$, *conditions* (4), (5), *and* (12) *be valid, and*

$$\lambda(x) := \max\{c_1(x), \ldots, c_n(x)\}, \qquad x \in K.$$

Assume that $F\colon K \to bd(Y)$ *fulfilling* (8) *is a solution of the functional inclusion*

$$\prod_{i=1}^{n} \Psi_i(x, F(\xi_i(x))) \subset F(x)G(x), \qquad x \in K. \tag{13}$$

Then there exists a unique function $f\colon K \to Y$ *such that*

$$\prod_{i=1}^{n} \Psi_i(x, f(\xi_i(x))) = f(x), \qquad x \in K, \tag{14}$$

and (10) *holds.*

Remark 2.3. Note that conditions (4) and (5) are valid in particular in the situation when $K = \mathbb{R}$, M and λ are nondecreasing, and

$$\xi_i(x) \le x, \qquad x \in K, i = 1, \ldots, n.$$

3. An auxiliary result

For the proofs of Theorems 2.1 and 2.2 we need the following auxiliary fixed point result that has been proved in [2]. To present it we must recall the basic notions from [2] ($\mathbb{R}$ stands for the set of real numbers).

Namely, given $a, b \in \mathbb{R}^K$ and $F, G \in n(Y)^K$, we write $a \le b$ provided

$$a(x) \le b(x), \qquad x \in K,$$

and $F \subset G$ provided $F(x) \subset G(x)$ for $x \in K$. We say that $\Lambda\colon {\mathbb{R}_+}^K \to {\mathbb{R}_+}^K$ is non-decreasing if $\Lambda a \le \Lambda b$ for every $a, b \in {\mathbb{R}_+}^K$ with $a \le b$.

In $bcl(Y)^K$ the Tychonoff topology (of pointwise convergence) is assumed, with the Hausdorff metric in $bcl(Y)$ and, for $F\colon K \to n(Y)$, we denote by $\mathrm{cl}\, F$ the multifunction defined by

$$(\mathrm{cl}\, F)(x) = \mathrm{cl}\, F(x), \qquad x \in K.$$

Next, we write

$$\Big(\lim_{n\to\infty} H_n\Big)(x) := \lim_{n\to\infty} H_n(x), \qquad x \in K,$$

for each sequence $(H_n)_{n\in\mathbb{N}}$ in $bcl(Y)^K$ that is convergent in $bcl(Y)^K$. We say that an operator $\alpha\colon n(Y)^K \to n(Y)^K$ is i.p. (inclusion preserving) if

$$\alpha F \subset \alpha G, \qquad F, G \in n(Y)^K, F \subset G;$$

α is l.p. (limit preserving) if

$$\alpha\Big(\lim_{n\to\infty} \mathrm{cl}\, H_n\Big) \subset \lim_{n\to\infty} \mathrm{cl}\,(\alpha H_n) \tag{15}$$

for each sequence $(H_n)_{n\in\mathbb{N}}$ in $bd(Y)^K$ such that the sequence $(\mathrm{cl}\, H_n)_{n\in\mathbb{N}}$ is convergent in $bcl(Y)^K$ and there exists

$$\lim_{n\to\infty} \mathrm{cl}\,(\alpha H_n) \in bcl(Y)^K.$$

Finally, $\tilde{\delta}\colon bd(Y)^K \to {\mathbb{R}_+}^K$ is given by the formula

$$\tilde{\delta} F(x) = \delta(F(x)), \qquad F \in bd(Y)^K, x \in K.$$

Now, we are in a position to present the mentioned above fixed point result that can be easily derived from [2, Theorem 1].

Theorem 3.1. *Let* $\Lambda\colon \mathbb{R}_+^K \to \mathbb{R}_+^K$ *be non-decreasing,* $\alpha\colon bd(Y)^K \to bd(Y)^K$ *be i.p. and l.p.,* $\mathcal{G}\colon bd(Y)^K \to n(Y)^K$, $F \in bd(Y)^K$, $\mathcal{G}F \in bd(Y)^K$,

$$\tilde{\delta}(\alpha H) \le \Lambda(\widetilde{\delta H}), \qquad H \in bd(Y)^K, \tag{16}$$

$$\alpha F \cup F \subset \mathcal{G}F, \tag{17}$$

and

$$\kappa(x) := \sum_{n=0}^{\infty} \Lambda^n(\tilde{\delta}(\mathcal{G}F))(x) < \infty, \qquad x \in K. \tag{18}$$

Then there exists a function $f\colon K \to Y$ *such that* $\widehat{f}$ *is a fixed point of the operator* α *(i.e.,* $\alpha\widehat{f} = \widehat{f}$*) and*

$$h\Big(\widehat{f}(x), F(x)\Big) \le \kappa(x), \qquad x \in K.$$

Moreover, if $G \in bd(Y)^K$ satisfies the conditions

$$G \subset \alpha G,$$

$$h\Big(G(x), F(x)\Big) \le \mu(x), \qquad x \in K,$$

with some $\mu \colon K \to \mathbb{R}_+$ such that

$$\liminf_{n\to\infty} \Lambda^n(\kappa + 2\mu)(x) = 0, \qquad x \in K, \tag{19}$$

then $G = \widehat{f}$.

4. Proofs

Now we present the proofs of Theorems 2.1 and 2.2. Let us start with a proof for Theorem 2.1.

Proof. Define $\alpha \colon bd(Y)^K \to bd(Y)^K$ by

$$\alpha H(x) := \Psi(x, H(\xi_1(x)), \dots, H(\xi_n(x))), \qquad H \in bd(Y)^K.$$

Then it is easily seen that it is i.p. For the convenience of readers we provide an elementary reasoning that α is also l.p.

So, let $(H_k)_{k\in\mathbb{N}}$ be a sequence in $bd(Y)^K$ such that the sequences $(\mathrm{cl}\, H_k)_{k\in\mathbb{N}}$ and $(\mathrm{cl}\,(\alpha H_k))_{k\in\mathbb{N}}$ are convergent in $bcl(Y)^K$. We show that

$$\mathrm{cl}\left[\alpha\Big(\lim_{k\to\infty} \mathrm{cl}\, H_k\Big)(x)\right] = \lim_{k\to\infty} \mathrm{cl}\Big(\alpha H_k(x)\Big), \qquad x \in K, \tag{20}$$

which actually is equivalent to the condition

$$\lim_{k\to\infty} h\Big(\alpha H_0(x), \alpha H_k(x)\Big) = 0, \qquad x \in K, \tag{21}$$

where

$$H_0 := \lim_{k\to\infty} \mathrm{cl}\, H_k.$$

Take $\epsilon > 0$, $x \in K$. There is $k_0 \in \mathbb{N}$ such that

$$h\Big(H_0(\xi_i(x)), H_k(\xi_i(x))\Big) < \epsilon, \qquad k > k_0, i = 1, \dots, n,$$

which yields

$$\begin{aligned} &h(\Psi(x, H_0(\xi_1(x)), \dots, H_0(\xi_n(x))), \Psi(x, H_k(\xi_1(x)), \dots, H_k(\xi_n(x))) \\ &\qquad \le \sum_{i=1}^{n} c_i(x) h(H_0(\xi_i(x), H_k(\xi_i(x))) \le \sum_{i=1}^{n} c_i(x)\epsilon \le \epsilon. \end{aligned}$$

Thus (21) holds. So, we see that α is l.p.

Next, $\Lambda \colon \mathbb{R}_+^K \to \mathbb{R}_+^K$, given by

$$\Lambda a(x) := \sum_{i=1}^{n} c_i(x) a(\xi_i(x)), \qquad a \in \mathbb{R}_+^K, x \in K,$$

is non-decreasing and

$$\begin{aligned}\delta((\alpha H)(x)) &= \delta\Big(\Psi(x, H(\xi_1(x))), \ldots, H(\xi_n(x)))\Big)\\ &\le \sum_{i=1}^{n} c_i(x)\delta(H(\xi_i(x))) = \Lambda(\tilde{\delta}H)(x)\end{aligned}$$

for every $x \in K$ and $H \in bd(Y)^K$, which means that (16) holds.

Let $\mathcal{G} : bd(Y)^K \to n(Y)^K$ be given by

$$\mathcal{G}H(x) := H(x)G(x), \qquad x \in K, H \in bd(Y)^K. \tag{22}$$

Note that, for every $x \in K$,

$$\Lambda(\tilde{\delta}(\mathcal{G}F))(x) \le \sum_{i=1}^{n} c_i(x)\delta(F(\xi_i(x))G(\xi_i(x))) \le \lambda(x)M(x)$$

and, in view of (4) and the equality

$$\Lambda^k(\tilde{\delta}(\mathcal{G}F))(x) = \sum_{i=1}^{n} c_i(x)\Lambda^{k-1}(\tilde{\delta}(\mathcal{G}F))(\xi_i(x)),$$

in a similar way we get by induction on k

$$\Lambda^k(\tilde{\delta}(\mathcal{G}F))(x) \le (\lambda(x))^k M(x), \qquad k \in \mathbb{N}.$$

Hence

$$\kappa(x) := \sum_{n=0}^{\infty} \Lambda^n(\tilde{\delta}(\mathcal{G}F))(x) \le \frac{M(x)}{1-\lambda(x)} =: \mu(x), \qquad x \in K.$$

Note that, in particular, this means that (19) holds, because Λ is additive and nondecreasing and

$$\Lambda^n\mu(x) \le (\lambda(x))^n\mu(x), \qquad x \in K.$$

Consequently, according to Theorem 3.1, there is a unique function $f: K \to Y$ such that $\widehat{f}$ is a fixed point of α and (10) is valid. □

Next, we present a proof for Theorem 2.2.

Proof. We argue as in the proof of Theorem 2.1, with $\Lambda: \mathbb{R}_+^K \to \mathbb{R}_+^K$ and $\alpha: bd(Y)^K \to bd(Y)^K$ given by

$$\alpha H(x) := \prod_{i=1}^{n} \Psi_i(x, H(\xi_i(x))), \qquad H \in n(Y)^K,$$

$$\Lambda a(x) := \max_{i=1,\ldots,n} c_i(x)a(\xi_i(x)), \qquad a \in \mathbb{R}_+^K, x \in K.$$

So, observe that (12) yields

$$\begin{aligned}\delta((\alpha H)(x)) &= \delta\Big(\prod_{i=1}^{n} \Psi_i(x, H(\xi_i(x)))\Big) \le \max_{i=1,\ldots,n} \delta(\Psi_i(x, H(\xi_i(x))))\\ &\le \max_{i=1,\ldots,n} c_i(x)\delta(H(\xi_i(x))) = \Lambda(\tilde{\delta}H)(x)\end{aligned}$$

for every $x \in K$, $H \in bd(Y)^K$, which means that (16) holds. Further, α is i.p. and, analogously as in the proof of Theorem 2.1, we can easily show that α is l.p.

Next, note that we have

$$\Lambda(\tilde{\delta}(\mathcal{G}F))(x) \leq \lambda(x) \max_{i=1,\dots,n} \delta(F(\xi_i(x))G(\xi_i(x))) \leq \lambda(x)M(x)$$

for every $x \in K$ with $\mathcal{G}\colon bd(Y)^K \to bd(Y)^K$ given by (22). Analogously we get

$$\Lambda^k(\tilde{\delta}(\mathcal{G}F))(x) \leq (\lambda(x))^k M(x), \qquad x \in K, k \in \mathbb{N},$$

by induction on k (in view of (4), (5) and (8)). Hence

$$\kappa(x) := \sum_{n=0}^{\infty} \Lambda^n(\tilde{\delta}(\mathcal{G}F))(x) \leq \frac{M(x)}{1-\lambda(x)} =: \mu(x), \qquad x \in K;$$

in particular, on account of (4) and (5), condition (19) is valid.

Consequently, according to Theorem 3.1, there exists a unique function $f\colon K \to Y$ such that $\widehat{f}$ is a fixed point of α and (10) holds. □

References

[1] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, On some recent developments in Ulam's type stability, Abstr. Appl. Anal. 2012 (2012), Article ID 716936, 41 pp.
[2] J. Brzdęk, M. Piszczek, Fixed points of some nonlinear operators in spaces of multifunctions and the Ulam stability, J. Fixed Point Th. Appl. 19 (2017), 2441-2448.
[3] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA, 27 (1941) 222–224.
[4] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, Boston, 1998.
[5] S.M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer Optimization and Its Applications vol. 48, Springer, New York-Dordrecht-Heidelberg-London, 2011.
[6] A. Khrennikov, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models, Kluwer Academic Publishers, Dordrecht, 1997.
[7] Z. Moszner, Stability has many names, Aequationes Math. 90 (2016) 983–999.
[8] K. Nikodem, D. Popa, On selections of general linear inclusions, Publ. Math. Debrecen 75 (2009) 239–249.
[9] M. Piszczek, The properties of functional inclusions and Hyers-Ulam stability, Aequationes Math. 85 (2013) 111d'ż˝-118.
[10] D. Popa, A stability result for a general linear inclusion, Nonlinear Funct. Anal. App. 3 (2004) 405–414.
[11] D. Popa, Functional inclusions on square-symetric grupoid and Hyers-Ulam stability, Math. Inequal. Appl. 7 (2004) 419–428.
[12] D. Popa, A property of a functional incusion connected with Hyers-Ulam stability, J. Math. Inequal. 4 (2009), 591–598.