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Scattering Analysis and Spectrum of Discrete Schrödinger Equations with Transmission Conditions

Elgiz Bairamov[a], Yelda Aygar[a], Dilara Karslıoğlu[a]

[a]*University of Ankara, Faculty of Science, Department of Mathematics, 06100, Ankara, Turkey*

Abstract. In this paper, we present an investigation about scattering analysis of an transmission boundary value problem (TBVP) which consists a discrete Schrödinger equation and transmission conditions. Discussing the Jost solution and scattering function of this problem, we find the properties of scattering function of this problem by using the scattering solutions. We also investigate the discrete spectrum of this boundary value problem. Furthermore, we apply the results on an example which is the special case of main TBVP and we discuss the existence of eigenvalues of this example.

1. Introduction

Scattering problems and analysis of these problems has been an important research topic in mathematical physics. Spectral analysis of scattering problems have meanings in a lot of branches of physics. Because classical scattering theory is a branch of Newtonian mechanics. It focuses on trajectory of a particle of mass moving with an initial velocity towards a scattering region. As a result of this, such problems play an important role in mathematical physics and have many applications in natural sciences. Studies of scattering problems first begin for continuous Schrödinger equations in literature [1, 6–8, 11, 13, 26]. In [13], the author investigates the properties of eigenvalues of Sturm–Liouville boundary value problem given by

$$-y'' + q(x)y = \lambda^2 y\,, \quad 0 \leq x < \infty, \tag{1}$$

and with the boundary condition

$$y\,(0) = 0, \tag{2}$$

where q is a real-valued function and λ is an eigenparameter. After investigation of this problem, Marchenko gets that the Jost function of (1) defined by

$$e(\lambda) := 1 + \int_0^\infty K(0,t)e^{i\lambda t}\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \ \mathrm{Im}\,\lambda \geq 0\}$$

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Email addresses: bairamov@science.ankara.edu.tr (Elgiz Bairamov), yaygar@ankara.edu.tr (Yelda Aygar), dilarakarslioglu@gmail.com (Dilara Karslıoğlu)

has a finite number of simple zeros in open half complex plain. If we show the zeros of Jost function of (1) by $i\lambda_k$, $k = 1, 2, ..., n$ under the assumption $0 < \lambda_1 < \lambda_2 < ... < \lambda_n$ and the norm of the Jost solution for $\lambda = i\lambda_k$ in $L_2(0, \infty)$ by m_k^{-1}, also the scattering function of (1)-(2) by

$$\mathcal{S}(\lambda) := \frac{\overline{e(\lambda)}}{e(\lambda)}, \quad \lambda \in (-\infty, \infty), \tag{3}$$

we collect the scattering data of boundary value problem (1)-(2) as the following set

$$\{\mathcal{S}(\lambda), \lambda \in (-\infty, \infty); \lambda_k, m_k, k = 1, 2, ..., n\}. \tag{4}$$

When the potential function q is given, the problem of getting scattering data given (4) and investigating the properties of scattering data is called the direct problem for scattering theory. Oppositely, finding the potential function q according to the scattering data is known inverse problem of scattering theory. There are a lot of papers about direct and inverse scattering problems for Sturm–Liouville equations, discrete Sturm–Liouville equations, Schrödinger and Dirac equations. Some of them are given by [5, 9, 10, 12] and references cited therein. But scattering theory of discrete Sturm–Liouville equation with transmission conditions has not been investigated yet. In this work, we will consider the following discrete Schrödinger equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} \setminus \{m_0 - 1, m_0, m_0 + 1\} \tag{5}$$

with the boundary condition

$$y_0 = 0, \tag{6}$$

and the transmission conditions

$$\begin{aligned} y_{m_0+1} &= \gamma_1 y_{m_0-1} \\ y_{m_0+2} - y_{m_0+1} &= \gamma_2(y_{m_0-1} - y_{m_0-2}) \quad ; \quad \gamma_1\gamma_2 \neq 0, \quad \gamma_1, \gamma_2 \in \mathbb{R} \end{aligned} \tag{7}$$

where λ is a spectral parameter, $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are real sequences satisfying the condition

$$\sum_{n\in\mathbb{N}} n\left(|1 - a_n| + |b_n|\right) < \infty\ . \tag{8}$$

Throughout the paper, we assume that $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Note that the transmission conditions increase the importance of scattering problem. Because TBVPs also have a host of applications to biological, physical and engineering problems. In the past few decades, many advances have been made in the theory of transmission differential and difference equations and transmission dynamical systems. Because of this, the theory undergoes extensive research and scientists formulate the mathematical research by using transmission equations to understand the daily life. In general such problems are related to discontinuous material properties such as heat and mass transfer, vibrating string problems when the string loaded with point masses. To deal with interior discontinuities, some conditions are imposed on discontinuous points. These points are called transmission conditions, impulsive conditions, interface conditions and point interactions [2, 3, 15–19, 21–25]. In literature, there are some various physics applications of this kind of problems. Related to the TBVP (5)-(7), we will consider the following TBVP

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = 2\cos z y_n, \quad n \in \mathbb{N} \setminus \{m_0 - 1, m_0, m_0 + 1\} \tag{9}$$

with the boundary condition

$$y_0 = 0, \tag{10}$$

and the transmission conditions

$$\begin{aligned} y_{m_0+1} &= \gamma_1 y_{m_0-1} \\ \triangle y_{m_0+1} &= \gamma_2 \nabla y_{m_0-1} \quad ; \quad \gamma_1\gamma_2 \neq 0, \quad \gamma_1, \gamma_2 \in \end{aligned} \tag{11}$$

where $\triangle$ is a forward difference operator, ∇ is a backward difference operator and defined by $\triangle u_n := u_{n+1} - u_n$ and $\nabla u_n := u_n - u_{n-1}$, respectively.

The set of this paper is summarized as follows: Section 2 contains the scattering solutions of TBVP (9)-(11). Using the scattering solutions, we get the Jost solution, scattering function and the properties of scattering function of (9)-(11). In Section 3, we present an example of a different TBVP which is a special case of (9)-(11). Discussing the Jost solution and scattering function of this example, we determine the region of eigenvalues for this example.

2. Scattering solution and scattering function

Let us take two semi-strips $\Pi_+ := \left\{z \in n\mathbb{C} : z = \eta + i\zeta, \zeta > 0, -\frac{\pi}{2} \leq \eta \leq \frac{3\pi}{2}\right\}$ and $\Pi := \Pi_+ \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. We will show the fundamental solutions of (9) for $z \in \Pi$ and $n = 0, 1, ..., m_0 - 1$ by $\{P_n(z)\}$ and $\{Q_n(z)\}$ satisfying the initial conditions

$$P_0(z) = 0, \qquad P_1(z) = 1$$

and

$$Q_0(z) = \frac{1}{a_0}, \qquad Q_1(z) = 0,$$

respectively. Since the wronskian of two solutions $y = \{y_n(z)\}$ and $u = \{u_n(z)\}$ of the difference equation (9) defined by

$$\begin{aligned} W[y, u] &:= a_n \{y_n(z)u_{n+1}(z) - y_{n+1}(z)u_n(z)\} \\ &:= a_n \left[y_{n-1}(z)\triangle u_n(z) - \triangle y_n(z)u_n(z)\right] \\ &:= a_{n-1} \left[y_{n-1}(z)\nabla u_n(z) - \triangle y_{n-1}(z)u_{n-1}(z)\right], \end{aligned} \tag{12}$$

we find $W[P_n(z), Q_n(z)] = -1$ for all $z \in \mathbb{C}$. It is evident that $P_n(z)$ and $Q_n(z)$ are entire functions of z. Furthermore, we show by $e(z) = \{e_n(z)\}$ $n = m_0 + 1, m_0 + 2, ...$ the bounded solution of (9) satisfying the condition $\lim_{n\to\infty} e^{-inz} e_n(z) = 1$, $z \in \Pi_+$. The representation of $e_n(z)$ is well-known in literature [10] as

$$e_n(z) = \rho_n e^{inz} \left(1 + \sum_{m=1}^{\infty} A_{nm} e^{imz}\right), \qquad n = m_0 + 1, m_0 + 2, ...$$

where ρ_n and A_{nm} are given in terms of the sequences $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{b_n\}_{n\in\mathbb{N}}$. The function $e_n(z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+$ and $e_n(z + 2\pi) = e_n(z)$ for all z in $\overline{\mathbb{C}}_+$. Now, we will define the following function by using $e_n(z)$, $P_n(z)$ and $Q_n(z)$

$$E_n(z) := \begin{cases} \alpha(z) P_n(z) + \beta(z) Q_n(z) & , \quad n = 0, 1, ..., m_0 - 1 \\ e_n(z) & , \quad n = m_0 + 1, m_0 + 2, ... \end{cases} \tag{13}$$

for z in Π. From the transmission conditions (11), we write

$$E_{m_0-1}(z) = \frac{1}{\gamma_1} E_{m_0+1}(z)$$

$$\nabla E_{m_0-1}(z) = \frac{1}{\gamma_2} \triangle E_{m_0+1}(z),$$

and

$$\begin{aligned} \frac{1}{\gamma_1} e_{m_0+1}(z) &= \alpha(z) P_{m_0-1}(z) + \beta(z) Q_{m_0-1}(z) \\ \frac{1}{\gamma_2} \triangle e_{m_0+1}(z) &= \alpha(z) \nabla P_{m_0-1}(z) + \beta(z) \nabla Q_{m_0-1}(z). \end{aligned} \tag{14}$$

Using (12) and (14), we obtain

$$\alpha(z) = \frac{a_{m_0-2}}{\gamma_1 \gamma_2} \left[\gamma_1 \Delta e_{m_0+1}(z) Q_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla Q_{m_0-1}(z) \right] \tag{15}$$

$$\beta(z) = -\frac{a_{m_0-2}}{\gamma_1 \gamma_2} \left[\gamma_1 \Delta e_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla P_{m_0-1}(z) \right] \tag{16}$$

for $z \in \overline{\mathbb{C}}_+$. The function $E(z) = \{E_n(z)\}$ is the Jost solution of TBVP (9)-(11), where $\alpha(z)$ and $\beta(z)$ are defined in (15), (16), respectively.

Lemma 2.1. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$

$$W[e_n(z), e_n(-z)] = -2i \sin z.$$

Proof. We know from [4] that the wronskian is independent from n, so we write

$$\begin{aligned} W[e_n(z), e_n(-z)] &= \lim_{n\to\infty} \{a_n[e_n(z) e_{n+1}(-z) - e_{n+1}(z) e_n(-z)]\} \\ &= \lim_{n\to\infty} a_n \left[e^{-iz} e_n(z) e^{-inz} e_{n+1}(-z) e^{i(n+1)z} \right] \\ &- \lim_{n\to\infty} a_n [e^{iz} e_{n+1}(z) e^{-i(n+1)z} e_n(-z) e^{inz}] \\ &= e^{-iz} - e^{iz} = -2i \sin z \end{aligned}$$

by using (8) and the definition of $e_n(z)$. □

Now, we will consider the following solution $F(z) = \{F_n(z)\}$ of (9)-(11)

$$F_n(z) := \begin{cases} P_n(z) & , \quad n = 0, 1, ..., m_0 - 1 \\ \delta(z) e_n(z) + d(z) e_n(-z) & , \quad n = m_0 + 1, m_0 + 2, ... \end{cases}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. Using (11), we get that

$$\delta(z) = -\frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 \Delta e_{m_0+1}(-z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(-z) \nabla P_{m_0-1}(z) \right] \tag{17}$$

and

$$d(z) = \frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 \Delta e_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla P_{m_0-1}(z) \right]. \tag{18}$$

We also obtain

$$d(z) = \delta(-z) = \overline{\delta(z)} \tag{19}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ because of $P_n(z) = P_n(-z)$ and $Q_n(z) = Q_n(-z)$.

Lemma 2.2. *The wronskian of the solutions $E(z)$ and $F(z)$ is given by*

$$W[E(z), F(z)] = \begin{cases} \beta(z) & , \quad n = 0, 1, 2, ..., m_0 - 1 \\ \dfrac{a_{m_0+1}}{a_{m_0-2}} \gamma_1 \gamma_2 \beta(z) & , \quad n = m_0 + 1, m_0 + 2, ... \end{cases}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$.

Proof. If we write the wronskian of $\{E_n(z)\}$ and $\{F_n(z)\}$ for $n = 0, 1, 2, ..., m_0 - 1$, we find

$$\begin{aligned} W[E, F] &= a_0 [E_0(z)F_1(z) - E_1(z)F_0(z)] \\ &= a_0 [(\alpha(z) P_0(z) + \beta(z) Q_0(z)) P_1(z) - (\alpha(z) P_1(z) + \beta(z) Q_1(z)) P_0(z)] . \end{aligned}$$

Since $P_0(z) = 0$, $P_1(z) = 1$, $Q_0(z) = \frac{1}{a_0}$ and $Q_1(z) = 0$, we get

$$W[E_n(z), F_n(z)] = \beta(z)$$

for $n = 0, 1, 2, ..., m_0 - 1$. Similarly, for $n = m_0 + 1, m_0 + 2, ...$, we obtain

$$\begin{aligned} W[E, F] &= a_{m_0+1} e_{m_0+1}(z) [\delta(z) e_{m_0+2}(z) + d(z) e_{m_0+2}(-z)] \\ &\quad - a_{m_0+1} e_{m_0+2}(z) [\delta(z) e_{m_0+1}(z) + d(z) e_{m_0+1}(-z)] . \end{aligned}$$

After applying the definition $d(z)$ given by (18) and $\beta(z)$ given by (16), we find $W[E_n(z), F_n(z)] = \frac{a_{m_0+1}}{a_{m_0-2}} \gamma_1 \gamma_2 \beta(z)$ for $n = m_0 + 1, m_0 + 2, ...$ This completes the proof of Lemma 2.3. □

Let the function $\widehat{e_n}(z)$ denote the unbounded solution of (9) for $n = m_0 + 1, m_0 + 2, ...$ providing the condition $\lim_{n\to\infty} e^{inz} \widehat{e_n}(z) = 1$, $z \in \overline{\mathbb{C}}_+$. It can be easily seen that for $n = m_0 + 1, m_0 + 2, ...$

$$W[e_n(z), \widehat{e_n}(z)] = -2i \sin z \quad , \quad z \in \Pi \setminus \{0, \pi\} .$$

For all $z \in \Pi$, we will define the following solution $G(z) = \{G_n(z)\}$ of (9)-(11)

$$G_n(z) = \begin{cases} P_n(z) & , \quad n = 0, 1, 2, ..., m_0 - 1 \\ q(z) e_n(z) + k(z) \widehat{e_n}(z) & , \quad n = m_0 + 1, m_0 + 2, ... \end{cases} . \tag{20}$$

Using the transmission conditions (11) for $G(z)$, we give the coefficients $q(z)$ and $k(z)$ by

$$q(z) = -\frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 \Delta \widehat{e}_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 \widehat{e}_{m_0+1}(z) \nabla P_{m_0-1}(z)\right] \tag{21}$$

and

$$k(z) = \frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 \Delta e_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla P_{m_0-1}(z)\right] .$$

Note that the function $G(z)$ is unbounded solution of TBVP (9)-(11). Moreover, we can write the following relations between the coefficients of the solutions $E(z)$, $F(z)$ and $G(z)$ for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$,

$$k(z) = d(z) = \overline{c(z)} = -\frac{a_{m_0+1} \gamma_1 \gamma_2}{a_{m_0-2} 2i \sin z} \beta(z) . \tag{22}$$

Theorem 2.3. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, $\beta(z) \neq 0$.

Proof. Let we assume that there exists a z_0 in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ such that

$$\beta(z_0) = 0.$$

According to equation (22), we get $d(z_0) = c(z_0) = 0$. In that occasion, it gives us $F_n(z_0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, but this is a trivial solution of TBVP (9)-(11). It says that there is a contradiction, so $\beta(z) \neq 0$ for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. □

Similarly to the Sturm–Liouville equation which is mentioned in (3), the function

$$\mathcal{S}(z) := \frac{\overline{\beta(z)}}{\beta(z)}, \qquad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\} \tag{23}$$

is the scattering function of the TBVP (9)-(11). It follows from (13) that

$$E_0(z) = \alpha(z) P_0(z) + \beta(z) Q_0(z) = \frac{\beta(z)}{a_0}$$

and

$$E_0(-z) = \frac{\beta(-z)}{a_0}.$$

From the last equations and (23), we write the scattering function

$$\mathcal{S}(z) = \frac{\overline{E_0(z)a_0}}{E_0(z)a_0} = \frac{\overline{E_0(z)}}{E_0(z)} = \frac{E_0(-z)}{E_0(z)} = \frac{\beta(-z)}{\beta(z)}$$

and

$$\mathcal{S}(z) = \frac{\gamma_1 \Delta e_{m_0+1}(-z) P_{m_0-1}(-z) - \gamma_2 e_{m_0+1}(-z) \nabla P_{m_0+1}(-z)}{\gamma_1 \Delta e_{m_0+1}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \nabla P_{m_0+1}(z)}, \tag{24}$$

by (17). It is clear from (24) that

$$\lim_{z \to 0} \mathcal{S}(z) = \mathcal{S}(0) = 1.$$

Theorem 2.4. *Scattering function satisfies*

$$\mathcal{S}(-z) = \mathcal{S}^{-1}(z) = \overline{\mathcal{S}(z)}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$.

Proof. By the definition of scattering function, we have $\mathcal{S}(-z) = \frac{E_0(z)}{E_0(-z)}$ and $\overline{\mathcal{S}(z)} = \frac{\overline{E_0(-z)}}{\overline{E_0(z)}}$.
Since $E_0(z) = \overline{E_0(-z)}$ and $\overline{E_0(z)} = E_0(-z)$, we get

$$\mathcal{S}(-z) = \mathcal{S}^{-1}(z) = \overline{\mathcal{S}(z)}.$$

It completes the proof of theorem. □

Theorem 2.5. *The set of eigenvalues of TBVP* (9)-(11) *is*

$$\sigma_d = \{\lambda \in \mathbb{C} : \lambda = 2\cos z,\ z \in \Pi_+,\ \beta(z) = 0\}.$$

Proof. From the definition of eigenvalues [20], the Jost solution $E_n(z)$ must be in $\ell_2(\mathbb{N})$. It is clear from (13) that the first part of $E_n(z) \in \ell_2(\mathbb{N})$ because contains finite values of n and the second part of $E_n(z)$ refers $e_n(z)$, in this case it is also in $\ell_2(\mathbb{N})$. On the other hand $E_n(z)$ must satisfy the boundary condition (10), in this occasion the coefficient of the function $Q_0(z)$ must be zero, i.e, $\beta(z) = 0$. □

Using (13) and (20), we also obtain

$$W[E(z), G(z)] = \begin{cases} \beta(z) & , \quad n = 0, 1, 2, ..., m_0 - 1 \\ \dfrac{a_{m_0+1}}{a_{m_0-2}} \gamma_1\gamma_2\beta(z) & , \quad n = m_0 + 1, m_0 + 2, ... \end{cases}$$

for all $z \in \Pi$. Theorem 2.6 shows that in order to investigate the quantitative properties of TBVP (9)-(11), it is necessary to get the quantitative properties of zeros of $\beta(z)$ in Π_+.

3. An Example

In this section, we construct a special example which consists a transmission boundary value problem and can be obtained from (9)-(11) as a special case. Using the results that we found in section 2, we find the Jost solution , scattering function and eigenvalues of this problem given in example.

Example 3.1. *Let us consider the following TBVP which is defined by the equation*

$$y_{n-1} + y_{n+1} = 2\cos z y_n, \quad n \in \mathbb{N} \setminus \{2, 3, 4\} \tag{25}$$

with the boundary condition

$$y_0 = 0 \tag{26}$$

and the transmission conditions

$$\begin{aligned} & y_4 = \gamma_1 y_2 \\ & \triangle y_4 = \gamma_2 \nabla y_2 \quad ; \quad \gamma_1\gamma_2 \neq 0, \quad \gamma_1, \gamma_2 \in \mathbb{R}, \end{aligned} \tag{27}$$

$\{P_n(z)\}$ *and* $\{Q_n(z)\}$, $n = 0, 1, 2$ *are again the fundamental solutions of* (25) *satisfying the same initial conditions. It is clear that* $e_n(z) = e^{inz}$ *for this problem. If we write* (25) *for the solution* $\{P_n(z)\}$ *and* $\{Q_n(z)\}$ *for* $n = 1$*, we obtain*

$$P_2(z) = \lambda, \quad Q_2(z) = -\frac{1}{a_0}.$$

Besides, from (13) *and* (16)*, we find* $\beta(z)$ *and Jost solution of this TBVP*

$$\beta(z) = a_1 e^{3iz} \left(\frac{e^{2iz} - e^{iz} + 1}{\gamma_1} - \frac{e^{3iz} - e^{2iz} + e^{iz} - 1}{\gamma_2} \right) \tag{28}$$

and

$$E_n(z) = \begin{cases} a_1 e^{4iz} (\alpha_1(z) P_n(z) + \beta_1(z) Q_n(z)) & , \quad n = 0, 1, 2 \\ e^{inz} & , \quad n = 4, 5, ... \end{cases},$$

respectively, where $\alpha_1(z) = \dfrac{1}{a_0\gamma_1} - \dfrac{e^{iz}-1}{a_0\gamma_2}$ *and* $\beta_1(z) = \dfrac{\lambda-1}{\gamma_1} - \dfrac{\lambda\left(e^{iz}-1\right)}{\gamma_2}$, $\lambda = 2\cos z$. *Using* (28), *we find the scattering function of* (25)-(27)

$$\mathcal{S}(z) = e^{-6iz}\left[\frac{\gamma_2\left(e^{-2iz}-e^{-iz}+1\right)-\gamma_1\left(e^{-3iz}-e^{-2iz}+e^{-iz}-1\right)}{\gamma_2\left(e^{2iz}-e^{iz}+1\right)-\gamma_1\left(e^{3iz}-e^{2iz}+e^{iz}-1\right)}\right]$$

for $z \in \left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \setminus \{0, \pi\}$. *From the equation* (28), *we get the set of eigenvalues of* (25)-(27)

$$\sigma_d = \{\lambda \in \mathbb{C} : \lambda = 2\cos z,\ z \in \Pi_+,\ \beta(z) = 0\}.$$

Since $\beta(z) = 0$ *and* $\lambda = 2\cos z$, *it follows from* (28) *that*

$$\frac{e^{2iz}-e^{iz}+1}{\gamma_1} - \frac{e^{3iz}-e^{2iz}+e^{iz}-1}{\gamma_2} = 0$$

or

$$\frac{e^{3iz}-e^{2iz}+e^{iz}-1}{e^{2iz}-e^{iz}+1} = \frac{\gamma_2}{\gamma_1}.$$

By using last equation for $\gamma_2 = a\gamma_1$, *we find*

$$e^{3iz} - (a+1)e^{2iz} + (a+1)e^{iz} - (a+1) = 0. \tag{29}$$

Case 1: If we solve (29) for $a = 1$, we get

$$e^{iz} \simeq 1.5437 \tag{30}$$

$$e^{iz} \simeq 0.22816 - i1.11514 \tag{31}$$

$$e^{iz} \simeq 0.22816 + i1.11514. \tag{32}$$

Since the roots of the equations (30), (31) and (32) do not belong to Π_+, the problem (25)-(27) has not eigenvalues in case 1.

Case 2: For $a = 2$ in (29), we find

$$e^{iz} \simeq 2.2599$$

$$e^{iz} \simeq 0.37004 - i1.09112$$

$$e^{iz} \simeq 0.37004 + i1.09112.$$

In a similar way, the last three equations has not roots in Π_+, so TBVP (25)-(27) has not eigenvalues, too.

Note that, it is clear from (29) that a can not be never -1, i.e., $\gamma_2 \neq -\gamma_1$.

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