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Fixed Point Results for F-contractions on Space with Two Metrics

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Abstract. In this paper, taking into account two metrics on a space, we present a new fixed point theorem for F-contractions. Our theorem includes both Agarwal and O'Regan's and Wardowski's results as properly. Also we provide a nontrivial example showing this fact.

1. Introduction and preliminaries

In 2012, Wardowski [13] introduced a new concept for contraction mappings as called F-contraction by considering a class of real valued functions. Let $\mathcal{F}$ be the set of all functions $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying the following conditions:

($F1$) F is strictly increasing, i.e., for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

($F2$) For each sequence $\{a_n\}$ of positive numbers

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} F(a_n) = -\infty,$$

($F3$) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Then a self mapping T of a metric space (X,d) is said to be F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y)). \tag{1}$$

Taking in Eq.(**1**) different functions $F \in \mathcal{F}$, one gets a variety of F-contractions, some of them are of a type known in the literature. For example, let $F_1 : (0,\infty) \to \mathbb{R}$ be given by the formula $F_1(\alpha) = \ln\alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each mapping $T : X \to X$ is an F-contraction such that

$$d(Tx,Ty) \le e^{-\tau} d(x,y),\ \text{for all } x, y \in X \text{ with } Tx \ne Ty. \tag{2}$$

Therefore every Banach contraction mapping with contractive constant $0 < L < 1$ is an F-contraction with $F_1(\alpha) = \ln\alpha$ and $\tau = -\ln L > 0$. Also by the condition ($F1$), every F-contraction is a contractive mapping and

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hence it is continuous. From the Banach and Edelstein fixed point theorems, we know that every Banach contraction mapping on a complete metric space has a unique fixed point and every contractive mapping on a compact metric space has a unique fixed point. That is, passing from Banach to Edelstein fixed point theorem, when the class of mapping is expending by contractive condition, the structure of the space is restricted. Now, it may come to mind, is there any change of structure of the space when investigating the existence of fixed points of F-contractions. Therefore, Wardowski [13] proved the following result with not restricted the structure of the space:

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

In the literature, there are many generalization of Theorem 1.1 (see [2–5, 7, 9–12]) which one of them as follows:

Theorem 1.2 ([9]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(M(x, y)), \tag{3}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\},$$

then T has a unique fixed point in X provided that T or F is continuous.

On the other hand, Agarwal and O'Regan [1] presented some fixed point results for generalized contractions on space with two metrics. Unlike the conventional fixed point theory studies, here it is accepted that the mapping is contraction or contraction type according to the one metric when the space is complete for the other metric. It can be find the fundamental version of these type fixed point results in [6, 8].

Let (X, d') be a complete metric space and d be another metric on X. If $x_0 \in X$ and $r > 0$ let

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\},$$

and let $\overline{B(x_0, r)}^{d'}$ denote the d'-closure of $B(x_0, r)$. In the following we will use the notation $d \not\geq d'$, which means that $d(x, y) \not\geq d'(x, y)$ for some $x, y \in X$.

Definition 1.3. *Let (X, d) and (Y, ρ) be two metric spaces and let $T : X \to Y$ be a mapping. Then T is said to be uniformly continuous on X, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(Tx, Ty) < \varepsilon$.*

Agarwal and O'Regan [1] presented the following results:

Theorem 1.4 ([1]). *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and $T : \overline{B(x_0, r)}^{d'} \to X$ be a mapping. Suppose there exists $q \in (0, 1)$ such that for $x, y \in \overline{B(x_0, r)}^{d'}$ we have*

$$d(Tx, Ty) \leq qM(x, y). \tag{4}$$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1 - q)r \tag{5}$$

$$\text{if } d \not\geq d' \text{ assume } T \text{ is uniformly continuous from } (B(x_0, r), d) \text{ into } (X, d'),$$

and

$$\text{if } d \neq d' \text{ assume } T \text{ is continuous from } \left(\overline{B(x_0, r)}^{d'}, d'\right) \text{ into } (X, d').$$

Then T has a fixed point. That is, there exists $x \in \overline{B(x_0, r)}^{d'}$ with $x = Tx$.

The following global result can easily be deduced from Theorem 1.4.

Theorem 1.5 ([1]). *Let (X, d') be a complete metric space, d another metric on X, and $T : X \to X$ be a mapping. Suppose there exists $q \in (0, 1)$ such that for $x, y \in X$ we have Eq.(4).*

In addition assume the following two properties hold:

$$\text{if } d \ngeq d' \text{ assume } T \text{ is uniformly continuous from } (X, d) \text{ into } (X, d'),$$

and

$$\text{if } d \neq d' \text{ assume } T \text{ is continuous from } (X, d') \text{ into } (X, d').$$

Then T has a fixed point.

In this paper, by considering the both Wardowski and Maia's techniques, we present a fixed point result for single valued mapping on a space with two metrics.

2. The Result

In this section we will consider $F \in \mathcal{F}$ as continuous.

Theorem 2.1. *Let (X, d') be a complete metric space, d another metric on X and $T : X \to X$ be a mapping. Suppose $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X, \ d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(M(x, y)).$$

In addition assume the following two properties hold:

$$\text{if } d \ngeq d' \text{ assume } T \text{ is uniformly continuous from } (X, d) \text{ into } (X, d'), \tag{6}$$

and

$$\text{if } d \neq d' \text{ assume } T \text{ is continuous from } (X, d') \text{ into } (X, d'). \tag{7}$$

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary and define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for $n \in \{1, 2, ...\}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \{0, 1, 2, ...\}$, then $Tx_{n_0} = x_{n_0}$. Therefore T has a fixed point. Now let $x_{n+1} \neq x_n$ and let $d_n = d(x_{n+1}, x_n)$ for $n \in \{0, 1, 2, ...\}$. Then $d_n > 0$ for all $n \in \{0, 1, 2, ...\}$. Now using Eq.(3), we have

$$\begin{aligned}
F(d_n) &= F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \\
&\leq F(M(x_n, x_{n-1})) - \tau \\
&= F\left(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\right\}\right) - \tau \\
&\leq F(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} - \tau \\
&= F(\max\{d_{n-1}, d_n\}) - \tau.
\end{aligned} \tag{8}$$

If $d_n \geq d_{n-1}$ for some $n \in \{1, 2, ...\}$, then from Eq.(8) we have $F(d_n) \leq F(d_n) - \tau$, which is a contradiction since $\tau > 0$. Therefore $d_n < d_{n-1}$ for all $n \in \{1, 2, ...\}$ and so from Eq.(8) we have

$$F(d_n) \leq F(d_{n-1}) - \tau.$$

Thus we obtain

$$\begin{aligned}
F(d_n) &\leq F(d_{n-1}) - \tau \\
&\leq (F(d_{n-2}) - \tau) - \tau \\
&\ \vdots \\
&\leq F(d_0) - n\tau.
\end{aligned} \tag{9}$$

Letting $n \to \infty$ in Eq.(9), we get $\lim_{n\to\infty} F(d_n) = -\infty$. Hence, from $(F2)$, we have $\lim_{n\to\infty} d_n = 0$. By $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From Eq.(9), the following holds for all $n \in \{1, 2, ...\}$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0. \tag{10}$$

By Eq.(10), we obtain that

$$\lim_{n\to\infty} n d_n^k = 0.$$

Hence, there exists $n_1 \in \{1, 2, ...\}$ such that $nd_n^k \leq 1$ for all $n \geq n_1$. Therefore, we have, for all $n \geq n_1$

$$d_n \leq \frac{1}{n^{1/k}}. \tag{11}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. By Eq.(11) and using the triangular inequality for the metric, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &= d_n + d_{n+1} + ... + d_{m-1} \\ &= \sum_{j=n}^{m-1} d_j \leq \sum_{j=n}^{\infty} d_j \leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}}. \end{aligned}$$

From the convergence of the series $\sum_{j=1}^{\infty} \frac{1}{j^{1/k}}$, we obtain $\lim_{n\to\infty} d(x_n, x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X, d).

Now we claim that $\{x_n\}$ is a Cauchy sequence with respect to d'.

If $d \geq d'$ this is trivial. In that case suppose that $d \ngeq d'$. Let $\varepsilon > 0$ be given. By Eq.(6), there exists $\delta(\varepsilon) > 0$ such that

$$d'(Tx, Ty) < \varepsilon \tag{12}$$

where $x, y \in X$ and $d(x, y) < \delta$. Since $\lim_{n\to\infty} d(x_n, x_m) = 0$, then there exists $N \in \{1, 2, ...\}$ such that

$$d(x_n, x_m) < \delta \tag{13}$$

for all $n, m \geq N$. Now Eq.(12) and Eq.(13) guarantee that

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon$$

for all $n, m \geq N$ and hence $\{x_n\}$ is a Cauchy sequence with respect to d'. Since (X, d') is a complete metric space, there exists $x \in X$ with $d'(x_n, x) \to 0$ as $n \to \infty$.

We claim that $x = Tx$.

If $d \neq d'$, then

$$\begin{aligned} 0 &\leq d'(x, Tx) \\ &\leq d'(x, x_n) + d'(x_n, Tx) \\ &= d'(x, x_n) + d'(Tx_{n-1}, Tx). \end{aligned}$$

Letting $n \to \infty$ and using Eq.(7), we attain $d'(x, Tx) = 0$. Therefore x is a fixed point of T.

Now suppose $d = d'$ and $x \neq Tx$. Thus, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(Tx_{n_k}, Tx) > 0$ for all $n_k \geq n_0$. (If not, there exists $n_1 \in \mathbb{N}$ such that $x_n = Tx$ for all $n \geq n_1$, which implies that $x_n \to Tx$. This is a contradiction, since $x \neq Tx$.). From $d(Tx_{n_k}, Tx) > 0$ for all $n_k \geq n_0$, by Eq.(3), we obtain

$$\begin{aligned}
\tau + F(d(x_{n_k+1}, Tx)) &= \tau + F(d(Tx_{n_k}, Tx)) \\
&\leq F(M(x_{n_k}, x)) \\
&= F\left(\max\left\{\begin{array}{c} d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx), \\ \frac{1}{2}\left[d(x_{n_k}, Tx) + d(x, x_{n_k+1})\right] \end{array}\right\}\right).
\end{aligned}$$

Taking the limit $k \to \infty$ and using the continuity of F we have

$$\tau + F(d(x, Tx)) \leq F(d(x, Tx)),$$

which is a contradiction. Thus x is a fixed point of T. □

Remark 2.2. *If we take $d = d'$ in Theorem 2.1, then Theorem 1.2 holds.*

Remark 2.3. *If we choose $F(\alpha) = \ln \alpha$ in Theorem 2.1, then Theorem 1.5 holds.*

Theorem 2.1 yields the following version of Theorem 1.1.

Corollary 2.4. *Let (X, d') be a complete metric space, d another metric on X and $T : X \to X$ be a mapping. Suppose $F \in \mathcal{F}$ (without the continuity of F) and there exists $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

In addition assume (6) and (7) properties hold. Then T has a fixed point in X.

The following example shows Theorem 2.1 is real generalization of Theorem 1.5.

Example 2.5. *Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$, $d'(x, y) = \left|x - y\right|$ and*

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ 1 + \left|x - y\right| & , \quad x \neq y \end{cases}$$

then (X, d') is complete metric space. Define a map $T : X \to X$,

$$Tx = \begin{cases} x_1 & , \quad x = x_1 \\ x_{n-1} & , \quad x = x_n,\ n \geq 2 \end{cases}.$$

Since

$$\begin{aligned}
\sup_{n>1} \frac{d(Tx_n, Tx_1)}{M(x_n, x_1)} &= \sup_{n>1} \frac{1 + |x_{n-1} - x_1|}{\max\left\{\begin{array}{c} 1 + |x_n - x_1|, 1 + |x_n - x_{n-1}|, \\ 1, \frac{1}{2}\left[1 + |x_n - x_1| + 1 + |x_{n-1} - x_1|\right] \end{array}\right\}} \\
&= \sup_{n>1} \frac{\frac{n(n-1)}{2}}{\max\left\{\frac{n(n+1)}{2}, 1 + n, 1, \frac{n^2}{2}\right\}} = 1
\end{aligned}$$

we can not find $q \in (0, 1)$ satisfying the inequality (4). Therefore Theorem 1.5 can not be applied to this example. Now consider for $\alpha > 0$, $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$, then contractive condition of Theorem 2.1 is equivalent to the following:

$$d(Tx, Ty) > 0,\ \frac{d(Tx, Ty)}{M(x, y)} e^{d(Tx,Ty) - M(x,y)} \leq e^{-1}.$$

First observe that for all $m, n \in \mathbb{N}$

$$d(Tx_m, Tx_n) > 0 \Longleftrightarrow (m > 2 \text{ and } n = 1) \text{ or } (m > n > 1).$$

Thus we must consider the following two cases:
Case 1: *For* $m > 2$ *and* $n = 1$*, we have*

$$\frac{d(Tx_m, Tx_1)}{M(x_m, x_1)} e^{d(Tx_m, Tx_1) - M(x_m, x_1)} = \frac{m-1}{m+1} e^{-m} < e^{-1}.$$

Case 2: *For* $m > n > 1$*, we have (note that* $M(x_m, x_n) = d(x_m, x_n)$*)*

$$\begin{aligned} \frac{d(Tx_m, Tx_n)}{M(x_m, x_n)} e^{d(Tx_m, Tx_n) - M(x_m, x_n)} &= \frac{d(x_{m-1}, x_{n-1})}{M(x_m, x_n)} e^{d(x_{m-1}, x_{n-1}) - M(x_m, x_n)} \\ &= \frac{1 + \frac{(m-n)(m+n-1)}{2}}{1 + \frac{(m-n)(m+n+1)}{2}} e^{n-m} < e^{-1}. \end{aligned}$$

Therefore the contractive condition of Theorem 2.1 holds. On the other hand, since $d \geq d'$*, then (6) is satisfied and since* $\tau_{d'}$ *is discrete topology, then (7) is satisfied. As a consequence Theorem 2.1 guarantees that* T *has a fixed point in* X*.*

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