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A Fixed Point Theorem for Uniformly Lipschitzian Mappings in Modular Vector Spaces

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Abstract. We give a fixed point theorem for uniformly Lipschitzian mappings defined in modular vector spaces which have the uniform normal structure property in the modular sense. We also discuss this result in the variable exponent space

$$\ell_{p(.)} = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\lambda\ x_n|^{p(n)} < \infty\ \text{ for some } \lambda > 0\Big\}.$$

1. Introduction

In 1950, Nakano [12] introduced the theory of modular vector spaces which was further developed by Musielak/Orlicz [15] [1959, p. 49]. It is well known that norms are modulars. An illuminating example [17] of a modular vector space is given by

$$X = \{(x_n) \in \mathbb{R}^{\mathbb{N}}; \sum_{n=1}^{\infty} |\lambda x_n|^n < \infty \quad \text{for some} \quad \lambda > 0\}.$$

Indeed, this example was given by Orlicz [16] and inspired Nakano in his definition of a modular.

In this paper, we study the existence of fixed points for uniformly Lipschitzian mappings in modular vector spaces. The key idea in our approach is the modular uniform normal structure property. We show that under certain settings a uniformly Lipschitzian map has a fixed point if its modular Lipschitz constant $K < (\tilde{N}_\rho(X_\rho))^{-1/2}$, where $\tilde{N}_\rho(X_\rho)$ is the modular uniform structure coefficient of the modular vector space X_ρ.

For more on metric fixed point theory, the reader may consult the book [6]. As for the metric fixed point theory of uniformly Lipschitzian mappings, the reader may consult the two papers [3, 10].

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2. Preliminaries

In this section, we give the basic definitions as well as the notations that will be used throughout. For more details one can consult the book by Khamsi and Kozlowski [7].

Let X be a linear vector space on the field $\mathbb{R}$.

Definition 2.1. *A function $\rho : X \to [0, \infty]$ is called modular if the following hold:*

(1) $\rho(x) = 0$ if and only if $x = 0$.

(2) $\rho(\alpha x) = \rho(x)$, for every scalar α with $|\alpha| = 1$ and $x \in X$.

(3) $\rho(\alpha x + (1-\alpha)y) \leq \rho(x) + \rho(y)$, for any $\alpha \in [0,1]$ and any $x, y \in X$.

If (3) *is replaced by*

$$\rho(\alpha x + (1-\alpha)y) \leq \alpha\rho(x) + (1-\alpha)\rho(y),$$

for any $\alpha \in [0,1]$ and $x, y \in X$, then ρ is called a convex modular.

Definition 2.2. *Let ρ be a convex modular defined on X. The set $X_\rho = \{x \in X : \lim_{\alpha\to 0} \rho(\alpha x) = 0\}$ is called a modular space . The Luxemburg norm $\|.\|_\rho : X_\rho \to [0, \infty)$ is defined by*

$$\|x\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}$$

Definition 2.3. *Let ρ be a modular defined on a vector space X.*

(a) We say that a sequence $\{x_n\} \subset X_\rho$ is ρ-convergent to $x \in X_\rho$ if and only if $\lim\limits_{n\to\infty} \rho(x_n - x) = 0$. Note that the ρ-limit is unique if it exists.

(b) We say that a sequence $\{x_n\} \subset X_\rho$ is ρ-Cauchy if $\lim\limits_{n,m\to\infty} \rho(x_n - x_m) = 0$.

(c) We say that the modular space X_ρ is ρ-complete if and only if any ρ-Cauchy sequence in X_ρ is ρ-convergent.

(d) A set $C \subset X_\rho$ is called ρ-closed if for any sequence of $\{x_n\} \subset C$ which ρ-convergences to x implies that $x \in C$.

(e) A set $C \subset X_\rho$ is called ρ-bounded if $diam_\rho(C) = \sup\{\rho(x - y) : x, y \in C\} < \infty$.

(f) ρ is said to satisfy the Fatou property if $\rho(x) \leq \liminf\limits_{n\to\infty} \rho(x_n)$ whenever $\{x_n\}$ ρ-converges to x, for any x, x_n in X_ρ.

The ρ-ball $B_\rho(x, r)$, where $x \in X_\rho$ and $r \geq 0$, is defined by

$$B_\rho(x, r) = \{y \in X_\rho; \rho(x - y) \leq r\}.$$

x and r are called respectively the center and the radius of the ρ-ball $B_\rho(x, r)$. Notice that ρ satisfies Fatou property if and only if the balls are ρ-closed.

Definition 2.4. *Let τ be a topology on X_ρ. We will say that X_ρ satisfies the strong τ-Opial property if*

$$\liminf_{n\to\infty} \rho(x_n - u) = \liminf_{n\to\infty} \rho(x_n - x) + \rho(x - u),$$

for any sequence $\{x_n\}$ which τ-converges to x and any $u \in X_\rho$.

Definition 2.5. *Let ρ be a modular defined on a vector space X. We say that ρ satisfies the Δ_2-type condition if there exists $K \neq 0$ such that*

$$\rho(2x) \leq K\rho(x),$$

for any $x \in X_\rho$.

Assume that ρ is convex. We define the growth function $\omega : [0, \infty) \to [0, \infty]$ by

$$\omega(t) = \sup\left\{\frac{\rho(tx)}{\rho(x)},\ 0 < \rho(x) < \infty\right\}.$$

The following properties of the growth function are direct consequence of the definition.

Lemma 2.6. *Let X_ρ be a modular vector space. Assume that ρ is convex and satisfies the Δ_2-type condition. Then the growth function ω has the following properties:*

(1) $\omega(1) = 1$ *and* $\omega(t) \leq t$ *for* $t \leq 1$.

(2) $\omega(t) < \infty$, $\forall\ t \in [0, \infty)$.

(3) $\omega : [0, \infty) \to [0, \infty)$ *is a convex, strictly increasing function. Thus it is continuous. Therefore the function inverse ω^{-1} of ω is a strictly increasing continuous function.*

(4) $\omega(\alpha\beta) \leq \omega(\alpha)\ \omega(\beta)$, *for any $\alpha, \beta \in [0, \infty)$, which implies*

$$\omega^{-1}(\alpha)\ \omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta),$$

for any $\alpha, \beta \in [0, \infty)$.

Moreover, the growth function can be used to give an upper bound for the associated Luxemburg norm by the formula

$$\|x\|_\rho \leq \frac{1}{\omega^{-1}\left(\dfrac{1}{\rho(x)}\right)},$$

for any $x \in X_\rho$.

The proof of this fundamental lemma follows from the similar results given in modular function spaces, see Lemmas 3.1 and 3.3 in [5].

Let τ be a topology on X_ρ. For any nonempty subset $A \subset X_\rho$, the τ-closure of A is defined by

$$cl_\tau(A) = \bigcap \Big\{C;\ C \text{ is } \tau\text{-closed and contains } A\Big\}.$$

Clearly $cl_\tau(A)$ is the smallest τ-closed subset of X_ρ which contains A. In a similar fashion, we may define $cl_\rho(A)$ the ρ-closure of A.

Example 2.7. *For a function $p : \mathbb{N} \to [1, \infty)$, define the vector space*

$$\ell_{p(.)} = \left\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\lambda\ x_n|^{p(n)} < \infty\ \ \textit{for some } \lambda > 0\right\}.$$

Consider the function $\rho : \ell_{p(.)} \to [0, \infty]$ defined by

$$\rho(x) = \rho((x_n)) = \sum_{n=0}^{\infty} |x_n|^{p(n)}.$$

Then ρ defines a convex modular on $\ell_{p(.)}$. Moreover, ρ satisfies the Δ_2-type condition if and only if $p^+ = \sup_{n\in\mathbb{N}} p(n) < \infty$ and $\omega(2) \leq 2^{p^+}$. The topology τ is the coordinatewise convergence. Using some of the ideas from Brezis and Lieb [1], we prove that $\ell_{p(.)}$ satisfies the strong τ-Opial condition. It is enough to prove the strong τ-Opial condition for sequences which τ-converge to 0 in $\ell_{p(.)}$. Fix $\varepsilon(0,1)$ and $p \in [1,\infty)$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \Big||1+x|^p - |x|^p\Big| - \varepsilon|x|^p.$$

Note that we have $\lim_{|x|\to\infty} f(x) = -\infty$. In fact, we have

$$x > \frac{1}{(1+\varepsilon)^{1/p} - 1} \implies f(x) \leq 0,$$

and

$$x < -\frac{1}{1-(1-\varepsilon)^{1/p}} \implies f(x) \leq 0.$$

Moreover, it is easy to check that

$$-\frac{1}{1-(1-\varepsilon)^{1/p^+}} \leq -\frac{1}{1-(1-\varepsilon)^{1/p}} \quad and \quad \frac{1}{(1+\varepsilon)^{1/p}-1} \leq \frac{1}{(1+\varepsilon)^{1/p^+}-1},$$

for $p \in [1,p^+]$, with $p^+ < \infty$. Since $f(0) = 1$, then we have

$$\sup_{x\in\mathbb{R}} f(x) = \sup_{x\in[x_-,x_+]} f(x),$$

where

$$x_- = -\frac{1}{1-(1-\varepsilon)^{1/p^+}} \quad and \quad x_+ = \frac{1}{(1+\varepsilon)^{1/p^+}-1},$$

for $p \in [1,p^+]$, with $p^+ < \infty$. Set $M = \max_{x\in[x_-,x_+]}\Big(1,|x|,|1+x|\Big)$. Then

$$f(x) = \Big||1+x|^p - |x|^p\Big| - \varepsilon|x|^p \leq |1+x|^p + |x|^p - \varepsilon|x|^p$$

which implies $f(x) \leq |1+x|^p + (1-\varepsilon)|x|^p \leq (2-\varepsilon)M$, for any $x \in [x_-,x_+]$. Therefore, we have

$$\Big||1+x|^p - |x|^p\Big| - \varepsilon|x|^p \leq (2-\varepsilon)M$$

for any $x \in \mathbb{R}$ and any $p \in [1,p^+]$, with $p^+ < \infty$. Set

$$C_\varepsilon = \sup\left\{\Big||1+x|^p - |x|^p\Big| - \varepsilon|x|^p,\ x\in\mathbb{R} \ \ and \ \ p\in[1,p^+]\right\} \geq 1.$$

Therefore, we have

$$\Big||1+x|^p - |x|^p\Big| \leq C_\varepsilon + \varepsilon|x|^p,$$

for any $x \in \mathbb{R}$ and any $p \in [1,p^+]$, with $p^+ < \infty$. Using this inequality, we get

$$\Big||a+b|^p - |b|^p\Big| \leq C_\varepsilon|a|^p + \varepsilon|b|^p, \tag{BL}$$

for any $a,b \in \mathbb{R}$ and any $p \in [1,p^+]$, with $p^+ < \infty$. Next, we use the inequality (BL) to prove that $\ell_{p(.)}$ satisfies the strong τ-Opial property, where $p(n) \in [1,p^+]$, with $p^+ < \infty$, for $n \in \mathbb{N}$. Note that we do not assume $p(n) \geq p^- > 1$, for any $n \in \mathbb{N}$. Let $\{x_n\} \subset \ell_{p(.)}$ which τ-converges to 0. Let $u \in \ell_{p(.)}$. Assume there exists $M > 0$ such that

$\max\Big(\rho(x_n), \rho(u)\Big) \leq M$, for any $n \in \mathbb{N}$. Fix $\varepsilon > 0$. Then there exists $N \geq 1$ such that $\sum\limits_{i=N+1}^{\infty} |u(i)|^{p(i)} < \varepsilon/C_\varepsilon$, where $u = (u(i))$. Since $\{x_n\}$ τ-converges to 0, we have

$$\lim_{n\to\infty} \sum_{i=0}^{N} |x_n(i) - u(i)|^{p(i)} = \sum_{i=0}^{N} |u(i)|^{p(i)}.$$

On the other hand using the inequality (BL), we have

$$\sum_{i=N+1}^{\infty} \Big| |x_n(i) - u(i)|^{p(i)} - |x_n(i)|^{p(i)} \Big| \leq \sum_{i=N+1}^{\infty} C_\varepsilon\ |u(i)|^{p(i)} + \varepsilon |x_n(i)|^{p(i)},$$

which implies

$$\sum_{i=N+1}^{\infty} \Big| |x_n(i) - u(i)|^{p(i)} - |x_n(i)|^{p(i)} \Big| \leq \varepsilon + \varepsilon\, M = (M+1)\varepsilon.$$

We have $\rho(x_n - u) - \rho(x_n) - \rho(u) = A + B$, where

$$A = \sum_{i=0}^{N} |x_n(i) - u(i)|^{p(i)} - |x_n(i)|^{p(i)} - |u(i)|^{p(i)}$$

and $B = \sum\limits_{i=N+1}^{\infty} |x_n(i) - u(i)|^{p(i)} - |x_n(i)|^{p(i)} - |u(i)|^{p(i)}$. Since $C_\varepsilon \geq 1$, we get

$$\begin{aligned} |B| &\leq \sum_{i=N+1}^{\infty} \Big| |x_n(i) - u(i)|^{p(i)} - |x_n(i)|^{p(i)} \Big| + \sum_{i=N+1}^{\infty} |u(i)|^{p(i)} \\ &\leq (M+1)\varepsilon + \varepsilon/C_\varepsilon \\ &\leq (M+1)\varepsilon + \varepsilon = (M+2)\varepsilon. \end{aligned}$$

Using $\lim\limits_{n\to\infty} \sum\limits_{i=0}^{N} |x_n(i) - u(i)|^{p(i)} - |x_n(i)|^{p(i)} - |u(i)|^{p(i)} = 0$, we get

$$\limsup_{n\to\infty} \Big| \rho(x_n - u) - \rho(x_n) - \rho(u) \Big| \leq (M+2)\varepsilon.$$

If we let $\varepsilon \to 0+$, we get

$$\limsup_{n\to\infty} \Big| \rho(x_n - u) - \rho(x_n) - \rho(u) \Big| = 0.$$

In other words, we proved that $\ell_{p(.)}$ satisfies the strong τ-Opial property.

The normal structure property played a major role early on in the study of the fixed point problem for nonexpansive mappings [8]. Before we give the definition of the modular normal structure, we will need the following notations. Let (X_ρ, ρ) be a modular vector space and C a nonempty ρ-bounded subset of X_ρ. Set

(1) $r_\rho(x, C) = \sup\{\rho(x - y) : y \in C\}$, for $x \in X_\rho$,
(2) $R_\rho(C) = \inf\{r_\rho(x, C) : x \in C\}$,
(3) $C_\rho(C) = \{x \in C : r_\rho(x, C) = R_\rho(C)\}$.

The number $R_\rho(C)$ is called the ρ-Chebyshev radius of C (in X) and $C_\rho(C)$ is called the ρ-Chebyshev center of C.

Definition 2.8. *A modular vector space X_ρ is said to have*

1. *ρ-normal structure property if for any nonempty ρ-bounded ρ-closed convex subset C of X_ρ not reduced to one point, we have $R_\rho(C) < diam_\rho(C)$;*
2. *ρ-uniformly normal structure if there exists a constant $c \in (0,1)$ such that for any nonempty ρ-bounded ρ-closed convex subset C of X_ρ not reduced to one point, we have $R_\rho(C) \leq c\ diam_\rho(C)$.*
3. *The normal structure coefficient $\tilde{N}_\rho(X_\rho)$ of X_ρ is the number defined by*

$$\tilde{N}_\rho(X_\rho) = \sup\ \frac{R_\rho(C)}{diam_\rho(C)},$$

where the supremum is taken over any ρ-bounded ρ-closed convex not reduced to one point subset $C \subset X_\rho$.

Clearly ρ-uniformly normal structure implies the ρ-normal structure. Notice that the modular vector space X_ρ has a ρ-uniform normal structure if and only if $\tilde{N}_\rho(X_\rho) < 1$. Historically, the main example of a Banach space which enjoys the uniform normal structure property is the class of uniformly convex spaces. Let us discuss this connection in the context of modular vector spaces. First, recall that since the beginning of the theory of modular vector spaces, the concept of modular uniform convexity was defined and investigated [9, 11–14, 18].

Definition 2.9. *[7, 13] Let (X_ρ, ρ) be a modular vector space. Let $r > 0$ and $\varepsilon > 0$. Define*

$$D(r,\varepsilon) = \Big\{(x,y);\ x,y \in X_\rho, \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq \varepsilon r\Big\}.$$

If $D(r,\varepsilon) \neq \emptyset$, let

$$\delta_\rho(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\left(\frac{x+y}{2}\right);\ (x,y) \in D(r,\varepsilon)\right\}.$$

If $D(r,\varepsilon) = \emptyset$, we set $\delta_\rho(r,\varepsilon) = 1$. We say that ρ satisfies the uniform convexity (UC) if for every $r > 0$ and $\varepsilon > 0$, we have $\delta_\rho(r,\varepsilon) > 0$.

Note, that for every $r > 0$, we have $D(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough. The following result is the modular analogue to the classical link between uniform convexity and uniform normal structure property.

Lemma 2.10. *Let (X_ρ, ρ) be a modular vector space. Assume*

$$\limsup_{\varepsilon \to 1-}\ \inf_{r>0} \delta_\rho(r,\varepsilon) > 0.$$

Then X_ρ has the ρ-uniform normal structure property and

$$\tilde{N}_\rho(X_\rho) \leq 1 - \limsup_{\varepsilon \to 1-}\ \inf_{r>0} \delta_\rho(r,\varepsilon).$$

Proof. Set $\eta = \limsup_{\varepsilon \to 1-}\ \inf_{r>0} \delta_\rho(r,\varepsilon)$. Let C be a ρ-bounded ρ-closed convex not reduced to one point subset of X_ρ. Hence $diam_\rho(C) > 0$. Fix $\varepsilon \in (0,1)$. There exists $x, y \in C$ such that $\rho(x-y) > diam_\rho(C)\varepsilon$. For any $z \in C$, we have $\rho(x-z) \leq diam_\rho(C)$ and $\rho(y-z) \leq diam_\rho(C)$. By definition of $\delta_\rho(diam_\rho(C),\varepsilon)$, we get

$$\rho\left(\frac{x+y}{2} - z\right) \leq diam_\rho(C)\ (1 - \delta_\rho(diam_\rho(C),\varepsilon)).$$

Hence

$$R_\rho(C) \leq r_\rho\left(\frac{x+y}{2}, C\right) \leq diam_\rho(C)\ (1 - \delta_\rho(diam_\rho(C),\varepsilon)),$$

which implies

$$\frac{R_\rho(C)}{diam_\rho(C)} \leq 1 - \inf_{r>0} \delta_\rho(r, \varepsilon).$$

Since the right hand side is independent of the subset C, we conclude that $\tilde{N}_\rho(X_\rho) \leq 1 - \inf_{r>0} \delta_\rho(r,\varepsilon)$. If we let $\varepsilon \to 1-$, we get $\tilde{N}_\rho(X_\rho) \leq 1 - \eta$. Since $\eta > 0$, we conclude that $\tilde{N}_\rho(X_\rho) < 1$, i.e., X_ρ has the ρ-uniform normal structure property. □

In the next example, we discuss the case of $\ell_{p(\cdot)}$ spaces (see Example 2.7) which satisfy the assumptions of Lemma 2.10.

Example 2.11. *Consider the function $p : \mathbb{N} \to [2, \infty)$ and the vector space*

$$\ell_{p(.)} = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\lambda\, x_n|^{p(n)} < \infty \ \text{ for some } \lambda > 0\Big\}.$$

The modular function ρ is defined by $\rho(x) = \rho((x_n)) = \sum_{n=0}^{\infty} |x_n|^{p(n)}$. Assume that $p^+ = \sup_{n\in\mathbb{N}} p(n) < \infty$. The following inequality was first used by Clarkson [2]

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\Big(|a|^p + |b|^p\Big),$$

for any $a, b \in \mathbb{R}$, provided $p \geq 2$. From this inequality, we easily deduce the following

$$\rho\left(\frac{x+y}{2}\right) + \frac{1}{2^{p_+}}\ \rho(x-y) \leq \frac{\rho(x)+\rho(y)}{2},$$

for any $x, y \in \ell_{p(.)}$. This inequality will imply

$$\delta_\rho(r,\varepsilon) \geq \frac{\varepsilon}{2^{p_+}},$$

for any $r > 0$ and $\varepsilon > 0$. Obviously, we have $\limsup_{\varepsilon\to 1-}\ \inf_{r>0} \delta_\rho(r,\varepsilon) \geq 1/2^{p_+} > 0$.

3. Main Result

Throughout this section, X_ρ stands for a complete modular vector space and ρ is convex. We will assume that ρ satisfies the Fatou property. Let τ be a topology on X_ρ. The following lemma is useful for the proof of the main result of this work.

Lemma 3.1. *Assume X_ρ satisfies the strong τ-Opial property. Let C be ρ-bounded and τ-sequentially compact nonempty subset of X_ρ. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C. Then there exists $y \in \bigcap_{n\geq 1} cl_\tau(conv\{y_i\}_{i\geq n}) \cap C$ such that*

$$\limsup_{n\to\infty} \rho(y - x_n) \leq \limsup_{i\to\infty}\ \limsup_{n\to\infty} \rho(y_i - x_n),$$

where $cl_\tau(conv(A))$ is the smallest convex τ-closed subset of X_ρ which contains A.

Proof. Since C is τ-sequentially compact and ρ-bounded subset, there is a subsequence $\{y_{\phi(n)}\}$ of $\{y_n\}$ such that $\{y_{\phi(n)}\}$ τ-converges to $y \in C$. Moreover, there exists a τ-convergent subsequence $\{x_{\psi(n)}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} \rho(x_{\psi(n)} - y) = \limsup_{n\to\infty} \rho(x_n - y)$. Set $x \in C$ be the τ-limit of $\{x_{\psi(n)}\}$. Fix $n \geq 1$. Then for any $m \geq n$, we have $y_{\phi(m)} \in cl_\tau(conv\{y_i\}_{i\geq n}) \cap C$ which is a τ-closed subset. Hence $y \in cl_\tau(conv\{y_i\}_{i\geq n}) \cap C$, for any $n \geq 1$. By using the strong τ-Opial property, we get

$$\begin{aligned} \limsup_{n\to\infty} \rho(y_i - x_n) &\geq \liminf_{n\to\infty} \rho(y_i - x_{\psi(n)}) \\ &= \liminf_{n\to\infty} \rho(x_{\psi(n)} - x) + \rho(x - y_i). \end{aligned}$$

Hence $\limsup_{i\to\infty} \limsup_{n\to\infty} \rho(y_i - x_n) \geq \liminf_{n\to\infty} \rho(x_{\psi(n)} - x) + \limsup_{i\to\infty} \rho(x - y_i)$. On the other hand,we have

$$\begin{aligned} \limsup_{i\to\infty} \rho(y_i - x) &\geq \liminf_{i\to\infty} \rho(y_{\phi(i)} - x) \\ &= \liminf_{i\to\infty} \rho(y_{\phi(i)} - y) + \rho(y - x), \end{aligned}$$

which implies

$$\begin{aligned} \limsup_{i\to\infty} \limsup_{n\to\infty} \rho(y_i - x_n) &\geq \liminf_{n\to\infty} \rho(x_{\psi(n)} - x) + \limsup_{i\to\infty} \rho(x - y_i) \\ &= \liminf_{n\to\infty} \rho(x_{\psi(n)} - x) + \rho(y - x), \\ &= \liminf_{n\to\infty} \rho(x_{\psi(n)} - y) \\ &= \lim_{n\to\infty} \rho(x_{\psi(n)} - y) \\ &= \limsup_{n\to\infty} \rho(x_n - y). \end{aligned}$$

□

Lemma 3.2. *Assume X_ρ satisfies the strong τ-Opial property. Let C be ρ-bounded and τ-sequentially compact convex nonempty subset of X_ρ. Let c be a constant such that $c > \tilde{N}_\rho(X_\rho)$. Then for any sequence $\{x_n\}$ in C, there exists $x \in C$ such that*

(i) $\limsup_{n\to\infty} \rho(x - x_n) \leq c\, diam_\rho(\{x_n\})$,

(ii) $\rho(y - x) \leq \limsup_{n\to\infty} \rho(y - x_n)$, *for all $y \in C$.*

Proof. Let $\{x_n\}$ be a sequence in C. Without loss of generality, we assume $diam_\rho(\{x_n\}) > 0$. Set $A_n = cl_\tau(conv\{x_i\}_{i\geq n})$, for any $n \geq 1$. Our assumptions on C imply that $A_n \subset C$, for any $n \geq 1$. Since C is τ-sequentially compact, $A = \bigcap_{n=1}^{\infty} A_n$ is a nonempty subset of C. For any $x \in A$ and $y \in C$, we have

$$\rho(y - x) \leq r_\rho(y, A) \leq r_\rho(y, A_n) \leq \sup_{i\geq n} \rho(y - x_i),$$

because the ρ-balls are τ-closed. Hence

$$\rho(y - x) \leq \limsup_{n\to\infty} \rho(y - x_i).$$

Hence (ii) holds for any $x \in A$. Next, we prove the existence of an $x \in A$ for which (i) holds. Let $\varepsilon > 0$ such that

$$\tilde{N}_\rho(X_\rho) diam_\rho(\{x_n\}) + \varepsilon \leq c\, diam_\rho(\{x_n\}).$$

By definition of the Chebyshev radius, for any $n \geq 1$, there exists $y_n \in A_n$ such that

$$\begin{aligned} r_\rho(y_n, A_n) &\leq R_\rho(A_n) + \varepsilon \\ &\leq \tilde{N}_\rho(X_\rho) diam_\rho(A_n) + \varepsilon \\ &\leq \tilde{N}_\rho(X_\rho) diam_\rho(\{x_i\}) + \varepsilon \\ &\leq c\, diam_\rho(\{x_i\}), \end{aligned}$$

which implies $\sup_{i\geq m} \rho(y_n - x_i) \leq c\, diam_\rho(\{x_i\})$, for any $m \geq n$. Hence

$$\limsup_{i\to\infty} \rho(y_n - x_i) \leq c\, diam_\rho(\{x_i\}).$$

Using Lemma 3.1, there exists $x \in A$ such that

$$\limsup_{n\to\infty} \rho(x - x_n) \leq \limsup_{n\to\infty}\ \limsup_{i\to\infty} \rho(y_n - x_i),$$

which implies $\limsup_{n\to\infty} \rho(x - x_n) \leq c\, diam_\rho(\{x_i\})$. □

Next, we give the main result of this work. This result was initially discovered in [4] in modular function spaces. For the metric version, the reader may refer to [3, 10].

Theorem 3.3. *Assume X_ρ satisfies the strong τ-Opial property and ρ satisfies the Δ_2-type condition. Assume $\tilde{N}_\rho(X_\rho) < 1$. Let C be ρ-bounded, ρ-closed and τ-sequentially compact convex nonempty subset of X_ρ. Let $T : C \to C$ be a uniformly Lipschitzian mapping with $K < (\tilde{N}_\rho(X_\rho))^{-1/2}$, then T has a fixed point.*

Proof. Without loss of generality, we may assume $K > 1$ since $(\tilde{N}_\rho(X_\rho))^{-1/2} > 1$. Pick $c \in (1, \tilde{N}_\rho(X_\rho))$, i.e., $1 < c < \tilde{N}_\rho(X_\rho)$, such that $1 < K < c^{-1/2}$. Fix $x_0 \in C$. Using Lemma 3.2, we construct inductively a sequence $\{x_n\}$ such that

(1) $\limsup_{n\to\infty} \rho(x_{i+1} - T^n(x_i)) \leq c\, diam_\rho(\{T^n(x_i)\})$,
(2) $\rho(x_{i+1} - y) \leq \limsup_{n\to\infty} \rho(y - T^n(x_i)))$, for any $y \in C$.

for any $i \in \mathbb{N}$. Set $D_i = \limsup_{n\to\infty} \rho(x_{i+1} - T^n(x_i))$, for any $i \in \mathbb{N}$. For $n \geq m$, we have

$$\begin{array}{rcl} \rho(T^n(x_i) - T^m(x_i)) & \leq & K\, \rho(x_i - T^{n-m}(x_i)) \\ & \leq & K\, \limsup_{s\to\infty} \rho(T^s(x_{i-1}) - T^{n-m}(x_i)) \\ & \leq & K^2\, \limsup_{s\to\infty} \rho(T^s(x_{i-1}) - (x_i)) \\ & \leq & K^2\, D_{i-1}, \end{array}$$

which implies $diam_\rho(\{T^n(x_i)\}) \leq K^2\, D_{i-1}$, for any $i \geq 1$. Hence

$$D_i = \limsup_{n\to\infty} \rho(x_{i+1} - T^n(x_i)) \leq c\, \delta_\rho(\{T^n(x_i)\}) \leq c\, K^2\, D_{i-1},$$

for any $i \geq 1$, which implies $D_i = (c\, K^2)^i\, D_0$, for any $i \in \mathbb{N}$. Set $h = c\, K^2 < 1$. On the other hand, we have

$$\begin{array}{rcl} \rho(x_{i+1} - x_i) & = & \rho\left(2\, \dfrac{x_{i+1} - T^n(x_i) + T^n(x_i) - x_i}{2}\right) \\ & \leq & \omega(2)\left(\rho(x_{i+1} - T^n(x_i)) + \rho(x_i - T^n(x_i))\right) \\ & \leq & \omega(2)\left(\rho(x_{i+1} - T^n(x_i)) + \limsup_{m\to\infty} \rho(T^m(x_{i-1}) - T^n(x_i))\right) \\ & \leq & \omega(2)\left(\rho(x_{i+1} - T^n(x_i)) + K\, \limsup_{m\to\infty} \rho(T^{m-n}(x_{i-1}) - x_i)\right) \\ & \leq & \omega(2)\left(\rho(x_{i+1} - T^n(x_i)) + K\, D_{i-1}\right), \end{array}$$

for any $i \geq 1$. If we let $n \to \infty$, we get

$$\rho(x_{i+1} - x_i) \leq \omega(2)\, (D_i + K\, D_{i-1}) \leq \omega(2) h^{i-1}\, K(c\, K + 1)\, D_0 = A\, h^i,$$

for any $i \in \mathbb{N}$, where $A = \omega(2)\, K(c\, K + 1)\, D_0/h$. Using the properties of the growth function ω (see Lemma 2.6), we get

$$\omega^{-1}\left(\frac{1}{A}\right)\, \omega^{-1}\left(\frac{1}{h}\right)^i \leq \omega^{-1}\left(\frac{1}{\rho(x_{i+1} - x_i)}\right)$$

which implies

$$\|x_{i+1} - x_i\|_\rho \leq \frac{1}{\omega^{-1}(A^{-1})}\, \left(\frac{1}{\omega^{-1}(h^{-1})}\right)^i,$$

for any $i \in \mathbb{N}$. Since $h < 1$, we have $1 = \omega^{-1}(1) < \omega^{-1}(h^{-1})$. Hence the series $\sum \|x_{i+1} - x_i\|_\rho$ is convergent which implies that $\{x_n\}$ is Cauchy in the Banach space $(X_\rho, \|.\|_\rho)$. Hence $\{x_n\}$ converges to some $x \in X_\rho$. Since ρ satisfies the Δ_2-type condition, then $\{x_n\}$ ρ-converges to x as well. Hence $x \in C$. Let us finish the proof of Theorem 3.3 by proving that x is in fact a fixed point of T. Indeed, we have

$$\begin{aligned}\rho(x - T(x)) &\leq \omega(3)\left(\rho(x - x_{i+1}) + \rho(x_{i+1} - T^n(x_i)) + \rho(T^n(x_i) - T(x))\right)\\ &\leq \omega(3)\Big(\rho(x - x_{i+1}) + \rho(x_{i+1} - T^n(x_i)) + K\,\rho(x - T^{n-1}(x_i))\Big)\end{aligned}$$

Since

$$\rho(x - T^{n-1}(x_i))) \leq \omega(2)\left(\rho(x_{i+1} - T^{n-1}(x_i)) + \rho(x_{i+1} - x)\right),$$

if we let $n \to \infty$, we get

$$\limsup_{n\to\infty} \rho(x - T^{n-1}(x_i))) \leq \omega(2)\left(\limsup_{n\to\infty} \rho(x_{i+1} - T^{n-1}(x_i)) + \rho(x_{i+1} - x)\right),$$

which implies

$$\rho(x - Tx) \leq \omega(3)\left(\rho(x - x_{i+1}) + D_i + K\,\omega(2)\,(D_i + \rho(x_{i+1} - x_i))\right),$$

for any $i \in \mathbb{N}$. Finally, if $i \to \infty$, we obtain $\rho(x - T(x)) = 0$, i.e. $T(x) = x$. □

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