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On Some Inequalities for Submanifolds of Bochner-Kaehler Manifolds

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Abstract. Chen established sharp inequalities between certain Riemannian invariants and the squared norm of mean curvature for submanifolds in real space form as well as in complex space form. In this paper we generalize Chen inequalities for submanifolds of Bochner-Kaehler manifolds. Moreover, we study CR-warped product submanifolds of Bochner-Kaehler manifold and establish an inequality for the Laplacian of the warping function, from which we conclude some obstructions to the existence of such immersions.

1. Introduction

In [7], Chen established sharp inequality for a submanifold in a real space form involving intrinsic invariants of the submanifolds and squared norm of mean curvature, the main extrinsic invariant and in [2], Chen obtained the same inequality for complex space form. After that many research articles have been published by different authors for different submanifolds and ambient spaces in complex as well as in contact version(see[4]). In this article we obtain these inequalities for submanifolds in Bochner-Kaehler manifold.

In [10] Bishop and O'Neil initiated the theory of warped product submanifold as a generalization of pseudo-Riemannian product manifold. In 2001, Chen studied warped product CR-submanifold in a Kaehler manifold $\overline{\mathcal{N}}$ and introduced the notion of CR-warped product[5]. He proved that there does not exist warped product CR-submanifold in the form $\mathcal{N}_\perp \times_f \mathcal{N}_\top$ other than CR-products such that $\mathcal{N}_\top$ is a holomorphic submanifold and $\mathcal{N}_\perp$ is a totally real submanifold of $\overline{\mathcal{N}}$. In this paper, we study warped product CR-submanifolds of Bochner-Kaehler $\overline{\mathcal{N}}$ in the form $\mathcal{N}_\top \times_f \mathcal{N}_\perp$, where $\mathcal{N}_\top$ is a holomorphic submanifold and $\mathcal{N}_\perp$ is a totally real submanifold of $\overline{\mathcal{N}}$. We establish the inequality for the Laplacian of the warping function f in terms of mean curvature for warped products isometrically immersed in Bochner-Kaehler manifold $\overline{\mathcal{N}}$. We also conclude some corollaries giving the obstructions to the existence of such immersions.

2. Preliminaries

Let $\mathcal{N}$ be a n-dimensional submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$ of dimension $2m$. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on $\mathcal{N}$ and $\overline{\mathcal{N}}$ respectively. Let J be the complex structure on $\overline{\mathcal{N}}$. Then the

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Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(X, Y), \tag{1}$$

$$\overline{\nabla}_X V = -B_V X + \nabla^{\perp}_X Y, \tag{2}$$

for all X, Y tangent to $\mathcal{N}$ and vector field V normal to $\mathcal{N}$. Where ω, $\nabla^{\perp}_X$, B_V denotes the second fundamental form, normal connection and the shape operator respectively. The second fundamental form and the shape operator are related by

$$g(\omega(X, Y), V) = g(B_V X, Y). \tag{3}$$

Let R be the curvature tensor of $\mathcal{N}$, Then the Gauss equation is given by [7]

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(\omega(X, W), \omega(Y, Z)) - g(\omega(X, Z), \omega(Y, W))$$

for any vector fields X, Y, Z, W tangent to $\mathcal{N}$.

Let $x \in \mathcal{N}$ and $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space $T_x\mathcal{N}$ and $\{e_{n+1}, ..., e_{2m}\}$ be the orthonormal basis of $T^{\perp}\mathcal{N}$. We denote by $\mathcal{H}$, the mean curvature vector at x, that is

$$\mathcal{H}(x) = \frac{1}{n}\sum_{i=1}^{n} \omega(e_i, e_i), \tag{4}$$

Also, we set

$$\omega^r_{ij} = g(\omega(e_i, e_j), e_r), \qquad i, j \in \{1, ..., n\}, \quad r \in \{n + 1, ..., 2m\}$$

and

$$||\omega||^2 = \sum_{i,j=1}^{n} (\omega(e_i, e_j), \omega(e_i, e_j)). \tag{5}$$

For any $x \in \mathcal{N}$ and $X \in T_x\mathcal{N}$, we put $JX = TX + FX$, where TX and FX are the tangential and normal components of JX, respectively.

We denote by

$$||T||^2 = \sum_{i,j=1}^{n} g^2(Te_i, e_j).$$

Let $\mathcal{N}$ be a Riemannian manifold. Denote by $\mathcal{K}(\pi)$ the sectional curvature of $\mathcal{N}$ of the plane section $\pi \subset T_x\mathcal{N}, x \in \mathcal{N}$. The scalar curvature ρ for an orthonormal basis$\{e_1, e_2, ..., e_n\}$ of the tangent space $T_x\mathcal{N}$ at x is defined by

$$\rho(x) = \sum_{i<j} K(e_i \wedge e_j).$$

The curvature tensor of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$ is given by [9]

$$\begin{aligned}\overline{R}(X, Y, Z, W) \;=\; & L(Y, Z)g(X, W) - L(X, Z)g(Y, W) + L(X, W)g(Y, Z) \\ & -L(Y, W)g(X, Z) + M(Y, Z)g(JX, W) - M(X, Z)g(JY, W) \\ & +M(X, W)g(JY, Z) - M(Y, W)g(JX, Z) \\ & -2M(X, Y)g(JZ, W) - 2M(Z, W)g(JX, Y)\end{aligned} \tag{6}$$

where

$$L(Y, Z) = \frac{1}{2n + 4}\overline{Ric}(Y, Z) - \frac{\overline{\rho}}{2(2n + 2)(2n + 4)}g(Y, Z), \tag{7}$$

$$M(Y, Z) = -L(Y, JZ), \tag{8}$$

$$L(Y, Z) = L(Z, Y), \qquad L(Y, Z) = L(JY, JZ), \qquad L(Y, JZ) = -L(JY, Z), \tag{9}$$

Ric and ρ are the Ricci tensor and scalar curvature of $\mathcal{N}$.

Definition 2.1. *The Kaehler manifold $\overline{\mathcal{N}}$ is said to be Bochner-Kaehler if its Bochner curvature tensor vanishes. These spaces are also known as Bochner-flat manifolds.*

Lemma 2.2. *[7] Let $n \geq 2$ and $x_1, x_2, ..., x_n$, b be real numbers such that*

$$(\sum_{i=1}^{n} x_i)^2 = (n-1)(\sum_{i=1}^{n} x_i^2 + b)$$

then $2x_1x_2 \geq b$, with equality holds if and only if

$$x_1 + x_2 = x_3 = ... = x_n.$$

In [1] A. Bejancu introduced the notion of CR-submanifolds, which is the generalization of invariant and anti-invariant submanifolds. In [3] B. Y. Chen introduced the notion of slant submanifolds as a generalization of CR-submanifolds.

Definition 2.3. *A submanifold $\mathcal{N}$ of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$ is said to be a slant submanifold if for any $x \in \mathcal{N}$ and $X \in T_x\mathcal{N}$, the angle between JX and $T_x\mathcal{N}$ is constant, i.e., the angle does not depend on the choice of $x \in \mathcal{N}$ and $X \in T_x\mathcal{N}$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of $\mathcal{N}$ in $\overline{\mathcal{N}}$.*

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively and when $0 < \theta < \frac{\pi}{2}$, then slant submanifold is called proper slant submanifold.

Definition 2.4. *A Riemannian manifold $\mathcal{N}$ is said to be Einstein manifold if the Ricci tensor is proportional to the metric tensor, that is, $Ric(X, Y) = \lambda g(X, Y)$ for some constant λ.*

Definition 2.5. *Let M and N be two Riemannian manifolds with Riemannian metrics g_M and g_N respectively and $f > 0$, a differentiable function on M. Consider the product manifold $M \times N$ with its projection $\pi : M \times N \to M$ and $\sigma : M \times N \to N$. The warped product of $\mathcal{N} : M \times_f N$ is the manifold $M \times N$ equipped with the Riemannian structure such that*

$$g(X, Y) = g(\pi_* X, \pi_* Y) + (f \circ g)^2 g(\sigma_* X, \sigma_* Y)$$

for any $X \in T_x\mathcal{N}$. The function f is called the warping function of the warped product.

Let $\mathcal{N} = \mathcal{N}_\top \times_f \mathcal{N}_\perp$ be the warped product CR-submanifolds of Bochner-Kaehler manifold $\overline{\mathcal{N}}$ such that the invariant distribution is $D = T\mathcal{N}_\top$ and anti-invariant distribution is $D^\perp = T\mathcal{N}_\perp$, where $f : \mathcal{N}_\top \longrightarrow \mathbb{R}$. Then the metric g on $\mathcal{N}$ is given by [5]

$$g(X, Y) = \langle \pi_* X, \pi_* Y \rangle + (f \circ \pi)^2 \langle \sigma_* X, \sigma_* Y \rangle$$

where π and σ are the projection maps from $\mathcal{N}$ onto $\mathcal{N}_\top$ and $\mathcal{N}_\perp$ respectively.

3. B. Y. Chen Inequalities

In this section, we obtain B. Y. Chen inequalities for submanifolds of a Bochner-Kaehler manifolds.
First we have,

Theorem 3.1. *Let $\mathcal{N}$ be a submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then, for each point $x \in \mathcal{N}$ and each plane section $\pi \subset T_x\mathcal{N}$, we have*

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n + 10 - 3n^2 + 3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2 - \frac{6}{2(2n+4)}\overline{Ric}(e_i, Je_j)g(e_i, Je_j). \tag{10}$$

Equality holds if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m}\}$ of $T^{\perp}\mathcal{N}$ such that the shape operators takes the following forms

$$B_{n+1} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi \end{pmatrix}, \alpha + \beta = \xi \tag{11}$$

and

$$B_r = \begin{pmatrix} \omega_{11}^r & \omega_{12}^r & 0 & \cdots & 0 \\ \omega_{12}^r & -\omega_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r = n+2, ..., 2m. \tag{12}$$

Proof. Using Gauss equation, the Riemannian curvature tensor of $\mathcal{N}$ is given by

$$\begin{aligned} R(X,Y,Z,W) &= L(Y,Z)g(X,W) - L(X,Z)g(Y,W) + L(X,W)g(Y,Z) \\ &\quad - L(Y,W)g(X,Z) + M(Y,Z)g(JX,W) - M(X,Z)g(JY,W) \\ &\quad + M(X,W)g(JY,Z) - M(Y,W)g(JX,Z) - 2M(X,Y)(JZ,W) \\ &\quad - 2M(Z,W)g(JX,Y) + g(\omega(X,W),\omega(Y,Z)) - g(\omega(X,Z),\omega(Y,W)) \end{aligned}$$

for any X, Y, Z, W $\in$ T$\mathcal{N}$.

$$\begin{aligned} \sum_{i,j} R(e_i,e_j,e_j,e_i) &= L(e_j,e_j)g(e_i,e_i) - L(e_i,e_j)g(e_j,e_i) + L(e_i,e_i)g(e_j,e_j) \\ &\quad - L(e_j,e_i)g(e_i,e_j) + M(e_j,e_j)g(Je_i,e_i) - M(e_i,e_j)g(Je_j,e_i) \\ &\quad + M(e_i,e_i)g(Je_j,e_j) - M(e_j,e_i)g(Je_i,e_j) - 2M(e_i,e_j)(Je_j,e_i) \\ &\quad - 2M(e_j,e_i)g(Je_i,e_j) + g(\omega(e_i,e_i),\omega(e_j,e_j)) - g(\omega(e_i,e_j),\omega(e_j,e_i)) \\ &= L(e_j,e_j)g(e_i,e_i) - L(e_i,e_j)g(e_j,e_i) + L(e_i,e_i)g(e_j,e_j) \\ &\quad - L(e_j,e_i)g(e_i,e_j) - L(e_j,Je_j)g(Je_i,e_i) + L(e_i,Je_j)g(Je_j,e_i) \\ &\quad - L(e_i,Je_i)g(Je_j,e_j) + L(e_j,Je_i)g(Je_i,e_j) + 2L(e_i,Je_j)(Je_j,e_i) \\ &\quad + 2L(e_j,Je_i)g(Je_i,e_j) + g(\omega(e_i,e_i),\omega(e_j,e_j)) - g(\omega(e_i,e_j),\omega(e_j,e_i)). \end{aligned} \tag{13}$$

Using (9), (4) and (5) in (13), we have

$$\begin{aligned} \sum_{i,j} R(e_i,e_j,e_j,e_i) &= 2nL(e_i,e_i) - 2L(e_i,e_j)g(e_i,e_j) + 6L(e_i,Je_j)g(e_i,Je_j) \\ &\quad + n^2\|\mathcal{H}\|^2 - \|\omega\|^2. \end{aligned}$$

Which simplifies to,

$$2\rho = 2(n-1)L(e_i,e_i)+6L(e_i,Je_j)g(e_i,Je_j)+n^2\|\mathcal{H}\|^2-\|\omega\|^2. \tag{14}$$

Combining (7) and (14), we have

$$\begin{aligned}2\rho &= \frac{2(n-1)}{2n+4}\overline{Ric}(e_i,e_i)-\frac{2(n-1)\overline{\rho}}{2(2n+2)(2n+4)}g(e_i,e_i)\\ &+\frac{6}{2n+4}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)-\frac{6\overline{\rho}}{2(2n+2)(2n+4)}g(e_i,Je_j)g(e_i,Je_j)\\ &+n^2\|\mathcal{H}\|^2-\|\omega\|^2.\end{aligned}$$

or

$$\begin{aligned}2\rho &= \frac{6n^2+2n-8-6\|T\|^2}{2(2n+2)(2n+4)}\overline{\rho}+\frac{6}{2n+4}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)\\ &+n^2\|\mathcal{H}\|^2-\|\omega\|^2.\end{aligned}$$

Denoting by

$$\epsilon = 2\rho-\Big(\frac{6n^2+2n-8-6\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho}-\frac{n^2(n-2)}{n-1}\|\mathcal{H}\|^2-\frac{6}{2n+4}\overline{Ric}(e_i,Je_j)g(e_i,Je_j),$$

we obtain

$$\epsilon = n^2\|\mathcal{H}\|^2-\|\omega\|^2-\frac{n^2(n-2)}{n-1}\|\mathcal{H}\|^2.$$

or

$$n^2\|\mathcal{H}\|^2=(n-1)(\epsilon+\|\omega\|^2). \tag{15}$$

For chosen orthonormal basis, the above equation takes the form

$$(\sum_{i=1}^{n}\omega_{ii}^{n+1})^2=(n-1)\left[\sum_{i=1}^{n}(\omega_{ii}^{n+1})^2+\sum_{i\neq j}(\omega_{ij}^{n+1})^2+\sum_{r=n+1}^{2m}\sum_{i,j=1}^{n}(\omega_{ij})^2+\epsilon\right]. \tag{16}$$

Using lemma 2.2 in (16), we have

$$2\omega_{11}^{n+1}\omega_{22}^{n+1}\geq\sum_{i\neq j}(\omega_{ij}^{n+1})^2+\sum_{r=n+1}^{2m}\sum_{i,j=1}^{n}(\omega_{ij}^{r})^2+\epsilon. \tag{17}$$

On the other hand, from Gauss equation we obtain

$$\mathcal{K}(\pi)=L(e_2,e_2)+L(e_1,e_1)+g(\omega(e_1,e_1),\omega(e_2,e_2)-g(\omega(e_1,e_2),\omega(e_2,e_1)). \tag{18}$$

Combing (7) and (18), we derive

$$\mathcal{K}(\pi)=\frac{4n+3}{(2n+2)(2n+4)}\overline{\rho}+\omega_{11}^{n+1}\omega_{22}^{n+1}+\sum_{r=n+2}^{2m}\omega_{11}^{r}\omega_{22}^{r}-\sum_{r=n+1}^{2m}(\omega_{12}^{r})^2. \tag{19}$$

Incorporating (17) in (19), we arrive at the inequality

$$\begin{aligned}\mathcal{K}(\pi) \quad \geq \quad & \frac{1}{2}\sum_{i\neq j}(\omega_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+1}^{2m}\sum_{i,j=1}^{n}(\omega_{ij}^{r})^2 + \frac{1}{2}\epsilon \\ & + \frac{4n+3}{(2n+2)(2n+4)}\overline{\rho} + \sum_{r=n+2}^{2m}\omega_{11}^{r}\omega_{22}^{r} - \sum_{r=n+1}^{2m}(\omega_{12}^{r})^2.\end{aligned}$$

Which implies that

$$\mathcal{K}(\pi) \geq \frac{4n+3}{(2n+2)(2n+4)}\overline{\rho} + \frac{1}{2}\epsilon.$$

or

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n+10-3n^2+3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2 - \frac{6}{2(2n+4)}\overline{Ric}(e_i, Je_j)g(e_i, Je_j). \tag{20}$$

If the equality in (10) at a point p holds, then the inequality (20) become equality. In this case, we have

$$\begin{cases}\omega_{1j}^{n+1} = \omega_{2j}^{n+1} = \omega_{ij}^{n+1} = 0, \qquad i \neq j > 2, \\ \omega_{ij}^{r} = 0, \forall i \neq j, \quad i, j = 3, ..., 2m, \quad r = n+1, ..., 2m, \\ \omega_{11}^{r} + \omega_{22}^{r} = 0, \forall r = n+2, ..., 2m, \\ \omega_{11}^{n+2} + \omega_{22}^{n+1} = ... = \omega_{11}^{m} + \omega_{22}^{m} = 0.\end{cases}$$

Now, if we choose e_1, e_2 such that $\omega_{12}^{n+1} = 0$ and we denote by $\alpha = \omega_{11}^{r}, \beta = \omega_{22}^{r}$, $\xi = \omega_{33}^{n+1} = ... = \omega_{33}^{r}$. Therefore by choosing the suitable orthonormal basis the shape operators take the desired forms. □

Corollary 3.2. *Let $\mathcal{N}$ be a submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$ which is Einstein. Then, for each point $x \in \mathcal{N}$ and each plane section $\pi \subset T_x\mathcal{N}$, we have*

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n+10-3n^2+3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2 - \frac{6\lambda}{2(2n+4)}\|T\|^2.$$

The equality at a point $x \in \mathcal{N}$ holds iff there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m}\}$ of $T^{\perp}\mathcal{N}$ such that shape operators of $\mathcal{N}$ in $\overline{\mathcal{N}}$ at x have the forms (11) and (12).

Similarly, in case if $\mathcal{N}$ is a slant submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$. We have the following theorem

Theorem 3.3. *Let $\mathcal{N}$ be a slant submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then, for each point $x \in \mathcal{N}$ and each plane section $\pi \subset T_x\mathcal{N}$, we have*

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n+10-3n^2+3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2 - \frac{6}{2(2n+4)}\overline{Ric}(e_i, Je_j)cos\theta.$$

Equality holds if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m}\}$ of $T^{\perp}\mathcal{N}$ such that the shape operator takes the following forms

$$B_{n+1} = \begin{pmatrix}\alpha & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi\end{pmatrix}, \alpha + \beta = \xi \tag{21}$$

and

$$B_r = \begin{pmatrix} \omega_{11}^r & \omega_{12}^r & 0 & \cdots & 0 \\ \omega_{12}^r & -\omega_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r = n+2, ..., 2m. \quad (22)$$

From this theorem, following corollaries can be easily deduced.

Corollary 3.4. *Let $\mathcal{N}$ be a slant submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$, which is Einstein . Then, for each point $x \in \mathcal{N}$ and each plane section $\pi \subset T_x\mathcal{N}$, we have*

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n + 10 - 3n^2 + 3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2 - \frac{6\lambda}{2(2n+4)}cos^2\theta.$$

The equality holds at a point $x \in \mathcal{N}$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m}\}$ of $T^{\perp}\mathcal{N}$ such that shape operators of $\mathcal{N}$ in $\overline{\mathcal{N}}$ at x have the forms (21) and (22).

Corollary 3.5. *Let $\mathcal{N}$ be a invariant submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then, for each point $x \in \mathcal{N}$ and each plane section $\pi \subset T_x\mathcal{N}$, we have*

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n + 10 - 3n^2 + 3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2 - \frac{6}{2(2n+4)}\overline{Ric}(e_i, Je_j).$$

The equality at a point $x \in \mathcal{N}$ holds iff there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m}\}$ of $T^{\perp}\mathcal{N}$ such that shape operators of $\mathcal{N}$ in $\overline{\mathcal{N}}$ at x have the forms (21) and (22).

Corollary 3.6. *Let $\mathcal{N}$ be a anti-invariant submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{N}}$. Then, for each point $x \in \mathcal{N}$ and each plane section $\pi \subset T_x\mathcal{N}$, we have*

$$\rho - \mathcal{K}(\pi) \leq \Big(\frac{7n + 10 - 3n^2 + 3\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} - \frac{n^2(n-2)}{2(n-1)}\|\mathcal{H}\|^2.$$

The equality at a point $x \in \mathcal{N}$ holds iff there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x\mathcal{N}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m}\}$ of $T^{\perp}\mathcal{N}$ such that shape operators of $\mathcal{N}$ in $\overline{\mathcal{N}}$ at x have the forms (21) and (22).

4. Warped Product CR-Submanifolds of Bochner-Kaehler Manifolds

Let $x : \mathcal{N}_{\top} \times_f \mathcal{N}_{\perp} \to \overline{\mathcal{N}}$ be an isometric immersion of a warped product CR-submanifold into Bochner-Kaehler manifold $\overline{\mathcal{N}}$. We denote $H_1 = \frac{1}{n_1}\sum_{i=1}^{n_1}\omega(e_i, e_i)$, where n_1 is the dimension of $\mathcal{N}_{\top}$ and $H_2 = \frac{1}{n_2}\sum_{s=n_1+1}^{n}\omega(e_s, e_s)$, where n_2 is the dimension of $\mathcal{N}_{\perp}$ [8]. The immersion x is said to be mixed totally geodesic if $\omega(X, Z) = 0$, for any vector fields X and Z tangent to $\mathcal{N}_{\top}$ and $\mathcal{N}_{\perp}$ respectively.

Furthermore, it is easy to see that the scalar curvature $\overline{\rho}$ of $\overline{\mathcal{N}}$ can be decomposed as $\overline{\rho} = \overline{\rho}_D + \overline{\rho}_{D^{\perp}}$, where $\overline{\rho}_D = \sum_{i=1}^{n_1}\overline{Ric}(e_i, e_i)$ and $\overline{\rho}_{D^{\perp}} = \sum_{s=n_1+1}^{n}\overline{Ric}(e_s, e_s)$.

Let $\mathcal{N}_{\top} \times_f \mathcal{N}_{\perp}$ be warped product CR-submanifolds of a Bochner-Kaehler manifold. Since $\mathcal{N}_{\top} \times_f \mathcal{N}_{\perp}$ is a warped product, from [8], we see that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z \quad (23)$$

for any vector fields X, Z tangent to $\mathcal{N}_\top$, $\mathcal{N}_\perp$ respectively. Also for X and Z unit vector fields, the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{24}$$

We now choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ such that $e_1, e_2, \ldots, e_{n_1}$ are tangent to $\mathcal{N}_\top$ and $e_{n_1+1}, \ldots, e_n$ are tangent to $\mathcal{N}_\perp$ and e_{n+1} is parallel to the mean curvature vector H. Then using (24), we find

$$\frac{\triangle f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s) \text{ for each } s \in \{n_1 + 1, \ldots, n\}. \tag{25}$$

Also from the equation of Gauss, we have

$$\begin{aligned} 2\rho \quad = \quad & \frac{6n^2 + 2n - 8 - 6\|T\|^2}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\overline{Ric}(e_i, Je_j)g(e_i, Je_j) \\ & + n^2\|\mathcal{H}\|^2 - \|\omega\|^2. \end{aligned}$$

Which implies that

$$n^2\|\mathcal{H}\|^2 = 2\rho + \|\omega\|^2 - \frac{6n^2 + 2n - 8 - 6\|T\|^2}{2(2n+2)(2n+4)}\overline{\rho} - \frac{6}{2n+4}\overline{Ric}(e_i, Je_j)g(e_i, Je_j). \tag{26}$$

Here we know that ρ denotes the scalar curvature of $\mathcal{N}_\top \times_f \mathcal{N}_\perp$ given by

$$\rho = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We set

$$\epsilon = 2\rho - \{\frac{6n^2 + 2n - 8 - 6\|T\|^2}{2(2n+2)(2n+4)}\}\overline{\rho} - \frac{6}{2n+4}\overline{Ric}(e_i, Je_j)g(e_i, Je_j) - \frac{n^2}{2}\|H\|^2.$$

Then (26) can be written as

$$n^2\|H\|^2 = 2(\epsilon + \|\omega\|^2) \tag{27}$$

The above equation can also be written as

$$\Big(\sum_{i=1}^{n} \omega_{ii}^{n+1}\Big)^2 = 2\{\epsilon + \sum_{i=1}^{n}(\omega_{ii}^{n+1})^2 + \sum_{i \ne j}(\omega_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(\omega_{ij}^{r})^2\}.$$

Let us suppose $a_1 = \omega_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} \omega_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} \omega_{tt}^{n+1}$. The above equation becomes

$$\begin{aligned} \Big(\sum_{i=1}^{3} a_i\Big)^2 = 2\{\epsilon + \sum_{i=1}^{3} a_i^2 + \sum_{1 \le i \ne j \le n}(\omega_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(\omega_{ij}^{r})^2 - \sum_{2 \le j \ne k \le n_1} \omega_{jj}^{n+1}\omega_{kk}^{n+1} \\ - \sum_{n_1+1 \le s \ne t \le n} \omega_{ss}^{n+1}\omega_{tt}^{n+1}\}. \end{aligned}$$

Thus a_1, a_2, a_3 satisfy lemma 2.2 for $n = 3$, i.e, we have

$$\Big(\sum_{i=1}^{3} a_i\Big)^2 = 2\Big(\sum_{i=1}^{3} a_i^2 + b\Big)$$

where

$$b = \epsilon + \sum_{1\leq i\neq j\leq n} (\omega_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n} (\omega_{ij}^{r})^2 - \sum_{2\leq j\neq k\leq n_1} \omega_{jj}^{n+1}\omega_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n} \omega_{ss}^{n+1}\omega_{tt}^{n+1}.$$

Then form the lemma 2.2, $2a_1a_2 \geq b$, with the equality holding if and only if $a_1 + a_2 = a_3$. In our case, we have from the above result

$$\sum_{1\leq j<k\leq n_1} \omega_{jj}^{n+1}\omega_{kk}^{n+1} + \sum_{n_1+1\leq s<t\leq n} \omega_{ss}^{n+1}\omega_{tt}^{n+1} \geq \frac{\epsilon}{2} + \sum_{1\leq \alpha<\beta\leq n} (\omega_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n} (\omega_{\alpha\beta}^{r})^2 \tag{28}$$

equality holds if and only if

$$\sum_{i=1}^{n_1} \omega_{ii}^{n+1} = \sum_{t=n_1+1}^{n} \omega_{tt}^{n+1}. \tag{29}$$

We know that

$$\rho = \sum_{1\leq \lambda<\mu\leq n} K(e_i\wedge e_j) = \sum_{1\leq i<j\leq n_1} K(e_j\wedge e_k) + \sum_{n_1+1\leq s<t\leq n} K(e_s\wedge e_t) + \sum_{s=n_1+1}^{n}\sum_{j=1}^{n_1} K(e_j\wedge e_s). \tag{30}$$

Also we know that

$$\frac{\triangle f}{f} = \sum_{j=1}^{n_1} K(e_j\wedge e_s) \quad \forall \quad s\in\{n_1+1,\ldots,n\}$$

which implies that

$$\sum_{s=n_1+1}^{n} \frac{\triangle f}{f} = \sum_{s=n_1+1}^{n}\sum_{j=1}^{n_1} K(e_j\wedge e_s)$$

or

$$n_2\frac{\triangle f}{f} = \sum_{s=n_1+1}^{n}\sum_{j=1}^{n_1} K(e_j\wedge e_s). \tag{31}$$

From (30) and (31), we get

$$n_2\frac{\triangle f}{f} = \rho - \sum_{1\leq j<k\leq n_1} K(e_j\wedge e_k) - \sum_{n_1+1\leq s<t\leq n} K(e_s\wedge e_t). \tag{32}$$

Now using the Gauss equation , we find

$$\sum_{1\leq i<j\leq n_1} K(e_i\wedge e_j) = \sum_{1\leq i<j\leq n_1} g(R(e_i,e_j)e_j,e_i)$$

The last equation combined with (13), gives

$$\begin{aligned}
n_2\frac{\triangle f}{f} &= \rho - 2\sum_{1\le i<j\le n_1} L(e_j,e_j)g(e_i,e_i) + 2\sum_{1\le i<j\le n_1} L(e_i,e_j)g(e_i,e_j) - 6\sum_{1\le i<j\le n_1} L(e_i,Je_j)g(e_i,Je_j)\\
&\quad - \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega^r_{ii}\omega^r_{jj} + \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega^r_{ij})^2 - 2\sum_{n_1+1\le s<t\le n} L(e_s,e_s)g(e_t,e_t)\\
&\quad +2\sum_{n_1+1\le s<t\le n} L(e_s,e_t)g(e_s,e_t) - 6\sum_{n_1+1\le s<t\le n} L(e_s,Je_t)g(e_s,Je_t)\\
&\quad - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega^r_{ss}\omega^r_{tt} + \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega^r_{st})^2
\end{aligned}$$

which can be further written as

$$\begin{aligned}
n_2\frac{\triangle f}{f} &= \rho - \sum_{1\le i\neq j\le n_1} L(e_j,e_j)g(e_i,e_i) + \sum_{1\le\neq j\le n_1} L(e_i,e_j)g(e_i,e_j) - 3\sum_{1\le i\neq j\le n_1} L(e_i,Je_j)g(e_i,Je_j)\\
&\quad - \sum_{n_1+1\le s\neq t\le n} L(e_s,e_s)g(e_t,e_t) + \sum_{n_1+1\le s\neq t\le n} L(e_s,e_t)g(e_s,e_t) - 3\sum_{n_1+1\le s\neq t\le n} L(e_{s,Je_t})g(e_s,Je_t)\\
&\quad - \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega^r_{ii}\omega^r_{tt} + \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega^r_{ij})^2 - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega^r_{ss}\omega^r_{tt}\\
&\quad + \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega^r_{st})^2. \qquad (33)
\end{aligned}$$

Since in the second and fifth term $i \neq j$ and $s \neq t$ and the basis $\{e_\lambda\}_{\lambda=1}^n$ is orthonormal, we have

$$\sum_{1\le i\neq j\le n_1} L(e_i,e_j)g(e_i,e_j) = \sum_{n_1+1\le s\neq t\le n} L(e_s,e_t)g(e_s,e_t) = 0.$$

Therefore, the last equation (33) becomes

$$\begin{aligned}
n_2\frac{\triangle f}{f} &= \rho - \sum_{1\le i,j\le n_1} L(e_j,e_j)g(e_i,e_i) - \sum_{n_1+1\le s,t\le n} L(e_s,e_s)g(e_t,e_t) + \sum_{k=1}^{n_1} L(e_k,e_k)\\
&\quad + \sum_{\omega=n_1+1}^{n} L(e_\omega,e_\omega) - 3\sum_{1\le i,j\le n_1} L(e_i,Je_j)g(e_i,Je_j) - 3\sum_{n_1+1\le s,t\le n} L(e_s,Je_t)g(e_s,Je_t)\\
&\quad - \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega^r_{ii}\omega^r_{jj} + \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega^r_{ij})^2 - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega^r_{ss}\omega^r_{tt}\\
&\quad + \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega^r_{st})^2
\end{aligned}$$

which implies that

$$\begin{aligned} n_2\frac{\triangle f}{f} &= \rho - n_1\sum_{j=1}^{n_1} L(e_j,e_j) - (n-n_1)\sum_{s=n_1+1}^{n} L(e_s,e_s) + \sum_{k=1}^{n_1} L(e_k,e_k) \\ &+ \sum_{\omega=n_1+1}^{n} L(e_\omega,e_\omega) - 3\{\sum_{i,j=1}^{n_1} L(e_i,Je_j)g(e_i,Je_j) - \sum_{s,t=n_1+1}^{n} L(e_s,Je_t)g(e_s,Je_t)\} \\ &- \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega_{ii}^r\omega_{jj}^r + \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega_{ij}^r)^2 - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega_{ss}^r\omega_{tt}^r \\ &+ \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega_{st}^r)^2 \end{aligned}$$

or

$$\begin{aligned} n_2\frac{\triangle f}{f} &= \rho - (n_1-1)\sum_{j=1}^{n_1} L(e_j,e_j) - (n_2-1)\sum_{s=n_1+1}^{n} L(e_s,e_s) \\ &-3\Big\{\sum_{i,j=1}^{n_1} L(e_i,Je_j)g(e_i,Je_j) - \sum_{s,t=n_1+1}^{n} L(e_s,Je_t)g(e_s,Je_t)\Big\} - \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega_{ii}^r\omega_{jj}^r \\ &+ \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega_{ij}^r)^2 - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega_{ss}^r\omega_{tt}^r + \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega_{st}^r)^2. \end{aligned}$$

Since $Je_t \in T\mathcal{N}^\perp$. This gives from the last equation

$$\begin{aligned} n_2\frac{\triangle f}{f} &= \rho - (n_1-1)\sum_{j=1}^{n_1} L(e_j,e_j) - (n_2-1)\sum_{s=n_1+1}^{n} L(e_s,e_s) \\ &-3\sum_{j=1}^{n_1} L(e_j,e_j) - \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega_{ii}^r\omega_{jj}^r \\ &+ \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega_{ij}^r)^2 - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega_{ss}^r\omega_{tt}^r + \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega_{st}^r)^2 \end{aligned}$$

which can further be simplified as

$$\begin{aligned} n_2\frac{\triangle f}{f} &= \rho - (n_1+2)\sum_{j=1}^{n_1} L(e_j,e_j) - (n_2-1)\sum_{s=n_1+1}^{n} L(e_s,e_s) \\ &- \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}\omega_{ii}^r\omega_{jj}^r + \sum_{r=n+1}^{2m}\sum_{1\le i<j\le n_1}(\omega_{ij}^r)^2 \\ &- \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\omega_{ss}^r\omega_{tt}^r + \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}(\omega_{st}^r)^2 \end{aligned}$$

Using (28) in the above equation, we derive

$$
\begin{aligned}
n_2\frac{\triangle f}{f} &\leq \rho-(n_1+2)\sum_{j=1}^{n_1}L(e_j,e_j)-(n_2-1)\sum_{s=n_1+1}^{n}L(e_s,e_s)-\sum_{r=n+2}^{2m}\sum_{1\leq i<j\leq n_1}\omega_{ii}^r\omega_{jj}^r\\
&-\sum_{r=n+2}^{2m}\sum_{n_1+1\leq s<t\leq n}\omega_{ss}^r\omega_{tt}^r+\sum_{r=n+1}^{2m}\sum_{1\leq i<j\leq n_1}(\omega_{ij}^r)^2+\sum_{r=n_1+1}^{2m}\sum_{n_1+1\leq s<t\leq n}(\omega_{st}^r)^2\\
&-\frac{\epsilon}{2}-\sum_{1\leq\alpha<\beta\leq n}(\omega_{\alpha\beta}^r)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(\omega_{\alpha\beta}^r)^2
\end{aligned}
$$

or

$$
\begin{aligned}
n_2\frac{\triangle f}{f} &\leq \rho-(n_1+2)\sum_{j=1}^{n_1}L(e_j,e_j)-(n_2-1)\sum_{s=n_1+1}^{n}L(e_s,e_s)+\sum_{r=n+2}^{2m}\sum_{1\leq i<j\leq n_1}\{(\omega_{ij}^r)^2-\omega_{ii}^r\omega_{jj}^r\}\\
&+\sum_{r=n+2}^{2m}\sum_{n_1+1\leq s<t\leq n}\{(\omega_{st}^r)^2-\omega_{ss}^r\omega_{tt}^r\}-\frac{\epsilon}{2}-\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(\omega_{\alpha\beta}^{n+1})^2\\
&-\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(\omega_{jt}^{n+1})^2
\end{aligned}
$$

which implies

$$
\begin{aligned}
n_2\frac{\triangle f}{f} &\leq \rho-(n_1+2)\sum_{j=1}^{n_1}L(e_j,e_j)-(n_2-1)\sum_{s=n_1+1}^{n}L(e_s,e_s)-\frac{\epsilon}{2}\\
&-\sum_{r=n+1}^{2m}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(\omega_{jt}^r)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\Big(\sum_{j=1}^{n_1}\omega_{jj}^r\Big)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\Big(\sum_{t=n_1+1}^{n}\omega_{tt}^r\Big)^2
\end{aligned}
$$

or

$$
n_2\frac{\triangle f}{f} \leq \rho-(n_1+2)\sum_{j=1}^{n_1}L(e_j,e_j)-(n_2-1)\sum_{s=n_1+1}^{n}L(e_s,e_s)-\frac{\epsilon}{2}.
$$

Last equation can be rewritten as

$$
\begin{aligned}
n_2\frac{\triangle f}{f} &\leq \rho-(n_1+2)\sum_{j=1}^{n_1}L(e_j,e_j)-(n_2-1)\sum_{s=n_1+1}^{n}L(e_s,e_s)\\
&+\Big(\frac{6n^2+2n-8-6\|T\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho}+\frac{3}{2n+4}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)+\frac{n^2}{4}\|\mathcal{H}\|^2-\rho.
\end{aligned}
$$

Using the definition of tensor L, we get from the last inequality

$$
\begin{aligned}
n_2\frac{\triangle f}{f} &\leq -\frac{(n_1+2)}{2n+4}\overline{\rho}_D+\frac{n_1(n_1+2)}{2(2n+2)(2n+4)}\overline{\rho}-\frac{(n_2-1)}{2n+4}\overline{\rho}_{D^\perp}+\frac{n_2(n_2-1)}{2(2n+2)(2n+4)}\overline{\rho}\\
&+\Big(\frac{6n^2+2n-8-6\|T\|^2}{4(2n+2)(2n+4)}\Big)\overline{\rho}+\frac{3}{2n+4}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)+\frac{n^2}{4}\|\mathcal{H}\|^2
\end{aligned}
$$

or

$$n_2\frac{\triangle f}{f} \leq -\frac{(n_1+2)}{2n+4}\overline{\rho}_D - \frac{(n_2-1)}{2n+4}\overline{\rho}_{D^\perp} - \frac{6\|T\|^2}{4(2n+2)(2n+4)}\overline{\rho} + \frac{n_1(n_1+2)}{2(2n+2)(2n+4)}\overline{\rho}$$
$$+\frac{n_2(n_2-1)}{2(2n+2)(2n+4)}\overline{\rho} + \Big(\frac{6n^2+2n-8}{4(2n+2)(2n+4)}\Big)\overline{\rho}$$
$$+\frac{3}{2n+4}\overline{Ric}(e_i, Je_j)g(e_i, Je_j) + \frac{n^2}{4}\|\mathcal{H}\|^2.$$

Let $\overline{\rho}_D \geq 0$ and $\overline{\rho}_{D^\perp} \geq 0$. We have

$$\frac{\triangle f}{f} \leq \frac{n_1(n_1+2)}{2n_2(2n+2)(2n+4)}\overline{\rho} + \frac{(n_2-1)}{2(2n+2)(2n+4)}\overline{\rho} + \Big(\frac{6n^2+2n-8}{4(2n+2)(2n+4)}\Big)\overline{\rho}$$
$$+\frac{3}{n_2(2n+4)}\overline{\rho}_D + \frac{n^2}{4}\|\mathcal{H}\|^2. \tag{34}$$

Hence, we have the following

Theorem 4.1. *Let $\mathcal{N} = \mathcal{N}_\top \times_f \mathcal{N}_\perp$ be an n-dimensional warped product CR-submanifold immersed in Bochner-Kaehler manifold with $\overline{\rho}_D \geq 0$ and $\overline{\rho}_{D^\perp} \geq 0$. Then the warping function f satisfies the following inequality*

$$\frac{\triangle f}{f} \leq \Big[\frac{4n^2 - 4n_1n_2 + 6n_1 - 8}{16n_2(n+1)(n+2)}\Big]\overline{\rho} + \frac{3}{2n_2(n+2)}\overline{\rho}_D + \frac{n^2}{4n_2}\|H\|^2.$$

Moreover, the equality holds, if and only if the immersion is mixed totally geodesic, the partial mean curvatures satisfy $n_1H_1 = n_2H_2$ and $\overline{\rho}_D = 0$, $\overline{\rho}_{D^\perp} = 0$. In the case of equality $\frac{\triangle f}{f} = \frac{n^2}{4n_2}\|H\|^2$.

Corollary 4.2. *Let $\mathcal{N} = \mathcal{N}_\top \times_f \mathcal{N}_\perp$ be a warped product CR-submanifold immersed in Bochner-Kaehler manifold such that the equality holds in the above theorem. Then there does not exist such immersions with harmonic warping function.*

Corollary 4.3. *Let $\mathcal{N} = \mathcal{N}_\top \times_f \mathcal{N}_\perp$ be a warped product CR-submanifold immersed in Bochner-Kaehler manifold $\overline{\mathcal{N}}$ such that the equality holds in the above theorem. Then there does not exist such immersions with warping function as an eigen function of the Laplacian on $\mathcal{N}_\top$ having corresponding eigenvalue $\lambda < 0$.*

Corollary 4.4. *Let $\mathcal{N} = \mathcal{N}_\top \times_f \mathcal{N}_\perp$ be a warped product CR-submanifold immersed in Bochner-Kaehler manifold such that the equality holds in the above theorem. Then there does not exist such minimal immersions with warping function as an eigen function of the Laplacian on $\mathcal{N}_\top$ having corresponding eigenvalue $\lambda \neq 0$.*

Theorem 4.5. *Let $\mathcal{N} = \mathcal{N}_\top \times_f \mathcal{N}_\perp$ be an n-dimensional warped product CR-submanifold immersed in Bochner-Kaehler manifold with $\overline{\rho}_D \leq 0$ and $\rho_{D^\perp} \leq 0$. Then the warping function f satisfies the following inequality*

$$\frac{\triangle f}{f} \leq -\frac{[(n_1+2)\overline{\rho}_D + (n_2-1)\overline{\rho}_{D^\perp}]}{2n_2(n+2)} - \frac{6\|T\|^2}{16n_2}(n+1)(n+2)\overline{\rho} + \frac{n^2}{4n_2}\|H\|^2$$

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