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A Note on the Weakest Taylor Term

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Abstract. We simplify the most general Taylor condition.

1. Introduction

By a *strong Mal'cev condition* we mean a finite set of identities in some language. Informally, a strong Mal'cev condition is *realized* in an algebra **A** (or variety $\mathcal{V}$) if there is a way to interpret the function symbols appearing in the condition as term operations of **A** (or $\mathcal{V}$) so that the identities in the Mal'cev condition become true equations in **A** (or $\mathcal{V}$). A Mal'cev condition is a sequence $\{C_n : n \in \omega\}$ of strong Mal'cev conditions such that any variety which realizes C_n must also realize C_{n+1} for all $n \in \omega$. We say that the variety $\mathcal{V}$ realizes the Mal'cev condition $\{C_n : n \in \omega\}$ if there exists an $n \in \omega$ such that $\mathcal{V}$ realizes C_n. We say that a varietal property P is a (strong) Mal'cev property if there exists a (strong) Mal'cev condition C such that for any variety $\mathcal{V}$, C is realized in $\mathcal{V}$ iff $\mathcal{V}$ has the property P. Also, the previous sentence is commonly relativized to locally finite varieties, so we say that some varietal property is a (strong) Mal'cev property of locally finite varieties if there exists a (strong) Mal'cev condition C such that for any locally finite variety $\mathcal{V}$, C is realized in $\mathcal{V}$ iff $\mathcal{V}$ has the property P.

W. Taylor in [6] proved that a variety realizes a nontrivial idempotent strong Mal'cev condition iff it realizes an idempotent linear strong Malcev condition in a one-operation language with identities in only two variables. We will call such strong Mal'cev condition a Taylor condition. If the variety $\mathcal{V}$ realizes a Taylor condition C, then any term which interprets the only operation symbol used in C is called a Taylor term for the variety $\mathcal{V}$.

In the definition of Taylor condition there is no mention of a single (uniform) Taylor condition which is realized in a variety iff any Taylor condition is realized. Until recently, it was thought no such condition exists. M. Siggers proved in [5] that there exists a Taylor condition (S) with a six-ary operation symbol such that for any locally finite variety $\mathcal{V}$, (S) is realized in $\mathcal{V}$ iff any Taylor condition is realized in $\mathcal{V}$. In other words, Siggers proved that existence of a Taylor term is a strong Mal'cev property of locally finite varieties. This astonishing development led to further investigations which simplified the Siggers condition to a four-ary operation [2], proved that congruence meet-semidistributivity is a strong Mal'cev property of locally finite varieties, but that most other natural properties are not strong Mal'cev properties even in the

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locally finite case in [3], and simplified the strong Mal'cev condition for congruence meet-semidistributivity, while proving its optimality [1]. The even more astonishing recent development is a result by M. Olšák [4] which proves that existence of a Taylor term is a strong Mal'cev property generally, not just in the locally finite case. Initially, Olšák proved that the weakest Taylor condition is the "double loop" term of arity twelve. Both the Siggers term and the Taylor terms found in [2] can be obtained from the double loop term by considering some variables as dummy variables. Motivated by this fact, we proceed to "remove the variables" from the double loop term in this note. After seeing a draft of our paper, Olšák further simplified the double loop term to the "weak 3-cube term" of arity 6. The weak 3-cube term and our Taylor term do not trivially imply each other in any direction (only an existential proof is known, not a constructive one).

2. Setup

Definition 2.1. *We say that a variety $\mathcal{V}$ has a double loop term t if t is a 12-ary term in the language of $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$:*

$$\begin{aligned} &t(x,x,x,x,x,x,x,x,x,x,x,x) \approx x \\ &t(x,x,y,y,x,x,y,y,x,x,y,y) \approx t(x,x,y,y,y,y,x,x,y,y,x,x) \\ &t(x,y,x,y,x,y,x,y,x,y,x,y) \approx t(y,x,y,x,x,y,x,y,y,x,y,x). \end{aligned} \tag{O}$$

Olšak's result announced in the Introduction states:

Theorem 2.2 (M. Olšák, [4]). *Any variety which has a Taylor term has a double loop term.*

Let $\mathcal{V}$ be a variety. By $\mathcal{W}$ we denote the idempotent reduct of $\mathcal{V}$, which is the variety whose clone is the clone of idempotent term operations of $\mathcal{V}$ and whose fundamental operations are the distinct elements of this clone. Since all Mal'cev conditions C which we will consider in this paper are idempotent, it follows that $\mathcal{V}$ realizes C iff $\mathcal{W}$ realizes C. In other words, without loss of generality, we may restrict ourselves to idempotent varieties.

In order to make our proofs easier to read we introduce a convention that the elements of A^W (mappings from W to A) are written as vector columns. This allows us to see better how to apply an operation which acts coordinatewise on several such vectors. When we describe the constraint $C \subseteq A^W$, we linearly order the elements of $W = \{x_{i_1}, \ldots, x_{i_k}\}$. Then we write $\rho_{i_1,\ldots,i_k} = R$, for some previously fixed $R \subseteq A^{|W|}$, which means that the uppermost coordinate of the vector column in R is the image of x_{i_1}, below it the image of x_{i_2}, and so on. In some cases, to save space we will use the transpose of the vector column, which will be denoted by $[a_1, \ldots, a_k]^T$.

Lemma 2.3. *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ has a double loop term, then $\mathcal{V}$ realizes the following strong Mal'cev condition:*

$$\begin{aligned} &f(x,x,x,x,x,x,x,x,x,x) \approx x \\ &f(x,x,y,y,x,x,y,y,x,x) \approx f(x,x,y,y,y,y,x,x,y,y) \\ &f(x,y,x,y,x,y,x,y,x,y) \approx f(y,x,y,x,x,y,x,y,y,x) \end{aligned} \tag{PV1}$$

Proof. As mentioned above, we may assume that $\mathcal{V}$ is idempotent. Let t be the double loop term in $\mathcal{V}$. Let $\mathbf{F}$ be the two-generated free algebra in $\mathcal{W}$, freely generated by x and y. Let

$$G = \mathrm{Sg}^{\mathbf{F}^4}\left(\begin{bmatrix} x\\x\\x\\y \end{bmatrix}, \begin{bmatrix} x\\x\\y\\x \end{bmatrix}, \begin{bmatrix} y\\y\\x\\y \end{bmatrix}, \begin{bmatrix} y\\y\\y\\x \end{bmatrix}, \begin{bmatrix} x\\y\\x\\x \end{bmatrix}, \begin{bmatrix} x\\y\\y\\y \end{bmatrix}, \begin{bmatrix} y\\x\\x\\x \end{bmatrix}, \begin{bmatrix} y\\x\\y\\y \end{bmatrix}, \begin{bmatrix} x\\y\\x\\y \end{bmatrix}, \begin{bmatrix} x\\y\\y\\x \end{bmatrix} \right).$$

We also define some elements of F, namely

$$\begin{aligned} a &= t(x,x,y,y,x,x,y,y,x,x,y,y), \\ \overline{a} &= t(y,y,x,x,x,x,y,y,x,x,y,y), \\ b &= t(x,y,x,y,x,y,x,y,x,y,x,y) \text{ and} \\ b' &= t(x,y,x,y,x,y,x,y,x,y,y,x). \end{aligned}$$

From (O) follow the equations

$$\begin{aligned} a &= t(x,x,y,y,y,y,x,x,y,y,x,x), \\ \overline{a} &= t(y,y,x,x,y,y,x,x,y,y,x,x) \text{ and} \\ b &= t(y,x,y,x,x,y,x,y,y,x,y,x). \end{aligned}$$

We prove that certain vectors are in G. Let $\overline{\varphi}$ be the automorphism of F which extends the map $\varphi(x) = y$ and $\varphi(y) = x$. Since $[x,y,x,x]^T, [y,x,x,x]^T \in G$, then for any $u \in F$, we get $[u, \overline{\varphi}(u), x, x]^T \in G$ by the idempotence. Similarly, from $[x,y,y,y]^T, [y,x,y,y]^T \in G$ and idempotence follows that for any $u \in F$, $[u, \overline{\varphi}(u), y, y]^T \in G$. Since $\overline{\varphi}(a) = \overline{a}$ and $\overline{\varphi}(\overline{a}) = a$, we get $[a,\overline{a},x,x]^T$, $[\overline{a},a,x,x]^T$, $[a,\overline{a},y,y]^T$, $[\overline{a},a,y,y]^T \in G$. Assuming that $v = v(x,y) \in F$, idempotence of the term v implies $[a,\overline{a},v,v]^T, [\overline{a},a,v,v]^T \in G$. For the appropriate choices of v we get

$$\begin{bmatrix} a \\ \overline{a} \\ b \\ b \end{bmatrix}, \begin{bmatrix} a \\ \overline{a} \\ b' \\ b' \end{bmatrix}, \begin{bmatrix} \overline{a} \\ a \\ b \\ b \end{bmatrix}, \begin{bmatrix} \overline{a} \\ a \\ b' \\ b' \end{bmatrix} \in G. \tag{1}$$

We proceed by directly computing that certain tuples are in G:

$$t\left(\begin{bmatrix} x \\ x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}\right) = \begin{bmatrix} a \\ a \\ b \\ b' \end{bmatrix} \in G.$$

$$t\left(\begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} a \\ \overline{a} \\ b \\ b' \end{bmatrix} \in G.$$

$$t\left(\begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} \overline{a} \\ a \\ b \\ b' \end{bmatrix} \in G.$$

$$t\left(\begin{bmatrix} y \\ y \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}\right) = \begin{bmatrix} \overline{a} \\ \overline{a} \\ b \\ b' \end{bmatrix} \in G.$$

Finally, we note that the generating set of G is closed under the transposition of the bottom two coordinates, which is an automorphism of $\mathbf{F}^4$. Therefore, G is also closed under the transposition of the bottom two coordinates. Thus we also get

$$\begin{bmatrix} a \\ a \\ b' \\ b \end{bmatrix}, \begin{bmatrix} a \\ \overline{a} \\ b' \\ b \end{bmatrix}, \begin{bmatrix} \overline{a} \\ a \\ b' \\ b \end{bmatrix}, \begin{bmatrix} \overline{a} \\ \overline{a} \\ b' \\ b \end{bmatrix} \in G. \tag{2}$$

Finally, we apply t to the twelve obtained vectors in G to obtain:

$$t\left(\begin{bmatrix}a\\a\\b\\b'\end{bmatrix},\begin{bmatrix}a\\a\\b'\\b\end{bmatrix},\begin{bmatrix}\bar{a}\\\bar{a}\\b\\b'\end{bmatrix},\begin{bmatrix}\bar{a}\\\bar{a}\\b'\\b\end{bmatrix},\begin{bmatrix}a\\\bar{a}\\b\\b\end{bmatrix},\begin{bmatrix}a\\\bar{a}\\b'\\b'\end{bmatrix},\begin{bmatrix}\bar{a}\\a\\b\\b\end{bmatrix},\begin{bmatrix}\bar{a}\\a\\b'\\b'\end{bmatrix},\begin{bmatrix}a\\\bar{a}\\b\\b'\end{bmatrix},\begin{bmatrix}a\\\bar{a}\\b'\\b\end{bmatrix},\begin{bmatrix}\bar{a}\\a\\b\\b'\end{bmatrix},\begin{bmatrix}\bar{a}\\a\\b'\\b\end{bmatrix}\right)\in G.$$

However, the first two coordinates of the resulting vector are equal by the identity

$$t(x,x,y,y,x,x,y,y,x,x,y,y)\approx t(x,x,y,y,y,y,x,x,y,y,x,x),$$

while the second two coordinates are equal according to the identity

$$t(x,y,x,y,x,y,x,y,x,y,x,y)\approx t(y,x,y,x,x,y,x,y,y,x,y,x).$$

So, let the resulting vector be $[c,c,d,d]^T$.

Therefore, there exists a term f in ten variables which, when applied to the ten generators of G, equals $[c,c,d,d]^T$. (This term is implicitly given by our construction, but we omit it to save space.) This term is computed in $\mathbf{F}^4$ coordinatewise, so

$$f^{\mathbf{F}}(x,x,y,y,x,x,y,y,x,x)=c=f^{\mathbf{F}}(x,x,y,y,y,y,x,x,y,y)$$

and

$$f^{\mathbf{F}}(x,y,x,y,x,y,x,y,x,y)=d=f^{\mathbf{F}}(x,y,x,y,y,x,y,x,y,x).$$

But since $\mathbf{F}$ is the $\mathcal{V}$-free algebra, this implies that

$$\mathcal{V}\models f(x,x,y,y,x,x,y,y,x,x)\approx f(x,x,y,y,y,y,x,x,y,y)$$

and

$$\mathcal{V}\models f(x,y,x,y,x,y,x,y,x,y)\approx f(x,y,x,y,y,x,y,x,y,x).$$

As idempotence of f is guaranteed in $\mathcal{V}$, f realizes the strong Mal'cev condition we wanted to prove. □

Theorem 2.4. *Let $\mathcal{V}$ be a variety with a double loop term. $\mathcal{V}$ also realizes the following strong Mal'cev condition:*

$$\begin{aligned}&f(x,x,x,x,x,x,x,x,x)\approx x\\&f(x,x,y,y,x,x,y,y,x)\approx f(x,x,y,y,y,y,x,x,y)\\&f(x,y,x,y,x,y,x,y,x)\approx f(y,x,y,x,x,y,x,y,y)\end{aligned}\tag{PV2}$$

Proof. We use the same technique as in the previous proof. Let f be the term which satisfies the identities in (PV1), provided by Lemma 2.3. Define

$$H=\mathrm{Sg}^{\mathbf{F}^4}\left(\begin{bmatrix}x\\x\\x\\y\end{bmatrix},\begin{bmatrix}x\\x\\y\\x\end{bmatrix},\begin{bmatrix}y\\y\\x\\y\end{bmatrix},\begin{bmatrix}y\\y\\y\\x\end{bmatrix},\begin{bmatrix}x\\y\\x\\x\end{bmatrix},\begin{bmatrix}x\\y\\y\\y\end{bmatrix},\begin{bmatrix}y\\x\\x\\x\end{bmatrix},\begin{bmatrix}y\\x\\y\\y\end{bmatrix},\begin{bmatrix}x\\y\\x\\y\end{bmatrix}\right).$$

As in the proof of Lemma 2.3, it suffices to prove that H contains a vector of the form $[u,u,v,v]^T$ for some $u,v\in F$. We define

$$\begin{aligned}&a=f(x,x,y,y,x,x,y,y,x,x),\\&b=f(x,y,x,y,x,y,x,y,x,y)\ \text{ and}\\&c=f(y,y,y,y,x,x,x,x,y,y).\end{aligned}$$

From (PV1) follow the equations

$$a = f(x, x, y, y, y, y, x, x, y, y) \text{ and}$$
$$b = f(y, x, y, x, x, y, x, y, y, x).$$

H contains

$$f\left(\begin{bmatrix}x\\y\\x\\x\end{bmatrix},\begin{bmatrix}x\\y\\x\\x\end{bmatrix},\begin{bmatrix}x\\y\\y\\y\end{bmatrix},\begin{bmatrix}x\\y\\y\\y\end{bmatrix},\begin{bmatrix}x\\x\\x\\y\end{bmatrix},\begin{bmatrix}x\\x\\x\\y\end{bmatrix},\begin{bmatrix}x\\x\\y\\x\end{bmatrix},\begin{bmatrix}x\\x\\y\\x\end{bmatrix},\begin{bmatrix}x\\y\\x\\y\end{bmatrix},\begin{bmatrix}x\\y\\x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\c\\a\\a\end{bmatrix}$$

and

$$f\left(\begin{bmatrix}y\\y\\x\\y\end{bmatrix},\begin{bmatrix}y\\y\\y\\x\end{bmatrix},\begin{bmatrix}y\\y\\x\\y\end{bmatrix},\begin{bmatrix}y\\y\\y\\x\end{bmatrix},\begin{bmatrix}y\\x\\x\\x\end{bmatrix},\begin{bmatrix}y\\x\\y\\y\end{bmatrix},\begin{bmatrix}y\\x\\x\\x\end{bmatrix},\begin{bmatrix}y\\x\\y\\y\end{bmatrix},\begin{bmatrix}y\\y\\x\\y\end{bmatrix},\begin{bmatrix}y\\y\\y\\x\end{bmatrix}\right)=\begin{bmatrix}y\\c\\b\\b\end{bmatrix}.$$

Finally, H contains

$$f\left(\begin{bmatrix}y\\c\\b\\b\end{bmatrix},\begin{bmatrix}y\\c\\b\\b\end{bmatrix},\begin{bmatrix}y\\c\\b\\b\end{bmatrix},\begin{bmatrix}y\\c\\b\\b\end{bmatrix},\begin{bmatrix}x\\c\\a\\a\end{bmatrix},\begin{bmatrix}x\\c\\a\\a\end{bmatrix},\begin{bmatrix}x\\c\\a\\a\end{bmatrix},\begin{bmatrix}x\\c\\a\\a\end{bmatrix},\begin{bmatrix}y\\c\\b\\b\end{bmatrix},\begin{bmatrix}y\\c\\b\\b\end{bmatrix}\right)=\begin{bmatrix}c\\c\\d\\d\end{bmatrix},$$

where $d = f(b, b, b, b, a, a, a, a, b, b)$. This is what we needed to prove, the rest is analogous as in Lemma 2.3. □

3. Final Remarks

Since we discovered the condition (PV2), M. Olšák also discovered improvements to his condition. Its new equivalent form is called the "weak 3-cube term" and entails idempotence and

$$s(x, y, y, y, x, x) \approx s(y, x, y, x, y, x) \approx s(y, y, x, x, x, y) \quad . \tag{S}$$

Since the weak 3-cube term has only six variables, it is more optimized than (PV2). However, it does not trivially imply (PV2), and our feeling is that there is some optimal condition which entails both our (PV2) and the weak 3-cube term.

What could that condition be? Well, in [4] it is proved (example attributed to A. Kazda) that the optimal condition from [2], $t(x, y, x, z) \approx t(y, x, z, y)$ plus idempotence (this is not a Taylor term since the equations use three variables), is not realized in all Taylor varieties, just in all locally finite ones. However, among its consequences there are three which use only two variables, which are realized in all locally finite Taylor varieties and which might be optimal Taylor conditions for all varieties. Those are (idempotence and)

$$t_1(x, x, y, y) \approx t_1(x, y, x, y) \approx t_1(y, y, y, x) \tag{C1}$$

$$\begin{aligned} &t_2(x, x, x, y) \approx t_2(x, x, y, x) \text{ and}\\ &t_2(x, y, x, x) \approx t_2(y, x, y, x)\end{aligned} \tag{C2}$$

and finally

$$\begin{aligned} &t_3(x, x, x, y) \approx t_3(x, x, y, x) \text{ and}\\ &t_3(x, y, x, x) \approx t_3(y, x, x, y).\end{aligned} \tag{C3}$$

Of the three, only the condition (C1), called the "weak 3-edge term" in [2] is known to syntactically imply both (S) and (PV2). The other two also imply (PV2) syntactically, but we only know they imply (S) by going through the whole Olšák's proof which is existential in parts, not constructive. So our conjecture is that the weak 3-edge term is the optimal weakest Taylor condition.

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