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The αAB-, βAB-, γAB- and NAB-duals for Sequence Spaces

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Abstract. Let $A = (a_{n,k})$ and $B = (b_{n,k})$ be two infinite matrices with real entries. The main purpose of this paper is to generalize the multiplier space for introducing the concepts of αAB-, βAB-, γAB-duals and NAB-duals. Moreover, these duals are investigated for the sequence spaces X and $X(A)$, where $X \in \{c_0, c, l_p\}$ for $1 \leq p \leq \infty$. The other purpose of the present study is to introduce the sequence spaces

$$X(A, \Delta) = \left\{ x = (x_k) \, : \, \left(\sum_{k=1}^{\infty} a_{n,k} x_k - \sum_{k=1}^{\infty} a_{n-1,k} x_k \right)_{n=1}^{\infty} \in X \right\},$$

where $X \in \{l_\infty, c, c_0\}$, and computing the NAB-(or Null) duals and βAB-duals for these spaces.

1. Introduction

Let ω denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. For $1 \leq p < \infty$, denote by l_p the space of all real sequences $x = (x_n) \in \omega$ such that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

For $p = \infty$, $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is interpreted as $\sup_{n\geq 1} |x_n|$. We write c and c_0 for the spaces of all convergent and null sequences, respectively. Also, bs and cs are used for the spaces of all bounded and convergent series, respectively. Kizmaz [8] defined the backward difference sequence space

$$X(\Delta) = \{x = (x_k) : \ \Delta x \in X\},$$

for $X \in \{l_\infty, c, c_0\}$, where $\Delta x = (x_k - x_{k-1})_{k=1}^{\infty}$, $x_0 = 0$. Observe that $X(\Delta)$ is a Banach space with the norm

$$\|x\|_\Delta = \sup_{k\geq 1} |x_k - x_{k-1}|.$$

In the summability theory, the β-dual of a sequence space is very important in connection with inclusion theorems. The idea of dual sequence space was introduced by Köthe and Toeplitz [9], and it is generalized

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to the vector-valued sequence spaces by Maddox [10]. For the sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y) = \{z = (z_k) \in \omega \; : \; (z_k x_k)_{n=1}^{\infty} \in Y \quad \forall x = (x_k) \in X\},$$

is called the multiplier space of X and Y. With the above notation, the α-, β- γ and N-duals of a sequence space X, which are respectively denoted by X^{α}, X^{β}, X^{γ} and X^{N}, are defined by

$$X^{\alpha} = M(X,l_1), \quad X^{\beta} = M(X,cs), \quad X^{\gamma} = M(X,bs), \quad X^{N} = M(X,c_0).$$

For a sequence space X, the matrix domain $X(A)$ of an infinite matrix A is defined by

$$X(A) = \{x = (x_n) \in \omega \; : \; Ax \in X\}, \tag{1}$$

which is a sequence space. The new sequence space $X(A)$ generated by the limitation matrix A from a sequence space X can be the expansion or the contraction and or the overlap of the original space X.

In the past, several authors studied Köthe-Toeplitz duals of sequence spaces that are the matrix domains in classical spaces l_p, l_{∞}, c and c_0. For instance, some matrix domains of the difference operator was studied in [4]. Domain of backward difference matrix in the space l_p was investigated for $1 \le p \le \infty$ by Başar and Altay in [3] and was studied for $0 < p < 1$ by Altay and Başar in [1]. Recently the Köthe-Toeplitz duals were computed for some new sequence spaces by Erfanmanesh and Foroutannia [5], [6] and Foroutannia [7]. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [2].

In this study, the concept of the multiplier space is generalized and the αAB-, βAB-, γAB- and NAB-duals are determined for the classical sequence spaces l_{∞}, c and c_0. Also the normed sequence space $X(\Delta)$ is extended to semi-normed space $X(A,\Delta)$, where $X \in \{l_{\infty}, c, c_0\}$. We consider some topological properties of this space and derive inclusion relations concerning with its. Moreover, we compute the NAB-(or Null) duals for the space $X(A,\Delta)$. The results are generalizations of some results of Malkowsky and Rakocevic [11], Kizmaz [8] and Erfanmanesh and Foroutannia [5].

2. The Generalized Multiplier Space and its Köthe-Toeplitz Duals and Null Duals

In this section, we introduce the generalization of multiplier space and present the new generalizations of Köthe-Toeplitz duals and Null duals of sequence spaces. Furthermore, we obtain these duals for the sequence spaces l_{∞}, c and c_0. Throughout this paper, let I be the identity matix.

Definition 2.1. *Suppose that $A = (a_{n,k})$ and $B = (b_{n,k})$ are two infinite matrices with real entries such that $\sum_{k=1}^{\infty} a_{n,k}x_k < \infty$ for all $x = (x_k) \in X$ and $n = 1, 2, \cdots$. For the sequence spaces X and Y, the set $M_{A,B}(X,Y)$ defined by*

$$M_{A,B}(X,Y) = \left\{ z \in \omega \; : \; \sum_{k=1}^{\infty} b_{n,k} z_k < \infty, \; \forall n \text{ and } \left(\sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k \right)_{n=1}^{\infty} \in Y, \; \forall x \in X \right\},$$

is called the generalized multiplier space of X and Y.

The αAB-, βAB-, γAB- and NAB-duals of a sequence space X, which are respectively denoted by $X^{\alpha AB}$, $X^{\beta AB}$, $X^{\gamma AB}$ and X^{NAB}, are defined by

$$X^{\alpha AB} = M_{A,B}(X,l_1), \quad X^{\beta AB} = M_{A,B}(X,cs), \quad X^{\gamma AB} = M_{A,B}(X,bs), \quad X^{NAB} = M_{A,B}(X,c_0).$$

It should be noted that in the special case $A = B = I$, we have $M_{A,B}(X,Y) = M(X,Y)$. So

$$X^{\alpha AB} = X^{\alpha}, \qquad X^{\beta AB} = X^{\beta}, \qquad X^{\gamma AB} = X^{\gamma}, \qquad X^{NAB} = X^{N}.$$

Let $E = (E_n)$ and $F = (F_n)$ be two partitions of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \qquad \max F_n < \min F_{n+1},$$

for $n = 1, 2, \cdots$. If the infinite matrices $A = (a_{n,k})$ and $B = (b_{n,k})$ are defined by

$$a_{n,k} = \begin{cases} 1 & if\ k \in E_n \\ 0 & otherwise, \end{cases} \tag{2}$$

and

$$b_{n,k} = \begin{cases} 1 & if\ k \in F_n \\ 0 & otherwise, \end{cases} \tag{3}$$

then $M_{A,B}(X,Y) = M_{E,F}(X,Y)$ and the new multiplier space $M_{A,B}(X,Y)$ is a generalization of the multiplier space $M_{E,F}(X,Y)$ introduced in [5].

Lemma 2.2. *Let $X, Y, Z \subset \omega$ and $\{X_\delta : \delta \in I\}$ be any collection of subsets of ω, then*
(*i*) *$X \subset Z$ implies $M_{A,B}(Z,Y) \subset M_{A,B}(X,Y)$,*
(*ii*) *$Y \subset Z$ implies $M_{A,B}(X,Y) \subset M_{A,B}(X,Z)$,*
(*iii*) *$X \subset M_{A,B}(M_{B,A}(X,Y),Y)$,*
(*iv*) *$M_{A,B}(X,Y) = M_{A,B}(M_{B,A}(M_{A,B}(X,Y),Y),Y)$,*
(*v*) *$M_{A,B}(\bigcup_{\delta\in I} X_\delta, Y) = \bigcap_{\delta\in I} M_{A,B}(X_\delta, Y)$.*

Proof. Parts (*i*) and (*ii*) are obvious, by using the definition of generalized multiplier space.

(*iii*) Let $x \in X$. We have $(\sum_{k=1}^{\infty} a_{n,k}z_k \sum_{k=1}^{\infty} b_{n,k}x_k)_{n=1}^{\infty} \in Y$ for all $z \in M_{B,A}(X,Y)$, and consequently $x \in M_{A,B}(M_{B,A}(X,Y),Y)$.

(*iv*) By applying (*iii*) with X replaced by $M_{B,A}(X,Y)$, we deduce that

$$M_{A,B}(X,Y) \subset M_{A,B}(M_{B,A}(M_{A,B}(X,Y),Y),Y).$$

Conversely, due to (*iii*), we have $X \subset M_{B,A}(M_{A,B}(X,Y),Y)$. So

$$M_{A,B}(M_{B,A}(M_{A,B}(X,Y),Y),Y) \subset M_{A,B}(X,Y),$$

by part (*i*).

(*v*) First, $X_\delta \subset \bigcup_{\delta\in I} X_\delta$ for all $\delta \in I$ implies

$$M_{A,B}(\bigcup_{\delta\in I} X_\delta, Y) \subset \bigcap_{\delta\in I} M_{A,B}(X_\delta, Y),$$

by part (*i*). Conversely, if $a \in \bigcap_{\delta\in I} M_{A,B}(X_\delta, Y)$, then $z \in M_{A,B}(X_\delta, Y)$ for all $\delta \in I$. So

$$\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)_{n=1}^{\infty} \in Y,$$

for all $\delta \in I$ and for all $x \in X_\delta$. This implies $(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k)_{n=1}^{\infty} \in Y$ for all $x \in \bigcup_{\delta\in I} X_\delta$, hence $z \in M_{A,B}(\bigcup_{\delta\in I} X_\delta, Y)$. Thus $\bigcap_{\delta\in I} M_{A,B}(X_\delta, Y) \subset M_{A,B}(\bigcup_{\delta\in I} X_\delta, Y)$. □

Remark 2.3. *If $A = B = I$, we have Lemma 1.25 from [11].*

Remark 2.4. *If two matrices A and B are defined by (2) and (3), then we obtain Lemma 2.1 from [5].*

If † denotes either of the symbols α, β, γ or N, from now on we will use the following notation

$$(X^{\dagger AB})^{\dagger AB} = X^{\dagger\dagger AB}.$$

Corollary 2.5. *Let $X, Y \subset \omega$ and $\{X_\delta : \delta \in I\}$ be any collection of subsets of ω, also † denotes either of the symbols α, β, γ or N, then*
(*i*) $X^{\alpha AB} \subset X^{\beta AB} \subset X^{\gamma AB} \subset \omega$; *in particular,* $X^{\dagger AB}$ *is a sequence space.*
(*ii*) $X \subset Z$ *implies* $Z^{\dagger AB} \subset X^{\dagger AB}$.
(*iii*) $X \subset X^{\dagger\dagger AA}$.
(*iv*) $X^{\dagger AA} = X^{\dagger\dagger\dagger AA}$.
(*v*) $(\bigcup_{\delta\in I} X_\delta)^{\dagger AB} = \bigcap_{\delta \in I} X_\delta^{\dagger AB}$.

Remark 2.6. *If $A = B = I$, we have Corollary 1.26 from [11].*

Remark 2.7. *If two matrices A and B are defined by (2) and (3), then we obtain Corollary 2.1 from [5].*

Below, we determine the generalized multiplier space for some sequence spaces. For this purpose, we recall the following theorem from [11]. Let X and Y be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix of real numbers $a_{n,k}$, where $n, k \in \mathbb{N} = \{1, 2, \cdots\}$. We say that A defines a matrix mapping from X into Y, and we denote it by $A : X \to Y$, if for every sequence $x \in X$ the sequence $Ax = \{(Ax)_n\}_{n=1}^\infty$ exists and is in Y, where $(Ax)_n = \sum_{k=1}^\infty a_{n,k}x_k$ for $n = 1, 2, \cdots$. By (X, Y), we denote the class of all infinite matrices A such that $A : X \to Y$. We consider the conditions

$$\sup_n \left(\sum_{k=1}^\infty |a_{n,k}| \right) < \infty, \tag{4}$$

$$\lim_{n\to\infty} a_{n,k} = 0 \quad (k = 1, 2, \cdots), \tag{5}$$

$$\lim_{n\to\infty} a_{n,k} = l_k \ \ for\ some\ l_i \in \mathbb{R} \ \ (i = 1, 2, \cdots), \tag{6}$$

$$\lim_{n\to\infty} \left(\sum_{k=1}^\infty a_{n,k} \right) = l \ \ for\ some\ l \in \mathbb{R}. \tag{7}$$

With the notation of (1), the spaces $l_\infty(A)$, $c(A)$ and $c_0(A)$ contain all of the sequences $x = (x_n)$ that $Ax = \{(Ax)_n\}$ are the bounded, convergent and null sequences, respectively.

Theorem 2.8. *([11], Theorem 1.36) We have*
(*i*) $A \in (l_\infty, l_\infty)$ *if and only if the condition (4) holds, in this case* $l_\infty \subset l_\infty(A)$;
(*ii*) $A \in (c_0, c_0)$ *if and only if the conditions (4) and (5) hold, in this case* $c_0 \subset c_0(A)$;
(*iii*) $A \in (c, c)$ *if and only if the conditions (4), (6) and (7) hold, in this case* $c \subset c(A)$;
(*iv*) $A \in (c_0, c)$ *if and only if the conditions (4) and (6) hold, in this case* $c_0 \subset c(A)$.

Theorem 2.9. *Let A be an invertible matrix. We have the following statements.*
(*i*) $M_{A,B}(c_0, X) = l_\infty(B)$, *where* $X \in \{l_\infty, c, c_0\}$ *and A satisfies the conditions (4) and (5);*
(*ii*) $M_{A,B}(l_\infty, X) = c_0(B)$, *where* $X \in \{c, c_0\}$ *and A satisfies the condition (4);*
(*iii*) *If in addition* $\sum_{k=1}^\infty a_{n,k} = R$ *for all n, then* $M_{A,B}(c, c) = c(B)$ *and A satisfies the conditions (4), (6) and (7).*

Proof. (*i*) Since $c_0 \subset c \subset l_\infty$, by applying Lemma 2.2(*ii*), we have

$$M_{A,B}(c_0, c_0) \subset M_{A,B}(c_0, c) \subset M_{A,B}(c_0, l_\infty).$$

So it is sufficient to verify $l_\infty(B) \subset M_{A,B}(c_0, c_0)$ and $M_{A,B}(c_0, l_\infty) \subset l_\infty(B)$. Suppose that $z \in l_\infty(B)$ and $x \in c_0$. Due to Theorem 2.8(*ii*) we have $x \in c_0(A)$, so

$$\lim_{n\to\infty} \left(\sum_{k=1}^\infty b_{n,k} z_k \sum_{k=1}^\infty a_{n,k} x_k \right) = 0, \tag{8}$$

this means that $z \in M_{A,B}(c_0, c_0)$. Thus $l_\infty(B) \subset M_{A,B}(c_0, c_0)$.

Now we assume $z \notin l_\infty(B)$. Then there is a subsequence $\left(\sum_{k=1}^{\infty} b_{n_j,k}z_k\right)_{j=1}^{\infty}$ of the sequence $(\sum_{k=1}^{\infty} b_{n,k}z_k)_{k=1}^{\infty}$ such that $\left|\sum_{k=1}^{\infty} b_{n_j,k}z_k\right| > j^2$ for $j = 1, 2, \cdots$. Since A is an invertible matrix, there exists a sequence $x = (x_k)$ such that

$$\sum_{k=1}^{\infty} a_{n_j,k}x_k = \frac{(-1)^j j}{\sum_{k=1}^{\infty} b_{n_j,k}z_k},$$

for all j. Hence

$$\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)_{n=1}^{\infty} \notin l_\infty,$$

this shows that $M_{A,B}(c_0, l_\infty) \subset l_\infty(B)$.

(*ii*) We have

$$M_{A,B}(l_\infty, c_0) \subset M_{A,B}(l_\infty, c),$$

by applying Lemma 2.2(*ii*). It is sufficient to prove $c_0(B) \subset M_{A,B}(l_\infty, c_0)$ and $M_{A,B}(l_\infty, c) \subset c_0(B)$. Suppose that $z \in c_0(B)$. By Theorem 2.8, we have $\lim_{n\to\infty} (\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k) = 0$ for all $x \in l_\infty$, that is $z \in M_{A,B}(l_\infty, c_0)$. Thus $c_0(B) \subset M_{A,B}(l_\infty, c_0)$.

Now we assume $z \notin c_0(B)$. Then there is a real number as $b > 0$ and a subsequence $\left(\sum_{k=1}^{\infty} b_{n_j,k}z_k\right)_{j=1}^{\infty}$ of the sequence $(\sum_{k=1}^{\infty} b_{n,k}z_k)_{n=1}^{\infty}$ such that $\left|\sum_{k=1}^{\infty} b_{n_j,k}z_k\right| > b$ for all for $j = 1, 2, \cdots$. We define the sequence x as in part (*ii*). We have $x \in l_\infty$ and

$$\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right)_{n=1}^{\infty} \notin c,$$

which implies $z \notin M_{A,B}(l_\infty, c)$. This shows that $M_{A,B}(l_\infty, c) \subset c_0(B)$.

(*iii*) Suppose that $z \in c(B)$. By applying Theorem 2.8(*iii*), we deduce that $\lim_{n\to\infty} (\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k)$ exists for all $x \in c$. So $z \in M_{A,B}(c, c)$ and $c(B) \subset M_{A,B}(c, c)$.

Conversely we assume $z \notin c(B)$. We define the sequence x by $x = (\frac{1}{R}, \frac{1}{R}, \cdots)$. It is obvious that $x \in c$ and $(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k)_{k=1}^{\infty} = (\sum_{k=1}^{\infty} b_{n,k}z_k)_{k=1}^{\infty} \notin c$. So $z \notin M_{A,B}(c, c)$, this shows $M_{A,B}(c, c) \subset c(B)$. □

Remark 2.10. *If $A = B = I$, we have Example 1.28 from [11].*

Remark 2.11. *If two matrices A and B are defined by (2) and (3), then we obtain Theorem 2.2 from [5].*

Corollary 2.12. *Suppose that* $\sup_n \sum_{k=1}^{\infty} |a_{n,k}| < \infty$, *we have* $c_0^{NAB} = l_\infty(B)$ *and* $l_\infty^{NAB} = c_0(B)$.

Proof. The desired result follows from Theorem 2.9. □

Theorem 2.13. *If matrix A satisfies the conditions in Theorem 2.9, then we have the following statements.*

(*i*) *$M_{A,B}(c_0(A), X) = l_\infty(B)$, where $X \in \{l_\infty, c, c_0\}$. In particular $(c_0(A))^{NAB} = l_\infty(B)$.*

(*ii*) *$M_{A,B}(l_\infty(A), X) = c_0(B)$, where $X \in \{c, c_0\}$. In particular $(l_\infty(A))^{NAB} = c_0(B)$.*

(*iii*) *If in addition $\sum_{k=1}^{\infty} a_{n,k} = R$ for all n, then $M_{A,B}(c(A), c) = c(B)$.*

Proof. We only prove the part (*i*), the other parts are proved similarly. Since $c_0 \subset c_0(A)$, according to Corollary 2.5(*ii*) and Theorem 2.9 we obtain

$$M_{A,B}(c_0(A), X) \subset M_{A,B}(c_0, X) = l_\infty(B).$$

The inclusion $l_\infty(B) \subset M_{A,B}(c_0(A), X)$ is gained by the relation (8). □

In the following, we obtain the αAB-, βAB- and γAB-duals for the sequence spaces l_∞, c and c_0.

Theorem 2.14. *Suppose that A is an invertible matrix that satisfies the condition (4), and $\dagger$ denote either of the symbols α, β or γ. We have*

$$c_0^{\dagger AB} = c^{\dagger AB} = l_\infty^{\dagger AB} = l_1(B).$$

In particular for $B = I$,

$$c_0^{\dagger AI} = c^{\dagger AI} = l_\infty^{\dagger AI} = l_1.$$

Proof. We only prove the statement for the case $\dagger = \beta$, the other cases prove similarly. Obviously $l_\infty^{\beta AB} \subset c^{\beta AB} \subset c_0^{\beta AB}$ by Corollary 2.5(*ii*). So it is sufficient to show that $l_1(B) \subset l_\infty^{\beta AB}$ and $c_0^{\beta AB} \subset l_1(B)$.

Now, let $z \in l_1(B)$ and $x \in l_\infty$ be given. Due to Theorem 2.8(*i*), we deduce that $x \in l_\infty(A)$. Hence

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k \right| \leq \sup_n \left| \sum_{k=1}^{\infty} a_{n,k} x_k \right| \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| < \infty, \tag{9}$$

which shows $(\sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k)_{n=1}^{\infty} \in cs$. Thus $z \in l_\infty^{\beta AB}$ and $l_1(B) \subset l_\infty^{\beta AB}$. On the other hand, for a given $z \notin l_1(B)$ we prove the existence of a sequence $x \in c_0$ with $(\sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k)_{n=1}^{\infty} \notin cs$, which implies $z \notin c_0^{\beta AB}$; thus altogether $c_0^{\beta AB} \subset l_1(B)$. Because $z \notin l_1(B)$, we may choose an index subsequence (n_j) in $\mathbf{N}$ with $n_0 = 0$ and

$$\sum_{n=n_{j-1}}^{n_j - 1} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| > j \qquad (j = 1, 2, \cdots).$$

Since A is an invertible matrix, there exists a sequence $x = (x_k)$ such that

$$\sum_{k=1}^{\infty} a_{n_j,k} x_k = \frac{1}{j} sgn \sum_{k=1}^{\infty} b_{n_j,k} z_k,$$

for all j. Hence $x \in c_0$ and

$$\sum_{n=n_{j-1}}^{n_j - 1} \left(\sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k \right) = \frac{1}{j} \sum_{n=n_{j-1}}^{n_j - 1} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| > 1,$$

for $j = 1, 2, \cdots$. Therefore $(\sum_{k=1}^{\infty} a_{n,k} x_k \sum_{k=1}^{\infty} b_{n,k} z_k)_{k=1}^{\infty} \notin cs$, and $z \notin c_0^{\beta AB}$. This completes the proof. □

Remark 2.15. *If $A = B = I$ and $\dagger$ denote either of the symbols α, β or γ. we have*

$$c_0^{\dagger} = c^{\dagger} = l_\infty^{\dagger} = l_1,$$

hence Theorem 1.29 from [11] is resulted.

Remark 2.16. *If two matrices A and B are defined by (2) and (3), then we obtain Theorem 2.3 from [5].*

In the next theorem, we examine the αAB-, βAB- and γAB-duals for the sequence spaces $l_\infty(A)$, $c(A)$ and $c_0(A)$.

Theorem 2.17. *Let A be a matrix which satisfies the conditions in Theorem 2.8. If $\dagger$ denote either of the symbols α, β or γ, then*

$$(c_0(A))^{\dagger AB} = (c(A))^{\dagger AB} = (l_\infty(A))^{\dagger AB} = l_1(B).$$

Proof. We only prove the statement for the case $\dagger = \beta$, the other case prove similarly. Obviously

$$(l_\infty(A))^{\beta AB} \subset (c(A))^{\beta AB} \subset (c_0(A))^{\beta AB},$$

by Corollary 2.5(*ii*). So it is sufficient to verify $(c_0(A))^{\beta AB} \subset l_1(B)$ and $l_1(B) \subset (l_\infty(A))^{\beta AB}$. By applying Corollary 2.5(*ii*) and Theorem 2.14, we deduce that $(c_0(A))^{\beta AB} \subset c_0^{\beta AB} = l_1(B)$. The other inclusion will gain by the relation (9). □

Theorem 2.18. *Suppose that A is an invertible matrix. If $1 < p < \infty$ and $q = p/(p-1)$, then $(l_p(A))^{\beta AB} = l_q(B)$. Moreover for $p = 1$, we have $(l_1(A))^{\beta AB} = l_\infty(B)$.*

Proof. We only prove the statement for the case $1 < p < \infty$, the case $p = 1$ will prove similarly. Let $z \in l_q(B)$ be given. By Hölder's inequality, we have

$$\left|\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} b_{k,j}z_j\right)\left(\sum_{j=1}^{\infty} a_{k,j}x_j\right)\right| \le \left(\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\infty} b_{k,j}z_j\right|^q\right)^{1/q}\left(\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\infty} a_{k,j}x_j\right|^p\right)^{1/p} < \infty, \tag{10}$$

for all $x \in l_p(A)$. This shows $z \in (l_p(A))^{\beta AB}$ and hence $l_q(B) \subset (l_p(A))^{\beta AB}$.
Now, let $z \in (l_p(A))^{\beta AB}$ be given. We consider the linear functional $f_n : l_p(A) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} b_{k,j}z_j\right)\left(\sum_{j=1}^{n} a_{k,j}x_j\right) \qquad \Big(x \in l_p(A)\Big),$$

for $n = 1, 2, \cdots$. Similar to (10), we obtain

$$|f_n(x)| \le \left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j}z_j\right|^q\right)^{1/q}\left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{k,j}x_j\right|^p\right)^{1/p},$$

for every $x \in l_p(A)$. So the linear functional f_n is bounded and

$$\|f_n\| \le \left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j}z_j\right|^q\right)^{1/q},$$

for all n. We now prove reverse of the above inequality. Since A is invertible, we define the sequence $x = (x_k)$ such that

$$\sum_{j=1}^{n} a_{k,j}x_j = \left(sgn\sum_{j=1}^{n} b_{k,j}z_j\right)\left|\sum_{j=1}^{n} b_{k,j}z_j\right|^{q-1},$$

for $1 \le k \le n$, and put the remaining elements zero. Obviously $x \in l_p(A)$, so

$$\|f_n\| \ge \frac{|f_n(x)|}{\|x\|_p} = \frac{\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j}z_j\right|^q}{\left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j}z_j\right|^q\right)^{1/p}} = \left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j}z_j\right|^q\right)^{1/q},$$

for $n = 1, 2, \cdots$. Since $z \in l_p(A)^{\beta AB}$, the map $f_z : l_p(A) \to \mathbb{R}$ defined by

$$f_z(x) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} b_{k,j}z_j\right)x_k \qquad \Big(x \in l_p(A)\Big),$$

is well-defined and linear, and also the sequence (f_n) is pointwise convergent to f_z. By using the Banach-Steinhaus theorem, it can be shown that $\|f_z\| \le \sup_n \|f_n\| < \infty$, so $\left(\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\infty} b_{k,j}z_j\right|^q\right)^{1/q} < \infty$ and $z \in l_q(B)$. This establishes the proof of theorem. □

Remark 2.19. *If $A = B = I$ and $1 < p < \infty$ and $q = p/(p-1)$. Then we have $l_p^\beta = l_q$. Moreover for $p = 1$, $l_1^\beta = l_\infty$.*

Definition 2.20. *A subset X of ω is said to be A-normal if $y \in X$ and $|\sum_{k=1}^{\infty} a_{n,k}x_k| \le |\sum_{k=1}^{\infty} a_{n,k}y_k|$ for $n = 1, 2, \cdots$, together imply $x \in X$. In the special case that $A = I$, the set X is called normal.*

Example 2.21. *The sequence spaces c_0 and l_∞ are normal, but they are not A-normal. Since if $x = (1, -1, 2, -2, \cdots)$, $y = (1, \frac{1}{2}, \cdots)$ and the matrix $A = (a_{n,k})$ is defined by*

$$a_{n,k} = \begin{cases} 1 & if\ k \in \{2n-1, 2n\} \\ 0 & otherwise. \end{cases}$$

We have $|\sum_{k=1}^{\infty} a_{n,k}x_k| \leq |\sum_{k=1}^{\infty} a_{n,k}y_k|$ and $y \in c_0, l_\infty$, while $x \notin c_0, l_\infty$.

Example 2.22. *The sequence spaces $c_0(A)$ and $l_\infty(A)$ are A-normal, but they are not normal. Because, if $x = (1, 1, 2, 2, \cdots)$ and $y = (1, -1, 2, -2, \cdots)$ and A is the matrix as in Example 2.21, then it is obvious that $|x_i| \leq |y_i|$, $y \in c_0(A)$ and $y \in l_\infty(A)$, while $x \notin c_0(A)$ and $x \notin l_\infty(A)$.*

Example 2.23. *The sequence spaces c and $c(A)$ are neither A-normal nor normal.*

Theorem 2.24. *Suppose that A is an invertible matrix and X is a A-normal subset of ω. We have*

$$X^{\alpha AB} = X^{\beta AB} = X^{\gamma AB}.$$

Proof. Obviously $X^{\alpha AB} \subset X^{\beta AB} \subset X^{\gamma AB}$, by Corollary 2.5($i$). To prove the statement, it is sufficient to verify $X^{\gamma AB} \subset X^{\alpha AB}$. Let $z \in X^{\gamma AB}$ and $x \in X$ be given. Since A is invertible, we define the sequence y such that

$$\sum_{k=1}^{\infty} a_{n,k}y_k = \left(sgn \sum_{k=1}^{\infty} b_{n,k}z_k\right)\left|\sum_{k=1}^{\infty} a_{n,k}x_k\right|,$$

for $n = 1, 2, \cdots$. It is clear $\left|\sum_{k=1}^{\infty} a_{n,k}y_k\right| \leq \left|\sum_{k=1}^{\infty} a_{n,k}x_k\right|$, for all n. Consequently $y \in X$, since X is A-normal. So

$$\sup_n \left|\sum_{k=1}^{n}\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}y_k\right)\right| < \infty.$$

Furthermore, by the definition of the sequence y, $\sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right| < \infty$. Since $x \in X$ was arbitrary, $z \in X^{\alpha AB}$. This finishes the proof of the theorem. □

Remark 2.25. *If $A = B = I$ and X be a normal subset of ω, we have*

$$X^\alpha = X^\beta = X^\gamma,$$

hence Remark 1.27 from [11] is gained.

Remark 2.26. *If two matrices A and B are defined by (2) and (3), then we obtain Theorem 2.4 from [5].*

3. The Difference Sequence Space $X(A, \Delta)$

Suppose that $A = (a_{n,k})$ is an infinite matrix with real entries. For every sequence space X, we define the generalized difference sequence space $X(A, \Delta)$ as follows:

$$X(A, \Delta) = \left\{x = (x_k) \ : \ \left(\sum_{k=1}^{\infty}(a_{n,k} - a_{n-1,k})x_k\right)_{n=1}^{\infty} \in X\right\},$$

where $X \in \{l_\infty, c, c_0\}$. The seminorm $\|.\|_{A,\Delta}$ on $X(A, \Delta)$ is defined by

$$\|x\|_{A,\Delta} = \sup_n \left|\sum_{k=1}^{\infty}(a_{n,k} - a_{n-1,k})x_k\right|. \tag{11}$$

It should be noted that the function $\|.\|_{A,\Delta}$ cannot be the norm. Since if $x = (1, -1, 0, 0, \cdots)$ and $A = (a_{n,k})$ is defined by,

$$a_{n,k} = \begin{cases} 1 & if\ k \in \{2n-1, 2n\} \\ 0 & otherwise, \end{cases}$$

then $\|x\|_{A,\Delta} = 0$ while $x \neq 0$. It is also significant that in the special case $A = I$, we have $X(A,\Delta) = X(\Delta)$ and $\|x\|_{A,\Delta} = \|x\|_\Delta$.

If the infinite matrix $\Delta = (\delta_{n,k})$ is defined by

$$\delta_{n,k} = \begin{cases} 1 & if\ k = n \\ -1 & if\ k = n-1 \\ 0 & otherwise, \end{cases}$$

with the notation of (1), we can redefine the spaces $l_\infty(A,\Delta)$, $c(A,\Delta)$ and $c_0(A,\Delta)$ as follows:

$$l_\infty(A,\Delta) = (l_\infty)_{\Delta A}, \qquad c(A,\Delta) = (c)_{\Delta A}, \qquad c_0(A,\Delta) = (c_0)_{\Delta A}.$$

The purpose of this section is to consider some properties of the sequence spaces $X(A,\Delta)$ and is to derive some inclusion relations related to them. We also characterize NAB-duals and βAB-duals of $X(A,\Delta)$ where $X \in \{l_\infty, c, c_0\}$.

Now, we may begin with the following theorem which is essential in the study.

Theorem 3.1. *The sequence spaces $X(A,\Delta)$ for $X \in \{l_\infty, c, c_0\}$ are complete semi-normed linear spaces with respect to the semi-norm defined by (11).*

Proof. This is a routine verification and so we omit the details. □

It can easily be checked that the absolute property does not hold on the space $X(A,\Delta)$, that is $\|x\|_{A,\Delta} \neq \||x|\|_{A,\Delta}$ for at least one sequence in this space which says that $X(A,\Delta)$ is the sequence space of non-absolute type, where $|x| = (|x_k|)$.

Theorem 3.2. *Let $A = (a_{n,k})$ be an invertible matrix. The space $X(A,\Delta)$ is linearly isomorphic to the space $X(\Delta)$, for $X \in \{l_\infty, c, c_0\}$.*

Proof. Consider the map

$$\begin{aligned} T : X(A,\Delta) &\longrightarrow X(\Delta) \\ x &\longrightarrow \left(\sum_{k=1}^{\infty} a_{n,k}x_k\right)_{n=1}^{\infty}, \end{aligned}$$

obviously the map T is linear, surjective and injective. □

In the following, we derive some inclusion relations concerning with the spaces X, $X(A)$, $X(\Delta)$ and $X(A,\Delta)$ where $X \in \{l_\infty, c, c_0\}$.

Theorem 3.3. *We have the following inclusions.*

(*i*) *If the condition (4) holds, then $l_\infty \subset l_\infty(A,\Delta)$.*

(*ii*) *If the conditions (4) and (5) hold, then $c_0 \subset c_0(A,\Delta)$.*

(*iii*) *If the conditions (4), (6) and (7) hold, then $c \subset c(A,\Delta)$.*

(*iv*) *We have $X(A) \subset X(A,\Delta)$ where $X \in \{l_\infty, c, c_0\}$.*

Proof. The parts (*i*), (*ii*) and (*iii*) obtain by applying Theorem 2.8.

(*iv*) Put $A = I$ in parts (*i*), (*ii*) and (*iii*), it can conclude that $X \subset X(\Delta)$. Let $x \in X(A)$ be given. We deduce that $(\sum_{k=1}^\infty a_{n,k}x_k)_{n=1}^\infty \in X$ so $(\sum_{k=1}^\infty a_{n,k}x_k)_{n=1}^\infty \in X(\Delta)$. Hence $x \in X(A,\Delta)$ and $X(A) \subset X(A,\Delta)$. □

Below, we compute NAB-dual of the difference sequence spaces $X(A,\Delta)$ where $X \in \{l_\infty, c, c_0\}$. In order to do this, we first give a preliminary lemma.

Lemma 3.4. *(i) If $x \in l_\infty(\Delta)$ then $\sup_k \left|\frac{x_k}{k}\right| < \infty$.*
(ii) If $x \in c(\Delta)$ then $\frac{x_k}{k} \to \xi \, (k \to \infty)$ where $\Delta x_k \to \xi \, (k \to \infty)$.
(iii) If $x \in c_0(\Delta)$ then $\frac{x_k}{k} \to 0 \, (k \to \infty)$.

Proof. The proof is trivial and so is omitted. □

Theorem 3.5. *Define the set d_1 as follows:*

$$d_1 = \left\{ z = (z_k) \ : \ \left(n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \in c_0 \right\},$$

then

$$c^{NAB}(A,\Delta) = l_\infty^{NAB}(A,\Delta) = d_1.$$

Proof. We first show that $c^{NAB}(A,\Delta) = d_1$. Suppose that $z \in c^{NAB}(A,\Delta)$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k = 0,$$

for all $x \in c(A,\Delta)$. Since A is invertible, we can choose the sequence x such that $\sum_{k=1}^{\infty} a_{n,k} x_k = n$ for all n, so $x \in c(A,\Delta)$ and hence $\lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k} z_k = 0$. Thus $c^{NAB}(A,\Delta) \subset d_1$. Now let $z \in d_1$. Since $(\sum_{k=1}^{\infty} a_{n,k} x_k)_{n=1}^{\infty} \in c(\Delta)$ for every $x \in c(A,\Delta)$, by previous lemma $\lim_{n\to\infty} \frac{\sum_{k=1}^{\infty} a_{n,k} x_k}{n} = \xi$, where $\xi = \lim_{n\to\infty} \sum_{k=1}^{\infty} (a_{n,k} - a_{n-1,k}) x_k$. Hence

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k = \lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k} z_k \frac{\sum_{k=1}^{\infty} a_{n,k} x_k}{n} = 0,$$

therefore $z \in c^{NAB}(A,\Delta)$ and $d_1 \subset c^{NAB}(A,\Delta)$.

Below, we prove that $l_\infty^{NAB}(A,\Delta) = d_1$. It is clear that $c(A,\Delta) \subset l_\infty(A,\Delta)$, so $l_\infty^{NAB}(A,\Delta) \subset c^{NAB}(A,\Delta) = d_1$. Now let $z \in d_1$ and $x \in l_\infty(A,\Delta)$. We have $(\sum_{k=1}^{\infty} a_{n,k} x_k)_{n=1}^{\infty} \in l_\infty(\Delta)$ and $\sup_n \left| \frac{\sum_{k=1}^{\infty} a_{n,k} x_k}{n} \right| < \infty$ by Lemma 3.4. So

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k} z_k \sum_{k=1}^{\infty} a_{n,k} x_k = \lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k} z_k \frac{\sum_{k=1}^{\infty} a_{n,k} x_k}{n} = 0,$$

This implies that $z \in l_\infty^{NAB}(A,\Delta)$. □

Remark 3.6. *If $A = B = I$, we have $c^N(\Delta) = l_\infty^N(\Delta) = \{z = (z_k) \ : \ (ka_k) \in c_0\}$, [8].*

Remark 3.7. *If two matrices A and B are defined by (2) and (3), then we obtain Theorem 3.4 from [5].*

Theorem 3.8. *Let $A = (a_{n,k})$ be an invertible matrix. We define the set d_2 as follows:*

$$d_2 = \left\{ z = (z_k) \ : \ \left(n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \in l_\infty \right\},$$

then $c_0^{NAB}(A,\Delta) = d_2$.

Proof. Suppose that $z \in d_2$. Since $(\sum_{k=1}^{\infty} a_{n,k}x_k)_{n=1}^{\infty} \in c_0(\Delta)$ for all $x \in c_0(A,\Delta)$, we have $\lim_{n\to\infty} \frac{\sum_{k=1}^{\infty} a_{n,k}x_k}{n} = 0$, by Lemma 3.4. So

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k = \lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k}z_k \frac{\sum_{k=1}^{\infty} a_{n,k}x_k}{n} = 0,$$

this implies that $z \in c_0^{NAB}(A,\Delta)$.

Now let $z \in c_0^{NAB}(A,\Delta)$ and $x \in c_0(A,\Delta)$ be given. By Theorem 3.2, there exists one and only one $y = (y_k) \in c_0$ such that $\sum_{k=1}^{\infty} a_{n,k}x_k = \sum_{j=1}^{n} y_j$. So

$$\lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k}z_k y_j = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k = 0,$$

for all $y = (y_k) \in c_0$. If we define the matrix $D = (d_{nj})_{n=1}^{\infty}$ by

$$d_{nj} = \begin{cases} \sum_{k=1}^{\infty} b_{n,k}z_k & for\ 1 \le j \le n \\ 0 & for\ j > n, \end{cases}$$

then $\lim_{n\to\infty} \sum_{j=1}^{\infty} d_{nj}y_j = 0$ for all $y \in c_0$. So $D = (d_{kj}) \in (c_0, c_0)$ and

$$\sup_n \left| n \sum_{k=1}^{\infty} b_{n,k}z_k \right| = \sup_n \left| \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k}z_k \right| = \sup_n \left| \sum_{j=1}^{\infty} d_{nj} \right| < \infty,$$

by Theorem 2.8(*ii*). This completes the proof of the theorem. □

Remark 3.9. *If $A = B = I$, we have $c_0^N(\Delta) = \{z = (z_k) : (ka_k) \in l_\infty\}$, hence Lemma 2 from [8] is resulted.*

Remark 3.10. *If two matrices A and B are defined by (2) and (3), then we obtain Theorem 3.6 from [5].*

In order to investigate the βAB-dual of the difference sequence space $c_0^N(\Delta)$, we need the following lemma.

Lemma 3.11. *([8], Lemma 1) Let $(z_k) \in l_1$ and if $\lim_{k\to\infty} |z_k x_k| = L$ exists for an $x \in c_0(\Delta)$, then $L = 0$.*

For the next result, we introduce the sequence (R_k) given by

$$R_k = \sum_{t=k}^{\infty} \sum_{j=1}^{\infty} b_{t,j}z_j.$$

Theorem 3.12. *Let $A = (a_{n,k})$ be an invertible matrix. If*

$$d_3 = \{z = (z_k) \in l_1(B) : (R_k) \in l_1 \cap c_0^N(\Delta)\},$$

then we have $c_0^{\beta AB}(A,\Delta) = d_3$

Proof. Suppose that $z \in d_3$ and $x \in c_0(A,\Delta)$, by using Abel's summation formula we have

$$\begin{aligned} & \sum_{n=1}^{m} \left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k \right) \\ = & \sum_{n=1}^{m} \left(\sum_{t=1}^{n} \sum_{j=1}^{\infty} b_{t,j}z_j \right) \left(\sum_{k=1}^{\infty} a_{n,k}x_k - \sum_{k=1}^{\infty} a_{n+1,k}x_k \right) + \left(\sum_{n=1}^{m} \sum_{k=1}^{\infty} b_{n,k}z_k \right) \sum_{k=1}^{\infty} a_{m+1,k}x_k \\ = & \sum_{n=1}^{m} (R_1 - R_{n+1}) \left(\sum_{k=1}^{\infty} a_{n,k}x_k - \sum_{k=1}^{\infty} a_{n+1,k}x_k \right) + (R_1 - R_{m+1}) \sum_{k=1}^{\infty} a_{m+1,k}x_k \\ = & \sum_{n=1}^{m+1} R_n \left(\sum_{k=1}^{\infty} a_{n,k}x_k - \sum_{k=1}^{\infty} a_{n-1,k}x_k \right) - R_{m+1} \sum_{k=1}^{\infty} a_{m+1,k}x_k. \end{aligned} \tag{12}$$

This implies that $\sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k)$ is convergent, so $z \in c_0^{\beta AB}(A,\Delta)$.

Conversely let $z \in c_0^{\beta AB}(A,\Delta)$, we show that $z \in d_3$. Obviously $z \in l_1(B)$. Suppose that $z \notin l_1(B)$, we can choose an index sequence (n_v) in $\mathbb{N}$ with

$$n_0 = 1 \quad and \quad \sum_{n=n_{v-1}}^{n_v-1}\left|\sum_{k=1}^{\infty} b_{n,k}z_k\right| > v \quad (v \in \mathbb{N}).$$

Since A is an invertible matrix, we may find $x = (x_k) \in c_0(A) \subset c_0(A,\Delta)$ such that

$$\sum_{k=1}^{\infty} a_{n,k}x_k = \frac{1}{v} sgn \sum_{k=1}^{\infty} b_{n,k}z_k \quad (n_{v-1} \le n < n_v \ \ and \ \ v \in \mathbb{N}),$$

hence

$$\sum_{n=n_{v-1}}^{n_v-1}\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right) = \frac{1}{v}\sum_{n=n_{v-1}}^{n_v-1}\left|\sum_{k=1}^{\infty} b_{n,k}z_k\right| > 1 \ \ (v \in \mathbb{N}),$$

therefore $(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k)_{n=1}^{\infty} \notin cs$ and $z \notin c_0^{\beta AB}(A,\Delta)$.

Let $x \in c_0(A,\Delta)$. Since A is invertible, by Theorem 3.2 there exist $y = (y_k) \in c_0$ such that $\sum_{k=1}^{\infty} a_{n,k}x_k = \sum_{j=1}^{n} y_j$, then by Abel's summation formula

$$\begin{aligned}
\sum_{n=1}^{m} R_n y_n &= \sum_{n=1}^{m}(R_n - R_{n+1})\left(\sum_{j=1}^{n} y_j\right) + \sum_{n=1}^{m} R_{m+1}y_n \\
&= \sum_{n=1}^{m}\left(\sum_{j=1}^{n} y_j\right)\left(\sum_{j=1}^{\infty} b_{n,j}z_j\right) + \sum_{n=1}^{m} R_{m+1}y_n.
\end{aligned}$$

So

$$\sum_{n=1}^{m}\left(\sum_{k=1}^{\infty} b_{n,k}z_k \sum_{k=1}^{\infty} a_{n,k}x_k\right) = \sum_{n=1}^{m}(R_n - R_{m+1})y_n = \sum_{n=1}^{m}\left(\sum_{i=n}^{m}\sum_{j=1}^{\infty} b_{i,j}z_j\right)y_n. \tag{13}$$

Now we define the matrix $D = (d_{n,k})$ by

$$d_{n,k} = \begin{cases} \sum_{i=k}^{n}\sum_{j=1}^{\infty} b_{i,j}z_j & for\ 1 \le k \le n \\ 0 & for\ k > n, \end{cases}$$

Since $\lim_{n\to\infty}\sum_{k=1}^{\infty} d_{n,k}y_k = \lim_{n\to\infty}\sum_{k=1}^{n} d_{n,k}y_k$ exists for all $y \in c_0$ by (13), then $D = (d_{n,k}) \in (c_0, c)$. This implies that

$$\sup_n \sum_{k=1}^{\infty}|d_{n,k}| = \sup_n \sum_{k=1}^{n}\left|\sum_{i=k}^{n}\sum_{j=1}^{\infty} b_{i,j}z_j\right| < \infty,$$

by Theorem 2.8(iv). Thus we conclude $\sum_{k=1}^{\infty}|R_k| < \infty$. Furthermore (12) implies that $\lim_{n\to\infty} R_{n+1}\sum_{k=1}^{\infty} a_{n+1,k}x_k$ exists for each $x \in c_0(A,\Delta)$. So by Lemma 3.11 we have $(R_n) \in c_0^N(\Delta)$, which completes the proof. □

Remark 3.13. *If $A = B = I$, we have $c_0^{\beta}(\Delta) = \{z = (z_k) \in l_1 : (R_k) \in l_1 \cap c_0^N(\Delta)\}$ where $R_k = \sum_{i=k}^{\infty} z_i$, hence Lemma 3 from [8] is resulted.*

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