Filomat 31:2 (2017), 321–334
DOI 10.2298/FIL1702321M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Lagrange's Theorem, Convex Functions and Gauss Map

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Abstract. As one of the main results we prove that if f has Lagrange unique property then f is strictly convex or concave (we do not assume continuity of the derivative), Theorem 2.1. We give two different proofs of Theorem 2.1 (one mainly using Lagrange theorem and the other using Darboux theorem). In addition, we give a few characterizations of strictly convex curves, in Theorem 3.5. As an application of it, we give characterization of strictly convex planar curves, which have only tangents at every point, by injective of the Gauss map. Also without the differentiability hypothesis we get the characterization of strictly convex or concave functions by two points property, Theorem 4.2.

1. Introduction and Notation

Convex functions play an important role in many areas of mathematics and in science, see for example [9, 11, 21, 22] and the cited literature there as well as in this paper. In particular, the question posted in [22], received significant attention and a large number of answers submitted to the question. For instance, a (strictly) convex function on an open set has no more than one minimum. Therefore, the properties of both monotone and convex functions as well as functionals defined by those functions in the calculus of variations, are well-understood.

In order to discus them we need the following definitions:

Let S be a vector space over the real numbers, or, more generally, some ordered field. This includes Euclidean spaces. A set C in S is said to be convex if, for all x and y in C and all t in the interval $[0, 1]$, the point $(1 - t)x + ty$ also belongs to C. In other words, every point on the line segment connecting x and y is in C. Furthermore, C is strictly convex if every point on the line segment connecting x and y other than the endpoints is inside the interior of C.

A function f defined on a subset of the real numbers with real values is called monotonic if and only if it is either entirely increasing or decreasing. It is called monotonically increasing (also increasing or nondecreasing), if for all x and y such that $x \leq y$ one has $f(x) \leq f(y)$, so f preserves the order. Likewise, a function is called monotonically decreasing (also decreasing or nonincreasing) if, whenever $x \leq y$, then $f(x) \geq f(y)$, so it reverses the order.

2010 *Mathematics Subject Classification*. Primary 26A48, 26A48, 26A51
Keywords. Lagrange's theorem, injective Gauss map, strictly convex
Received: 10 February 2016; Accepted: 20 April 2016
Communicated by Vladimir Rakočević
M. Mateljević is supported by Ministry of Education, Science and Technological Development of the Republic of Serbia, grant No. 174032.
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If the order $\leq$ in the definition of monotonicity is replaced by the strict order $<$, then one obtains a stronger requirement. A function with this property is called strictly increasing. Again, by inverting the order symbol, one finds a corresponding concept called strictly decreasing. Functions that are strictly increasing or decreasing are one-to-one (because for x not equal to y, either $x < y$ or $x > y$ and so, by monotonicity, either $f(x) < f(y)$ or $f(x) > f(y)$, thus $f(x)$ is not equal to $f(y)$.)

By G_f we denote the graph of f.

Let X be a convex set in a real vector space and let $f : X \to \mathbb{R}$ be a function.

f is called convex if:

$\forall x_1, x_2 \in X, \forall t \in [0,1] : \qquad f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$.

f is called strictly convex if:

$\forall x_1 \neq x_2 \in X, \forall t \in (0,1) : \qquad f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$.

A function f is said to be (strictly) concave if $-f$ is (strictly) convex.

For $x_1, x_2 \in (a,b)$, define

$$k = k(x_1,x_2) = k_f(x_1,x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}\,.$$

The quantity $k_f(x_1,x_2)$ is related to notion of convexity and it has geometric interpretation as "slope" of the line defined by $A_1 = (x_1, f(x_1))$ and $A_2 = (x_2, f(x_2))$.

In this setting, if there is $c \in (x_1,x_2)$ such that $k_f(x_1,x_2) = f'(c)$, we say that c is Lagrange point with respect to the secant A_1A_2 (or points x_1, x_2). c is Lagrange point for f, if it is Lagrange point with respect to some secant of graph f.

Example $f(x) = x^3$ shows that 0 is not a Lagrange point.

Frequently, we use in the text short notations for properties and statements. For example we denote the implication (B) implies (A) shortly by (Ba). We advice the interested reader either to use the scheme below or to make his own scheme of some notations and statements in order to follow the exposition.

The scheme of short notations:

(A^*): the function f is strictly convex, (A_*): the function f is strictly concave on $[a,b]$,

(A): The function f is strictly convex or strictly concave on $[a,b]$; thus (A) is equivalent to ((A^*) or (A_*))

(B): f has Lagrange unique property for secants

(C): every line intersects the graph of f at most two points (two points property),

(A'): The function f is convex or concave on $[a,b]$

(B'): f' is injective,

(I-1): $(A) \Rightarrow (B') \Rightarrow (B)$, (I-2): $(B) \Rightarrow (C)$ and (I-3): $(A) \Rightarrow (C)$

$(Ba) : (B) \Rightarrow (A)$,$(Ca) : (C) \Rightarrow (A)$, $(Bb') : (B) \Rightarrow (B')$,

(S-1): $(A) \Rightarrow (B') \Rightarrow (B) \Rightarrow (C)$, (S-2): $(B) \Rightarrow (C) \Rightarrow (A')$,

(I-4): if f' is injective then (B"): it is strictly increasing or decreasing,

(S-3): $(B) \Rightarrow (B') \Rightarrow (B'')$.

1.1. *Background and short review*

Our consideration in section 2 is based on the mean value theorem which states, roughly: that for a given planar arc between two endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints. It is one of the most important results not only in differential calculus, but in mathematical analysis as well. That is also useful in proving the fundamental theorem of calculus (sometimes referred to as the second fundamental theorem of calculus or the Newton-Leibniz axiom: the definite integral of a function can be computed by using any one of its infinitely-many antiderivatives). The mean value theorem follows from the more specific statement of Rolle's theorem, and can be used to prove the more general statement of Taylor's theorem (with Lagrange form of the remainder term).

More precisely, the mean value theorem (referred also as Lagrange's theorem) states:

Theorem 1.1. *If a function f is continuous on the closed interval $[a,b]$, where $a < b$, and differentiable on the open interval (a,b), then there exists a point c in (a,b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

For some versions of the Lagrange mean value theorem without the assumption of continuity and differentiability of functions we refer the interested reader to [12] and literature cited there.

While writing the first versions of the manuscript and discussions with colleagues we found that the subject is related to Darboux's theorem:

Theorem 1.2 (Darboux's theorem). *Let I be an open interval, $f : I \to \mathbb{R}$ a real-valued differentiable function. Then f' has the intermediate value property: If a and b are points in I with $a < b$, then for every y between $f'(a)$ and $f'(b)$, there exists an x in $[a, b]$ such that $f'(x) = y$.*

A proof is based on application of Fermat's theorem to the function $\phi(t) = f(t) - yt$, which attains a local maximum at a point $x \in (a, b)$.

Another proof based solely on the mean value theorem and the intermediate value theorem is due to L. Olsen [17].

Suppose that (H-1): f is continuous function on closed interval $[a, b]$ and (H-2): differentiable on open interval (a, b).
It is well known, if f convex on $[a, b]$, then f' is increasing. Hence, if $f'(x_1) = f'(x_2)$ for some $x_1 < x_2$, then f' is a constant k on $[x_1, x_2]$ and therefore $f(x) = k(x - x_1) + f(x_1)$.

Thus if (A^*): f is strictly convex, then (B^*): f' is strictly increasing. It is convenient to write this result in the form (I-0), where

(I-0): $(A^*) \Rightarrow (B^*)$.

In a similar way, we can prove if (A_*): f is strictly concave, then (B_*): f' is strictly decreasing.

Thus if (A): f is strictly convex or concave, then (B'): f' is injective. In particular, (A) implies

(B): for every points y, z in $[a, b]$ there is a unique c such that $f(z) - f(y) = f'(c)(z - y)$ (Lagrange unique property).

Thus, we have (I-1): $(A) \Rightarrow (B') \Rightarrow (B)$.

Note that it is not immediately clear that (B) is equivalent by (B'): namely, $\neg(B')$ means that there are different points c_1 and c_2 such that $f'(c_1) = f'(c_2)$, but in general there are examples of function f such that c_1 and c_2 are not Lagrange points.

In connection with (B) consider the statement

(C): every line intersect the graph at most two points (two points property).

$\neg(C)$: there is a line which intersect graph at least three point.

If we denote these points by $M_k = (x_k, f(x_k)$ and suppose that $x_1 < x_2 < x_3$, then an application of Lagrange's theorem shows that $f'(c_1) = k(x_1, x_2) = k(x_2, x_3) = f'(c_2)$, $x_1 < c_1 < x_2 < c_2 < x_3$, and it means that $\neg(B)$ is true.

Since if f is strictly convex then M_2 is strictly below the secant M_1M_3, and we also conclude that $\neg(A)$ holds.

Since $\neg(C)$ implies $\neg(B)$ and $\neg(A)$, by contraposition law[1)] (B) implies (C) and (A)implies (C).

Thus, we have (I-2): $(B) \Rightarrow (C)$ and (I-3): $(A) \Rightarrow (C)$ (without hypothesis (H-2) that derivative exists).

Thus, by (I-1) and (I-2), we have (S-1): $(A) \Rightarrow (B') \Rightarrow (B) \Rightarrow (C)$.

At a first glance it is not clear how to use the hypothesis(B) and therefore it seems that the following questions are natural problems and more intriguing than the implications $(A) \Rightarrow (B) \Rightarrow (C)$:

Question 1. Whether (B) implies (A)?

Question 2. Whether (B) implies (B'): f' is injective?

Question 3. Whether (Ca): (C) implies (A)?

By contraposition law, an affirmative answer to Question 3 is equivalent to the statement: $\neg(A)$ implies $\neg(C)$: there is a line which intersect graph at least three point.

Note that in the first versions of the paper, motivated by a problem in [2], we considered only differentiable curves and our beginning considerations are basically contained in Section 2. Then we observe that our approach allows to omit the hypothesis differentiability and it gets the characterization by two points property, Theorem 4.2. In section 4 we give an affirmative answer to Question 3. In section 2 and 3, using

[1)] In logic, contraposition is a law that says that a conditional statement is logically equivalent to its contrapositive.

different methods, we give affirmative answer to Question 1 and 2. Our approach, 2, is elementary and using contraposition law we first prove (B) implies that f is convex or concave. Namely, if f is neither convex nor concave we construct a line which intersect graph at least at three point (thus $\neg(C)$ is true) and therefore $\neg(A) \Rightarrow \neg(C)$ holds which is equivalent to (Ca). (Recall that an application of Lagrange's theorem shows that the condition B is not fulfilled). By (Ca) we close the chain of the implications given by (S-1) and and we have that conditions $(A), (B'), (B)$ and (C) are equivalent.

Application of Darboux's theorem to characterizations of convexity are the subject of section 3. First, using clear geometric interpretation we give short and direct proof of the implication: (B) implies (B') (Proposition 3.2). An intrigue corollary of Darboux's theorem is that (I-4): if (B'): f' is injective then, (B"): it is strictly increasing or decreasing. Hence, directly follows (A) without appeal to (Ca).

Note that in this setting (we suppose only differentiability of f) we have (S-3): $(B) \Rightarrow (B') \Rightarrow (B'')$ and by the way we get Proposition 3.4: If f' is injective on (a, b), then it is continuous on (a, b).

This Proposition is intriguing because it states an interesting phenomenon: that differentiability and injectivity of f' imply continuity of it.

After writing the manuscript a question appeared in [20], which is related to the subject.

2. Convex Functions and Lagrange's Theorem

Any convex function $f(x)$ on [a, b] is continuous on (a, b) and has a finite right derivative $f_+(x)$ and a left derivative $f_-(x)$ at each point $x \in (a; b)$. Moreover, for all $x \in (a; b)$, $f'_-(x) \leq f'_+(x)$, the equality occurring and yielding the derivative $f'(x)$ everywhere, except possibly a countable number of points inside (a, b). Wherever it exists, $f'(x)$ is a non-decreasing function of x.

Let $f(x)$ be a continuous function on a closed interval $[a, b]$ and differential on the open interval (a, b). In addition, if f is convex or concave, by Darboux's theorem, we conclude that f' is continuous (see Proposition 3.4). This result may exist in some forms in the literature, but we have not found it up to now.

2.1. Lagrange unique property

In the beginning versions of paper we basically prove that (A): The function is strictly convex or strictly concave on $[a, b]$ if and only if
(B): f has Lagrange unique property: For every secant of graph f there is a unique Lagrange point. Later we extract Proposition 4.1.

Theorem 2.1. *Let $f(x)$ be a continuous function on a closed interval $[a, b]$ and differential on an open interval (a, b).*
(I1) [2] Then (A): The function is strictly convex or strictly concave on $[a, b]$ if and only if
(B): for every α, β ($a \leq \alpha < \beta \leq b$) there is exactly one $\xi \in (\alpha, \beta)$ (which depends on α and β) such that

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(\xi).$$

(I2) The conditions (A), (B'), (B) and (C): every line intersect the graph at most two points (two points property), are equivalent.

[2] Part of (I1) is formulated as the problem in N. Bourbaki, 2007, *Fonctions d'une variable réelle: Théorie élémentaire*, Springer,[2]

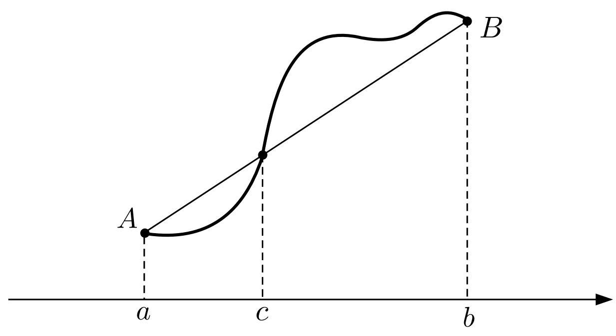

Figure 1: Secant intersection the curve

Proof. Let us prove that (Ba): (B) implies (A). In the proof, we will use contraposition law at several places.

Step 1. Let g be a straight line containing point $A(a, f(a))$ and $B(b, f(b))$.

Let us first prove that (B) implies (C′): g does not intersect the graph of f over (a, b).

Suppose (B).

If $f(c) = g(c)$ for some $c \in (a, b)$ then

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1) = \frac{f(b) - f(a)}{b - a} \quad \text{and}$$
$$\frac{f(b) - f(c)}{b - c} = f'(\xi_2) = \frac{f(b) - f(a)}{b - a}.$$

Hence, for $\alpha = a, \beta = b$ there are two points ξ_1 and ξ_2 with mentioned characteristics, so f does not meet stated requirements.

Thus, by the contraposition low, (C′) is true.

Therefore, since f and g are continuous function, then

(i) $f(x) < g(x)$, $\forall x \in (a, b)$ or

(ii) $f(x) > g(x)$, $\forall x \in (a, b)$.

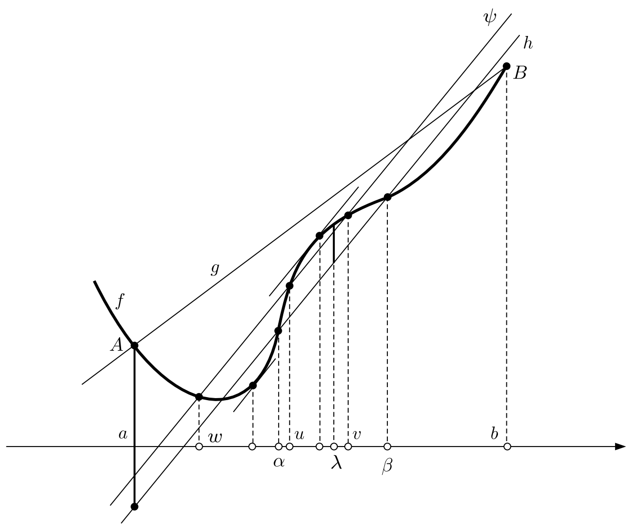

Figure 2: Graph of a function beneath the secant

Step 2. Let us assume that condition (i), is valid, i.e. that

$$f(x) < g(x), \quad \forall x \in (a, b).$$

We are now proving that $(B) \Rightarrow (A')$, where (A') : f is convex (and then using it that (A) holds). It is a corollary of implications (S-2): $(B) \Rightarrow (C) \Rightarrow (A')$.

We first prove (Ca′): C implies A'.

Let us assume the opposite, that $(\neg A')$: f is not convex. Then there are $\alpha, \beta \in [a, b]$, $\alpha < \beta$ such that

$$f(\lambda) > h(\lambda) \quad \text{for some} \quad \lambda \in (\alpha, \beta),$$

where h is a linear function whose graph contains points $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$.

For the line h we have $h(a) < f(a)$ or $h(b) < f(b)$, because otherwise we would have $f(\lambda) > g(\lambda)$ which contradicts the assumption (Namely, if $h(a) \geq f(a) = g(a)$ and $h(b) \geq f(b) = g(b)$, then also $h(x) \geq g(x)$, $a \leq x \leq b$. Hence, on the basis of $f(\lambda) > h(\lambda)$ $f(\lambda) > g(\lambda)$.)

Let $h(a) < f(a)$, as shown in Figure 3. So

$$\delta_1 \equiv f(\lambda) - h(\lambda) > 0$$
$$\delta_2 \equiv f(a) - h(a) > 0.$$

Let $0 < \delta < \min\{\delta_1, \delta_2\}$.

We place the line ψ which is parallel to line h such that

$$\psi(\lambda) = h(\lambda) + \delta.$$

It is straightforward to check that $\psi(a) < f(a)$, $\psi(\alpha) > h(\alpha) = f(\alpha)$, $\psi(\lambda) < f(\lambda)$ and $\psi(\beta) > f(\beta)$.

Hence line ψ intersects f at at least one point $(u, f(u))$ where $\alpha < u < \lambda$ (because $\psi(\alpha) > f(\alpha)$, $\psi(\lambda) < f(\lambda)$); and the line ψ also intersects f at some point $(v, f(v))$ where $\lambda < v < \beta$ and at some point $(w, f(w))$ where $a < w < \alpha$.

Thus $\neg A'$ implies $\neg C$, which is equivalent to (Ca′): C implies A'.

Now, we prove that (Bc): B implies C, which is equivalent $\neg C$ implies $\neg B$.

Suppose $\neg C$. Thus there are a line L and at least three different points u, v and w in (a, b) such that the points $(u, f(u))$, $(v, f(v))$ and $(w, f(w))$ belongs L.

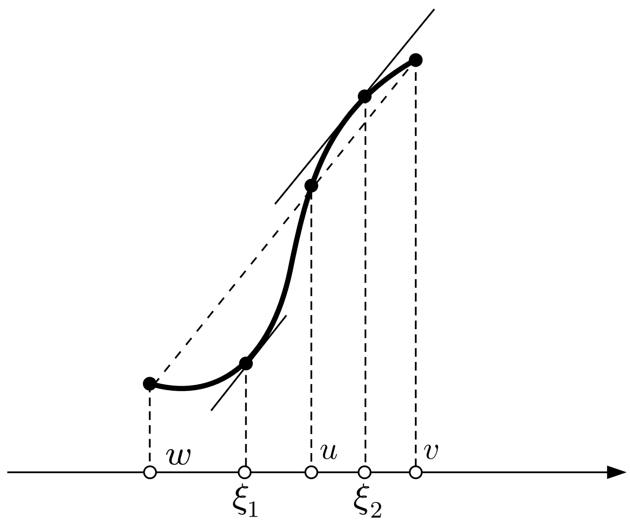

Figure 3:

Hence, there are two points $\xi_1 \in (w,u)$ and $\xi_2 \in (u,v)$ such that

$$f'(\xi_1) = \frac{f(u)-f(w)}{u-w} = \frac{f(v)-f(w)}{v-w} \quad \text{and}$$
$$f'(\xi_2) = \frac{f(v)-f(u)}{v-u} = \frac{f(v)-f(w)}{v-w}.$$

Therefore, on the interval $[w,v]$ there are two points for which

$$f'(\xi_i) = \frac{f(v)-f(w)}{v-w}, \quad i = 1,2;$$

which contradicts the initial assumption (B) that there is only one such ξ. By contraposition law, we conclude that (Bc) holds. Together with (Ca'), we conclude that (B) implies

(A'): function f is *convex*.

Step 3. It is left to prove that f is *strictly convex*.

If it is not true then by Step2, we can suppose that f is a convex function which is not strictly convex.

Since f is not strictly convex, then there are two different points α,β, $\alpha < \beta$, such the graph of f over (α,β) is not strictly below the secant AB, where $A = (\alpha, f(\alpha))$ and $(\beta, f(\beta))$. By convexity hypothesis it is below the secant AB. Therefore there is at least one point $\gamma \in (\alpha,\beta)$ such that the point $C = (\gamma, f(\gamma))$ is on the secant AB. Then there are three different points α,β and γ in (a,b) such that the points $(\alpha, f(\alpha))$, $(\beta, f(\beta))$, $(\gamma, f(\gamma))$ are on the some line. Hence f satisfy the condition (¬C) so the initial assumption about a unique point ξ would not be fulfilled.

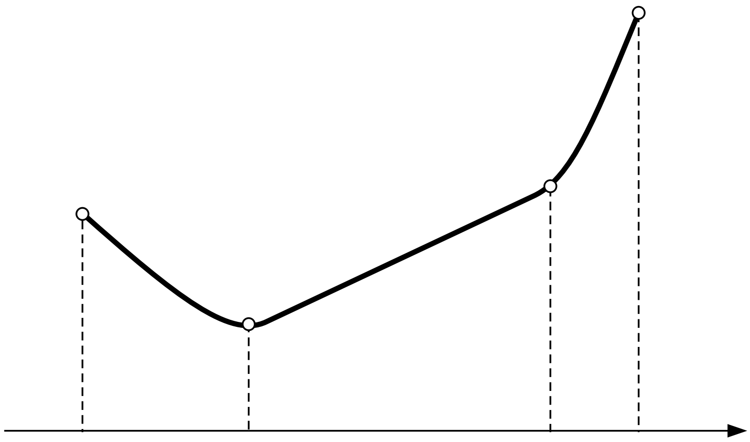

Figure 4: An example of a convex function graph

Let us prove the opposite: (A) implies (B). Thus we need to prove: if f strictly convex (or strictly concave), then for every α, β ($a \le \alpha < \beta \le b$) there is exactly one ξ (which depends on α and β) such that

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(\xi).$$

We use again contraposition law. Let us assume that f is strictly convex and that there are two points ξ_1, ξ_2, $\alpha < \xi_1 < \xi_2 < \beta$, for which

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(\xi_1) = f'(\xi_2).$$

Hence,

$$f'(\xi_1) \le f'(\xi) \le f'(\xi_2)$$

for every $\xi \in (\xi_1, \xi_2)$ so $f'(x) = \text{const}$ for every $x \in [\xi_1, \xi_2]$, because $f'(\xi_1) = f'(\xi_2)$. Function f is a straight line on an interval (ξ_1, ξ_2). Therefore, f is not strictly convex, which is a contradiction.

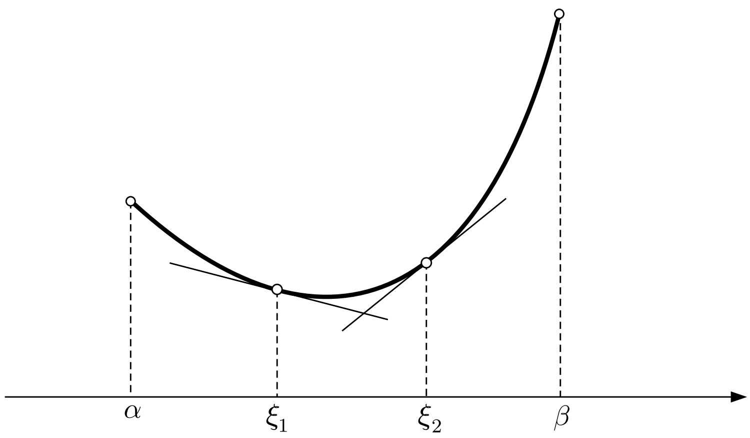

Figure 5: A strictly convex function

□

Remark 2.2. *a Note that in the proof of (Ca′) we use only the hypothesis (H-1): f is continuous function on closed interval $[a, b]$ and we do not use the hypothesis(H-2): f is differentiable on open interval (a, b). Hence, a small modification of the above proof of the implication (Ba): (B) implies (A), shows that (C) implies (A), where (C): every line intersect the graph at most two points (two points property). The proof that (A) implies (C) is straightforward and details are left to the interested readers (see section 4).*

2.2. Inflection points

In order to discuss the inflection points we first need the following results.

Recall that in the introduction section we outlined proof of (I-0): if (A^*): f is strictly convex, then (B^*): f' is strictly increasing.

Lemma 2.3. *Suppose (H1), (H2) and (A*). Let $c \in (a, b)$ and L the tangent line at $(c, f(c))$ over (a, b). Then (i): $f(x) > L(x)$ for $x \in (a, b) \setminus \{c\}$.*

Proof. By hypothesis we have two cases $x \in (a, c)$ or $x \in (c, b)$. Consider the case $x \in (a, c)$. Since $L(x) = f'(c)(x - c) + f(c)$ and $f(x) - L(x) = f(x) - f(c) - f'(c)(x - c)$, an application of Lagrange's theorem yields $f(x) - L(x) = [f'(c_1) - f'(c)](x - c)$, where $a < c_1 < c$. By (I-0): $(A^*) \Rightarrow (B^*)$. Hence $f'(c_1) < f'(c)$ and it gives (i). □

We left the interested reader to prove the following proposition.

Proposition 2.4. *Suppose (H1) and (H2). The f is strictly convex (concave) if and only if for every $c \in (a, b)$ the tangent line L_c at $(c, f(c))$ over (a, b) is strictly above (below) the graph of f except at $(c, f(c))$.*

Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$. We say that $c \in (a, b)$ is the inflection point of f if there is $\delta > 0$ such that (i) f is strictly convex on $(c - \delta, c)$ and f is strictly concave on $(c, c + \delta)$ or (i) holds by $-f$ instead of f. In addition if f is differentiable on (a, b), then by Proposition 2.4, c is the inflection point if and only if (1) or (2).

A good model for the statement (II2) of the next theorem is the function $f(x) = x^3$ with the inflection point at 0. Every line $y = kx$, $0 < k < \infty$, intersect the graph at three points.

Theorem 2.5. *Let $c \in (a,b)$ be the inflection point of the function $f : [a,b] \to \mathbb{R}$ which is continuous on $[a,b]$ and differentiable on (a,b).*

(II1) Then for some α, β ($a < \alpha < \beta < b$) there are two points $\xi, \eta \in (\alpha, \beta)$ ($\xi \neq \eta$) such that

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(\xi) = f'(\eta).$$

(II2) There is a subinterval (a_1, b_1), $a_1 < c < b_1$, of $[a,b]$ such that $f'(I_1) = f'(I_2)$, where $I_1 = (a_1, c)$ and $I_2 = (c, b_1)$ and if we suppose without loss of generality that $c = 0$, $f(c) = 0$ and define $k(x) = f(x)/x$, then for each $x_1 \in I_1$ and $x_2 \in I_2$ there are $x_1' \in (a,c)$ and $x_2' \in (c,b)$ such that $f'(x_1) = k(x_1')$ and $f'(x_2) = k(x_2')$.

Proof. Let the line $g(x)$ ($x \in (a,b)$) be the tangent of the function f at point c. Then there is $\delta > 0$ such that

$$\begin{aligned} &f(x) - g(x) > 0 \quad \big(x \in (c-\delta, c)\big) \\ &f(x) - g(x) < 0 \quad \big(x \in (c, c+\delta)\big) \end{aligned} \tag{1}$$

or

$$\begin{aligned} &f(x) - g(x) < 0 \quad \big(x \in (c-\delta, c)\big) \\ &f(x) - g(x) > 0 \quad \big(x \in (c, c+\delta)\big) \end{aligned} \tag{2}$$

where δ is chosen so that $(c-\delta, c+\delta) \subseteq (a,b)$.

Let us assume that (1) is valid (see Fig. 6).

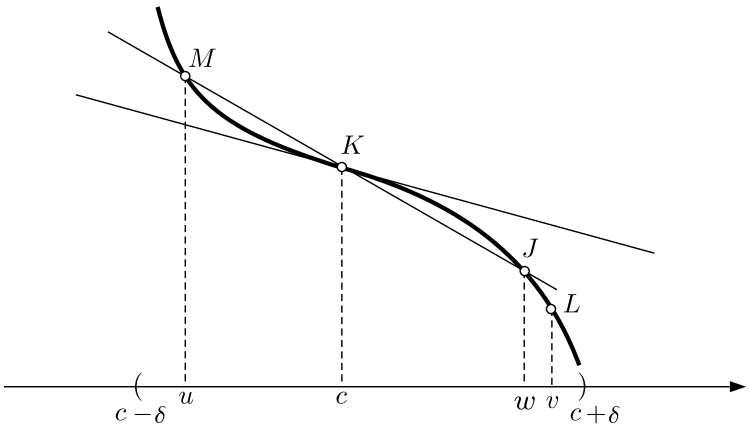

Figure 6: Inflection

Let points u and v chose so that

$$c - \delta < u < c < v < c + \delta.$$

Let h be a line that contains points $M(u, f(u))$ and $K(c, f(c))$.

(i) If the point $L(v, f(v))$ is below the line h, i.e. $f(v) < h(v)$ then there is $w \in (c, v)$ so that $f(w) = h(w)$. (This case is shown in Fig. 6).

(ii) If $f(v) > h(v)$, then the line h should be places so as to contain points $(v, f(v))$ and $K(c, f(c))$. In that case $h(u) > f(u)$, so $h(\lambda) = f(\lambda)$ for some $\lambda \in (u, c)$.

Let us assume that (i) is valid. The explanation of the existence of point w. Line KL has less the slope then the line containing points K and $(x, f(x))$ for some $x \in (c, v)$ since the line $g(x)$ is the tangent of curve f at point c.

The remaining conclusion is:

$$\frac{f(c)-f(u)}{c-u} = \frac{f(w)-f(c)}{w-c}$$

so there are different points ξ and η where $\xi \in (u,c)$, $\eta \in (c,w)$ for which

$$\frac{f(c)-f(u)}{c-u} = f'(\xi) \quad \text{and} \quad \frac{f(w)-f(c)}{w-c} = f'(\eta)$$

according to Lagrange's theorem.

Outline of (II2). Recall, for $x_1, x_2 \in (a,b)$, define

$$k = k(x_1,x_2) = k_f(x_1,x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Without loss of generality we can suppose that f is strictly concave on $[a,c]$, and f is strictly convex on $[c,b]$ and that $c = 0$ and $f(c) = 0$. Define $k(x) = f(x)/x$. If $k_2 = k(b) \geq k_1 = |k(a)|$, then for every $x \in [a,c)$ there is $z \in (c,b)$ such that $k(x) = |k(z)|$.

If f has derivative on $I = (0,b)$, then $k(x) \leq f'(x)$ on I and there is $b_1 \in I$ such that $f'(b_1) = k_2$. Thus for every $x \in I_1 := (0,b_1)$ there is $x' \in I$ such that $x' > x$ and $f'(x) = k(x')$. The rest of the proof is straightforward. □

3. Darboux and Lagrange Theorem

In this section using Darboux theorem we give another proof of Theorem 2.1, see also [13].

Throughout this section we suppose (H1): f is continuous function on closed interval $[a,b]$ and (H2): f differentiable on open interval (a,b).

Lemma 3.1. *Let $x_1 < x_2$ in (a,b) and $f'(x_1) = f'(x_2)$. Then there are at least two points c_1 and c_2 in (x_1,x_2) such that $f(x_2)-f(x_1) = f'(c_1)(x_2-x_1)$ and $f(x_2)-f(x_1) = f'(c_2)(x_2-x_1)$.*

Proof. Set $M_i = (x_i, f(x_i))$, 1=1,2 , and denote by L_i tangent line at M_i. Since L_1 is paralell to L_2, we can define a parallelogram P by lines $L_1, L_2, x = x_1$ and $x = x_2$. >From geometric interpretation it is clear that the diagonal (segment) $[M_1,M_2]$ of parallelogram P intersects graph of f over (x_1,x_2) at least a point $M_3 = (x_3, f(x_3))$, $x_1 < x_3 < x_2$. By application of Lagrange theorem there are points $c_1 \in (x_1,x_3)$ and $c_2 \in (x_3,x_2)$ such that $f(x_3)-f(x_1) = f'(c_1)(x_3-x_1)$ and $f(x_2)-f(x_3) = f'(c_2)(x_2-x_3)$. □

Note that the above lemma states $\neg(B') \Rightarrow \neg(B)$. By contraposition low, from the above lemma we get $(B) \Rightarrow (B')$, that is:

Proposition 3.2. *If (B): f satisfies Lagrange unique property for secants, then (B'):f' is injective.*

Lemma 3.3. *If (B'): f' is injective on (a,b), then (B''): f' is strictly increasing or decreasing.*

Proof. Contrary, suppose that $\neg(B'')$. Thus there are points $x_1 < x_2 < x_3$ such that $f'(x_1) < f'(x_2) > f'(x_3)$.

By Darboux's theorem, we can choose c_* and c^* such that $x_1 < c_* < x_2 < c^* < x_3$ and $f'(c_*) = f'(c^*)$. Thus $\neg(B')$ and this yields a contradiction. □

By Proposition 3.2 and Lemma 3.3, we have

(S-3): $(B) \Rightarrow (B') \Rightarrow (B'')$.

By (I-1) we have

(S-4): $(A) \Rightarrow (B) \Rightarrow (B') \Rightarrow (B'')$.

It is not difficult to prove that $(B'') \Rightarrow (A)$.

We need to consider two cases: Case 1. (B^*): f' is strictly increasing.

Case 2. (B^*): f' is strictly decreasing.

We will consider Case 1. Take $a \le x_1 < x_2 < x_3 \le b$. By Lagrange's theorem, there are points $c_1 \in (x_1, x_2)$ and $c_2 \in (x_2, x_3)$ such that $k_f(x_1, x_2) = f'(c_1)$ and $k_f(x_2, x_3) = f'(c_2)$. By hypothesis (B^*), $f'(c_1) < f'(c_2)$ and therefore $k_f(x_1, x_2) < k_f(x_2, x_3)$ and we have (A^*). Thus we have $(B^*) \Rightarrow (A^*)$. We left to the reader to treat Case 2 in a similar way.

Using the implication $(B'') \Rightarrow (A)$, we can close the chain of implications $(A) \Rightarrow (B'') \Rightarrow (A)$.

By (S-4), we have a more delicate result,

(S-5): $(A) \Rightarrow (B) \Rightarrow (B') \Rightarrow (B'') \Rightarrow (A)$.

Although the following result is a corollary of Darboux's theorem, we state it as a separate result, because we believe that it has an independent interest.

Proposition 3.4. *If f' is injective on (a, b), then it is continuous on (a, b).*

Proof. By Lemma 3.3, f' is strictly increasing or decreasing on (a, b). Without loss of generality we can suppose that f' is strictly increasing. By Darboux's theorem, then for every c and d such that $a < c < d < b$, we have $f'\big((c, d)\big) = (f'(c), f'(d))$. Hence, f' is continuous on (a, b). □

Now, we can summarize the above considerations as follows.

Theorem 3.5. *Suppose that f is continuous function on closed interval $[a, b]$ and differentiable on open interval (a, b). Then the following conditions are equivalents:*

(A): f is strictly convex or f is strictly concave

(B): for every points y, z in $[a, b]$ there is a unique c such that $f(z) - f(y) = f'(c)(z - y)$ (Lagrange unique property for secants).

(C) every line intersect the graph at most two points (two points property).

(D) The line segment between any two points on the graph lies above (or below) the graph.

(B′): f' is injective

(B″) f' is strictly decreasing or strictly increasing.

By (S-5), (A), (B), (B′) and (B″) are equivalent. The rest of the proof is strait-forward and we leave it to the interested reader.

3.1. Gauss map and convexity

In this subsection we only outline some results, and we plan in a future work further to develop this subsection and connect better to the bulk of the paper[3)]; see also [13]. The Gauss map (named after Carl F. Gauss) maps a surface in Euclidean space $\mathbb{R}^n$ to the unit sphere $\mathbb{S}^{n-1}$. But here we need only definition for curves.

Namely, given a curve X lying in $\mathbb{R}^2$, the Gauss map is a continuous map $N : X \to \mathbb{S}$ such that $N(p)$ is a unit vector orthogonal to X at p, namely the normal vector to X at p.

Theorem 3.6 ([13]). *A convex planar curve which has only tangents at every point is strictly convex if and only if (i) Gauss map is injective.*

Proof. Suppose first that the curve is given as the graph of a function f on (a, b) then (i) implies that (ii) f has Lagrange unique property. If (ii) is not true then there are points a_1, b_1 and Lagrange points c_1, c_2 such that $f(b_1) - f(a_1) = f'(c_1) = f'(c_2)$. Then $n_{M_1} = n_{M_2}$, where $M_k = (c_k, f(c_k))$. Proof that (ii) implies (i). By (I-0), if f is strictly convex then f' is strictly increasing and (i) holds.

In general if C is strictly convex closed curve, then there are two supporting lines parallel to y-axis which touch C at points A and B. The points A and B divide C on two curves which are graphs of functions f_1 and f_2. □

[3)]S. J. Colley, Editor-Elect, The American Mathematical Monthly, who read a previous version (in a privative communication) of this paper, gave similar comments.

We shortly discuss a similar characterization of strictly convex surfaces: If $z = f(x_1, x_2, \cdots, x_n)$ defined a hyper surface W in $\mathbb{R}^{n+1}$ and Gauss map is injective, then W is strictly convex. The Gauss map (named after Carl F. Gauss) maps a surface in Euclidean space $\mathbb{R}^3$ to the unit sphere $\mathbb{S}^2$. Namely, given a surface X lying in $\mathbb{R}^3$, the Gauss map is a continuous map $N : X \to \mathbb{S}^2$ such that $N(p)$ is a unit vector orthogonal to X at p, namely the normal vector to X at p.

The Gauss map can be defined (globally) if and only if the surface is orientable, in which case its degree is half the Euler characteristic.

For example, it seems that we can prove (cf. [13]):

Theorem 3.7. *Let $z = f(x, y)$ be a real-valued differentiable function defined on a domain D in $\mathbb{R}^2$. Then it defines a surface W in $\mathbb{R}^3$. If its Gauss map N is injective, then W is strictly convex.*

Here $N(p)$ denotes a unit vector orthogonal to W at p, namely the normal vector to W at p.

If $p = (p_1, p_2, p_3)$, $X(p) = (f'_x, f'_y, 1)$ and $N(p) = X(p)/|X(p)|$. Since we compute f'_x, f'_y at (p_1, p_2), here we can consider that Gauss map is defined on D.

Theorem 3.8. *Let D be a domain in $\mathbb{R}^n$ and $z = f(x_1, x_2, \cdots, x_n)$, $x = (x_1, x_2, \cdots, x_n) \in D$ a differentiable function, and W a hyper surface in $\mathbb{R}^{n+1}$ defined by the graph of f. Then if the Gauss map of W is injective, then W is strictly convex.*

4. Concluding Remarks; Convex Functions and Two Points Property

Recall from the proof of Theorem 2.1, see Remark 2.2, we can extract:

Proposition 4.1. *Let $f(x)$ be a continuous function on a closed interval $[a, b]$. (I1) If f is not convex, then $\neg(C)$: there is a line which intersect graph at least three point.*

(I2) If f is not strictly convex, then $\neg(C)$.

(I3) (C) implies (A)

Note (I2) implies (I1). By contraposition law (I2) is equivalent with (I3).

By (S-1) and (I3), we get:

Theorem 4.2. *Let $f(x)$ be a continuous function on a closed interval $[a, b]$.*

(A): The function is strictly convex or strictly concave on $[a, b]$ if and only if

(C): every line intersect the graph at most two points (two points property)

Proof. It is left to prove that (A) implies (C), which is by contraposition low equivalent to $\neg(C)$ implies $\neg(A)$. Suppose that $\neg(C)$ and therefore there are there different points M_1, M_2 and M_3 on the graph, say that M_2 is between M_1 and M_3. We have a contradiction with convexity property (A) and the proof is done. □

A. Mohammadi, at the Research Gate, cf. [20], asked

Question 4. Let B be a closed set in n-dimensional Euclidean space. What other properties B should have in order to be guaranteed that there exist the closed convex set A such that $\partial A = B$. How about infinite dimensional spaces?

It seems that

Question 5. How can one characterize the boundary of a strictly convex set?

is related to the original question (but it is easier then Question 4).

For example, one can check the following:

Proposition 4.3. *If A is a closed Jordan curve in plane, then the following conditions are equivalent*

(1) A is the boundary of a strictly convex set

(2): every line intersect A at most two points (two points property).

(3): A is union of two sub-arcs which are graphs of strictly convex functions

4.1. Further results

It seems to us that it is natural to consider possible generalizations of this result, concerning convex sets, to several dimensions. For example, the first author, cf. [13, 20], suggested the following answer to Mohammadi's question: Suppose that (i) B is homeomorphic to $n-1$ dimensional sphere and (ii) for every point x in B there is a closed half-spaces $L(x)$ (sets of point in space that lie on and to one side of a hyperplane) which contains B. It seems at a first glance that properties (i) and (ii) guarantee that there exist the closed convex set A such that $\partial A = B$.

We also discussed on the Belgrade Analysis Seminar (BAS) some results of this type and in particular, the first author presented the content of this paper and suggested some generalizations of the results obtained in this paper to several dimensions. Here are some possibilities:

(S-1) Under hypothesis of Theorem 3.8, prove that the Gauss map is continuous.

(S-2) Let domain G be homeomorphic to 3-dimensional ball and let S be the boundary of G which is homeomorphic to 2-dimensional sphere S^2. Then S is a strictly convex surface if and only if the intersection with every plane is empty, a point or a strictly convex closed curve.
It seems natural to consider what are appropriate generalization of Theorem 4.2 in several variables? We give a possibility.

(S-3) Let f be a continuous function defined on a (connected) subdomain D of $\mathbb{R}^n$. Then the function is either strictly convex or strictly concave if and only if every straight line intersects its graph at most two points(for $n = 1$, this is Theorem 4.2). More generally, we can also consider hyper surfaces instead of surfaces defined by the graphs of functions.
Our first impression was that it is a straightforward of 1-dimensional statement. If we remember correctly the following procedure has been suggested for a proof of (S3): Let $I = [a, b]$ be a segment contained in D and g the restriction of f on I. Apply to 1-dimensional statement on g and use continuity of f. It turns out that we need additional details, see [13] and M. Pavlović, Technical Report: Remarks on convex functions (to be continued) posted on Researchgate (RG).

Acknowledgement. The manuscript [13] with this article have attracted much attention, in particular on RG. J. Peters, S. Zlobec and M. Cankaya wrote feedbacks, and in particular J. Peters gave encouraging comments. The first author also discussed the subject with D. Minda, S. J. Colley and in particular with M. Pavlović, and they all gave some interesting and useful comments from their perspectives. M. Svetlik helped very much in the preparation of this manuscript and he is also taking the notes on the Belgrade Analysis Seminar. We are indebted to all of those people for encouraging to work on the subject and for useful comments.

References

[1] S. Alexander, M. Ghomi, The convex hull property of non compact hyper surfaces with positive curvature: http://www.math.uiuc.edu/ sba/nchp.pdf
[2] N. Bourbaki, *Fonctions d'une variable réelle: Théorie élémentaire*, Springer, 2007.
[3] Bronshtein, I. N.; Semendyayev, K. A.; Musiol, G.; and Muehlig, H. Handbook of Mathematics, 4th ed. New York: Springer-Verlag, 2004.
[4] Biflecnode, Curvature, Differentiable, Extremum, First Derivative Test, Local Maximum, Local Minimum, Second Derivative Test, Stationary Point
[5] J. J. Castro, On Convex Curves Which Have Many Inscribed Triangles of Maximum Area, The American Mathematical Monthly Vol. 122, No. 10 (December 2015), pp. 967-971
[6] U. Eckhardt, W. Scherl, Z. Yu, *Representations of plane curves by means of descriptors in Hough space*: http://www.math.uni-hamburg.de/home/eckhardt/HOUGH.pdf
[7] G. H. Hardy, *A Course of Pure Mathematics*, Cambridge University Press, 1945.
[8] M. A. Khan, *Majorization theorem for convexifable functions*, Math. Commun. **18** (2013), 61-65.
[9] V. Klee, What is a convex set ?
https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/VictorKlee.pdf
[10] S. Kurepa, *Matematička analiza*, I i II deo, Školska knjiga, Zagreb, 1997.
[11] P. Loh, Convexity, 2013, www.math.cmu.edu/ ploh/.../convexity-soln.pdf
[12] M. Mateljević, M. Svetlik, M. Albijanić, N. Savić, *Generalizations of the Lagrange mean value theorem and applications*, FILOMAT, **27:4** (2013), 515-528.

[13] M. Mateljević, *Note on Convexity V2*, Sep 2015;
Note on Convex functions and sets (In preparation), March 2016;
https://www.researchgate.net/profile/Miodrag_Mateljevic.
[14] M. Mateljević, https://www.researchgate.net/profile/Miodrag_Mateljevic/questions
[15] D. S. Mitrinović (saradnik P. M. Vasić), *Analitičke nejednakosti*, Beograd, 1970.
[16] D. S. Mitrinović, *Matematika u obliku metodičke zbirke zadataka sa rešenjima* III deo, Gradjevinska knjiga, Beograd, 1988.
[17] L. Olsen, *A New Proof of Darboux's Theorem*, Vol. 111, No. 8 (Oct., 2004) (pp. 713-715), The American Mathematical Monthly
[18] N. Merentes, K. Nikodem, Remarks on strongly convex functions, Aequationes Mathematicae 80(1):193-199, September 2010
[19] N. Merentes, K. Nikodem, Strong convexity and separation theorems, Aequationes Mathematicae 2015
[20] A. Mohammadi, https://www.researchgate.net/post/How_can_one_characterize_the_ boundary_of_a_convex_ set?
[21] C. P. Niculescu, L. E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics vol. 23, Springer-Verlag, New York, 2006. xvi+256 pp. ISBN 0-387-24300-3
[22] J. F. Peters, What are applications of convex sets and the notion of convexity in mathematics and science?, https://www.researchgate.net/profile/James_Peters, Question 85
[23] Simon, Chapter 21, p.505- 522, www.rmi.ge/ kade/.../Week6Concave.pdf
[24] V. Soltan, Lectures on Convex Sets, World Scientific 2015.
[25] Weisstein, Eric W. "Inflection Point." From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/InflectionPoint.html
[26] S. Zlobec, *Characterizing zero-derivative points*, Journal of Global Optimization **46(1)**(2010), 155-161.
[27] S. Zlobec, *On The Derivative Of Smooth Meaningful Functions*, Croatian Operational Research Review **2** (2011).
[28] S. Zlobec, *Separation property of continuously differentiable functions*, Math. Commun. **19**(2014), 57-64.
[29] Is there a book in English where one can find characterizations of zero-derivative (stationary) points ? - ResearchGate. Available from: https://www.researchgate.net/post
[30] Б. А. М. Зорич, *Математически анализ*, ФАЗИС, Наука, 1997.