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C-Normal Topological Property

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Abstract. A topological space X is called C-*normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \upharpoonright C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. We investigate this property and present some examples to illustrate the relationships between C-normality and other weaker kinds of normality.

Introduction

In this paper, we investigate C-normal topological property which was presented by Arhangel'skii in 2012 when he was visiting Mathematics Department, King Abdulaziz University at Jeddah, Saudi Arabia. We prove that both submetrizability and local compactness imply C-normality but the converse is not true in general. We present some examples to show that C-normality and mild normality are independent of each other. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space, and a Tychonoff space is a T_1 completely regular space. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinals α such that $\alpha < \gamma$. The first infinite ordinal is ω_0 and the first uncountable ordinal is ω_1.

1. C-Normal Topological Property

Definition 1.1. (Arhangel'skii, 2012) A topological space X is called C-*normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \upharpoonright C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$.

Obviously, any normal space is C-normal, just by taking $X = Y$ and f to be the identity function. But the converse is not true in general. For example, the square of the Sorgenfrey line is C-normal which is not normal. It is C-normal because it is submetrizable. Recall that a topological space (X, τ) is called *submetrizable* if there exists a metric d on X such that the topology τ_d on X generated by d is coarser than $\mathcal{T}$, i.e. $\mathcal{T}_d \subseteq \mathcal{T}$, see [2].

Theorem 1.2. *Every submetrizable space is C-normal.*

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Proof. Let $\mathcal{T}'$ be a metrizable topology on X such that $\mathcal{T}' \subseteq \mathcal{T}$. Then $(X, \mathcal{T}')$ is normal and the identity function $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}')$ is a continuous bijective. If C is any compact subspace of $(X, \mathcal{T})$, then the restriction of the identity function on C onto $id_X(C)$ is a homeomorphism because C is compact, $id_X(C)$ is Hausdorff being a subspace of the metrizable space $(X, \mathcal{T}')$, and every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism [1, 3.1.13]. □

Arhangel'skii introduced the notion of epinormality on his visit to Saudi Arabia as mentioned earlier. A topological space (X, τ) is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4. By a similar proof as that of Theorem 1.2 above, we can prove the following corollary.

Corollary 1.3. *Every epinormal space is C-normal.*

Any indiscrete space which has more than one element is an example of a C-normal space which is not epinormal.

Corollary 1.4. *If X is a T_1 space such that the only compact subsets are the finite subsets, then X is C-normal.*

Proof. Let X be a T_1 space such that the only compact subsets are the finite subsets. Let $Y = X$ and consider Y with the discrete topology. Then the identity function from X onto Y works. □

Here is an example of a non-C-normal space,

Example 1.5. $\mathbb{R}$ with the particular point topology $\mathcal{T}_p$, see [8], where the particular point is $p \in \mathbb{R}$, is not C-normal. Recall that $\mathcal{T}_p = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : p \in U\}$. It is well-known that $(\mathbb{R}, \mathcal{T}_p)$ is neither T_1 nor normal space and if $A \subseteq \mathbb{R}$, then $\{\{x, p\} : x \in A\}$ is an open cover for A, thus a subset A of $\mathbb{R}$ is compact if and only if it is finite. To see that $(\mathbb{R}, \mathcal{T}_p)$ is not C-normal, suppose that $(\mathbb{R}, \mathcal{T}_p)$ is C-normal. Let Y be a normal space and $f : \mathbb{R} \longrightarrow Y$ be a bijective such that the restriction $f \upharpoonright C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace C of $(\mathbb{R}, \mathcal{T}_p)$. For the space Y, we have only two cases:

Case 1: Y is T_1. Take $C = \{x, p\}$, where $x \neq p$; then C is a compact subspace of $(\mathbb{R}, \mathcal{T}_p)$. By assumption $f \upharpoonright C : C \longrightarrow f(C) = \{f(x), f(p)\}$ is a homeomorphism. Since $f(C)$ is a finite subspace of Y and Y is T_1, then $f(C)$ is a discrete subspace of Y. Thus, we obtain that $f_{|_C}$ is not continuous which is a contradiction as $f_{|_C}$ is a homeomorphism.

Case 2: Y is not T_1. We claim that the topology on Y is the particular point topology with $f(p)$ as its particular point. To prove this claim, we suppose not, then there exists a non-empty open set $U \subset Y$ such that $f(p) \notin U$. Pick $y \in U$ and let $x \in \mathbb{R}$ be the unique real number such that $f(x) = y$. Consider $\{x, p\}$. Note that $x \neq p$ because $f(x) = y \in U$, $f(p) \notin U$, and f is one-to-one. Consider $f \upharpoonright \{x, p\} : \{x, p\} \longrightarrow \{y, f(p)\}$. Now, $\{y\}$ is open in the subspace $\{y, f(p)\}$ of Y because $\{y\} = U \cap \{y, f(p)\}$, but $f^{-1}(\{y\}) = \{x\}$ and $\{x\}$ is not open in the subspace $\{x, p\}$ of $(\mathbb{R}, \mathcal{T}_p)$, which means $f \upharpoonright \{x, p\}$ is not continuous, a contradiction, and our claim is proved. But any particular point space consisting of more than one point cannot be normal, so we get a contradiction as Y is assumed to be normal.

Therefore, $(\mathbb{R}, \mathcal{T}_p)$ is not C-normal.

Theorem 1.6. *If X is a compact non-normal space, then X cannot be C-normal.*

Proof. Let X be a compact non-normal space. Suppose that X is C-normal, then there exist normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \upharpoonright C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. Since X is compact, then $X \cong Y$, and this is a contradiction as Y is normal and X is not. Therefore, X cannot be C-normal. □

From the above theorem, we conclude that $\mathbb{R}$ with the finite complement topology is not C-normal.

Theorem 1.7. *C-normality is a topological property.*

Proof. Let X be a C-normal space and let $X \cong Z$. Let Y be a normal space and let $f : X \longrightarrow Y$ be a bijective function such that the restriction $f \upharpoonright C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. Let $g : Z \longrightarrow X$ be a homeomorphism. Then Y and $f \circ g : Z \longrightarrow Y$ satisfy the requirements. □

A space X is called *locally compact* if X is Hausdorff and for each $x \in X$ and each open neighborhood V of x there exists an open neighborhood U of x such that $x \in U \subseteq \overline{U} \subseteq V$ and $\overline{U}$ is compact.

Theorem 1.8. *Every locally compact space is C-normal.*

Proof. Let X be any locally compact topological space. By [1, 3.3.D] and [5], there exists a T_2 compact space Z and a bijective continuous $f : X \longrightarrow Z$. Since f is continuous, then for any compact subspace $A \subseteq X$ we have that $f \upharpoonright A : A \longrightarrow f(A)$ is a homeomorphism because $1-1$, onto, and continuity are inherited from f, and $f \upharpoonright A$ is closed as A is compact and $f(A)$ is Hausdorff. □

Corollary 1.9. *Since every locally compact space is C-normal, Deleted Tychonoff Plank [8] is a Tychonoff non-normal space which is C-normal.*

The converse of Theorem 1.8 is not true in general as shown by the following example.

Example 1.10. Recall that the Dieudonné Plank, [8], is defined as follows: Let

$$X = ((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_1, \omega_0 \rangle\}.$$

Write $X = A \cup B \cup N$, where $A = \{\langle \omega_1, n\rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0\rangle : \alpha < \omega_1\}$, and $N = \{\langle \alpha, n\rangle : \alpha < \omega_1$ and $n < \omega_0\}$. The topology $\mathcal{T}$ on X is generated by the following neighborhood system: For each $\langle \alpha, n\rangle \in N$, let $\mathcal{B}(\langle \alpha, n\rangle) = \{\{\langle \alpha, n\rangle\}\}$. For each $\langle \omega_1, n\rangle \in A$, let $\mathcal{B}(\langle \omega_1, n\rangle) = \{V_\alpha(n) = (\alpha, \omega_1] \times \{n\} : \alpha < \omega_1\}$. For each $\langle \alpha, \omega_0\rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0\rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. It is well-known that the Dieudonné Plank is Tychonoff non-normal space which is not locally compact, [8]. Now, a subset $C \subseteq X$ is compact if and only if C satisfies all of the following conditions:

(i) $C \cap A$ and $C \cap B$ are both finite;
(ii) If $\langle \omega_1, n\rangle \in C$, then the set $(\omega_1 \times \{n\}) \cap C$ is finite;
(iii) The set $\{\langle \alpha, n\rangle \in C : \langle \alpha, \omega_0\rangle \notin C\}$ is finite.

Now, define $Y = X = A \cup B \cup N$. Generate a topology $\mathcal{T}'$ on Y by the following neighborhood system: Elements of $B \cup N$ have the same local base as in X. For each $\langle \omega_1, n\rangle \in A$, let $\mathcal{B}(\langle \omega_1, n\rangle) = \{\{\langle \omega_1, n\rangle\}\}$. Then Y is T_4 space because it is paracompact. Consider the identity function $id : X \longrightarrow Y$. Let $C \subset X$ be any compact subspace. Then $id \upharpoonright C : C \longrightarrow id(C) = C$ is a bijectition. Let $\langle a, b\rangle$ be any element in C. If $\langle a, b\rangle \in N$, then $\{\langle a, b\rangle\}$ which is open in C as a subspace of X and Y will give that $id \upharpoonright C$ is continuous. If $\langle a, b\rangle \in B$ and W is any basic open set of $\langle a, b\rangle$ in C as a subspace of Y, then W is also a basic open set of $\langle a, b\rangle$ in C as a subspace of X, hence $id \upharpoonright C$ is continuous. If $\langle a, b\rangle \in A$, then the smallest open neighborhood of $\langle a, b\rangle$ in C as a subspace of Y is $\{\langle a, b\rangle\}$. Since C is compact in X, then, by item (ii) above, the set $(\omega_1 \times \{b\}) \cap C$ is finite. Write $(\omega_1 \times \{b\}) \cap C = \{\langle \alpha_1, b\rangle, ..., \langle \alpha_m, b\rangle\}$. Pick $\beta < \omega_1$ such that $\alpha_i < \beta$ for each $i \in \{1, ..., m\}$. Then $V_\beta(b)$ is a basic open set of $\langle a, b\rangle$ in X, hence $V_\beta(b) \cap C = \{\langle a, b\rangle\}$ is an open neighborhood of $\langle a, b\rangle$ in C as a subspace of X. Thus $id \upharpoonright C$ is continuous. From the three cases, we conclude $id \upharpoonright C$ is continuous. Since C is compact as a subspace of X and C is Hausdorff as a subspace of Y, we conclude that $id \upharpoonright C$ is a homeomorphism. Therefore, the Dieudonné Plank X is C-normal.

2. *C*-Normality and Other Properties

Recall that, see [3] and [4], a subset E of a space X is called a *closed domain* (called also, *regularly closed*, *κ-closed*) if $E = \overline{\mathrm{int} E}$, and a topological space X is called *mildly normal* [6] (called also, *κ-normal* [7]) if any two disjoint closed domains E and F of X, there exist two disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$. In general, C-normality and mild normality do not imply each other. $(\mathbb{R}, \mathcal{T}_p)$ is not C-normal, as we proved in 1.5, but it is mildly normal as the only closed domains are $\mathbb{R}$ and the empty set. C-normality does not imply mild normality as shown by the following example.

Example 2.1. Let $\mathbb{P}$ denote the irrationals and $\mathbb{Q}$ denote the rationals. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $p_n = \langle p, \frac{1}{n} \rangle \in \mathbb{R}^2$. For each $p \in \mathbb{P}$, fix a sequence $(p_n^\star)_{n\in\mathbb{N}}$ of rationals such that $p_n' = \langle p_n^\star, 0\rangle \longrightarrow \langle p, 0\rangle$, where the convergence is taken in $\mathbb{R}^2$ with its usual topology $\mathcal{U}$. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $A_n(\langle p, 0\rangle) = \{p_k' : k \geq n\}$ and $B_n(\langle p, 0\rangle) = \{p_k : k \geq n\}$. Now, for each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $U_n(\langle p, 0\rangle) = \{\langle p, 0\rangle\} \cup A_n(\langle p, 0\rangle) \cup B_n(\langle p, 0\rangle)$.

Let $X = \{\langle x, 0\rangle \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{\langle p, \frac{1}{n}\rangle = p_n : p \in \mathbb{P} \text{ and } n \in \mathbb{N}\}$. For each $q \in \mathbb{Q}$, let $\mathcal{B}(\langle q, 0\rangle) = \{\{\langle q, 0\rangle\}\}$. For each $p \in \mathbb{P}$, let $\mathcal{B}(\langle p, 0\rangle) = \{U_n(\langle p, 0\rangle) : n \in \mathbb{N}\}$. For each $p \in \mathbb{P}$ and each $n \in \mathbb{N}$, let $\mathcal{B}(p_n) = \{\{p_n\}\}$. Denote by $\mathcal{T}$ the unique topology on X that has $\{\mathcal{B}(\langle x, 0\rangle), \mathcal{B}(p_n) : x \in \mathbb{R}, p \in \mathbb{P}, n \in \mathbb{N}\}$ as its neighborhood system. Let $Z = \{\langle x, 0\rangle : x \in \mathbb{R}\}$. That is, Z is the x-axis. Then $(Z, \mathcal{T}_Z) \cong (\mathbb{R}, \mathcal{RS})$, where $\mathcal{RS}$ is the Rational Sequence Topology, see [8]. Since Z is closed in X and $(\mathbb{R}, \mathcal{RS})$ is not normal, then X is not normal, but, since any basic open set is closed-and-open and X is T_1, then X is zero-dimensional, hence Tychonoff. Now, Let $E \subset \mathbb{P}$ and $F \subset \mathbb{P}$ be closed disjoint subsets that cannot be separated in $(\mathbb{R}, \mathcal{RS})$. Let $C = \cup\{B_1(\langle p, 0\rangle) : p \in E\}$ and $D = \cup\{B_1(\langle p, 0\rangle) : p \in F\}$. Then C and D are both open in $(X, \mathcal{T})$ and $\overline{C}$ and $\overline{D}$ are disjoint closed domains that cannot be separated, hence X is not mildly normal. But X is submetrizable, hence C-normal.

Let X be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in X' by x' and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U \text{ is open in } X \text{ with } x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate* of X.

Theorem 2.2. *If X is C-normal, then its Alexandroff Duplicate $A(X)$ is also C-normal.*

Proof. Let X be any C-normal space. Pick a normal space Y and a bijective function $f : X \longrightarrow Y$ such that $f \restriction C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. Consider the Alexandroff duplicate spaces $A(X)$ and $A(Y)$ of X and Y respectively. Since Y is normal, then $A(Y)$ is also normal. Define $g : A(X) \longrightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$. If $a \in X'$, let b be the unique element in X such that $b' = a$, then define $g(a) = (f(b))'$. Then g is a bijective function. Now, a subspace $C \subseteq A(X)$ is compact if and only if $C \cap X$ is compact in X and for each open set U in X with $C \cap X \subseteq U$, we have that $(C \cap X') \setminus U'$ is finite. Let $C \subseteq A(X)$ be any compact subspace. We show $g \restriction C : C \longrightarrow g(C)$ is a homeomorphism. Let $a \in C$ be arbitrary. If $a \in C \cap X'$, let $b \in X$ be the unique element such that $b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$ we have that $\{a\}$ is open in C and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in C \cap X$. Let W be any open set in Y such that $g(a) = f(a) \in W$. Consider $H = (W \cup (W' \setminus \{f(a)'\})) \cap g(C)$ which is a basic open neighborhood of $f(a)$ in $g(C)$. Since $f \restriction C \cap X : C \cap X \longrightarrow f(C \cap X)$ is a homeomorphism, then there exists an open set U in X with $a \in U$ and $f \restriction C \cap X(U \cap C) \subseteq W$. Now, $(U \cup (U' \setminus \{a'\})) \cap C = G$ is open in C such that $a \in G$ and $g \restriction C(G) \subseteq H$. Therefore, $g \restriction C$ is continuous. Now, we show that $g \restriction C$ is open. Let $K \cup (K' \setminus \{k'\})$, where $k \in K$ and K is open in X, be any basic open set in $A(X)$, then $(K \cap C) \cup ((K' \cap C) \setminus \{k'\})$ is a basic open set in C. Since $X \cap C$ is compact in X, then $g \restriction C(K \cap (X \cap C)) = f \restriction X \cap C(K \cap (X \cap C))$ is open in $Y \cap f(C \cap X)$ as $f \restriction X \cap C$ is a homeomorphism. Thus $K \cap C$ is open in $Y \cap f(X \cap C)$. Also, $g((K' \cap C) \setminus \{k'\})$ is open in $Y' \cap g(C)$ being a set of isolated points. Thus $g \restriction C$ is an open function. Therefore, $g \restriction C$ is a homeomorphism. □

Theorem 2.3. *C-normality is an additive property.*

Proof. Let X_α be a C-normal space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is C-normal. For each $\alpha \in \Lambda$, pick a normal space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_\alpha \restriction C_\alpha : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each compact subspace C_α of X_α. Since Y_α is normal for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is normal, [1, 2.2.7]. Consider the function sum [1, 2.2.E], $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is compact in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact, then $(\oplus_{\alpha\in\Lambda} f_\alpha)_{|_C}$ is a homeomorphism because $f_\alpha \restriction C \cap X_\alpha$ is a homeomorphism for each $\alpha \in \Lambda_0$. □

The following problems are still open.

Problem 2.4.
1. *Is C-normality hereditary with respect to closed subspaces?*
2. *If X is a Dowker space, is then $X \times I$ C-normal? (Arhangel'skii)*
3. *Does there exist a Tychonoff non-normal space which is not C-normal? (Arhangel'skii)*

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