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On a Simultaneous Generalization of β-Normality and Almost Normality

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Abstract. A new generalization of normality called almost β-normality is introduced and studied which is a simultaneous generalization of almost normality and β-normality. A topological space is called almost β-normal if for every pair of disjoint closed sets A and B one of which is regularly closed, there exist disjoint open sets U and V such that $\overline{U \cap A} = A$, $\overline{U \cap B} = B$ and $\overline{U} \cap \overline{V} = \phi$.

1. Introduction and Preliminaries

Normality plays a prominent role in general topology and behaves differently from other separation axioms in terms of subspaces and products. Several generalized notions of normality such as (weakly) θ-normal [8, 9], almost normal [13], κ-normal (mildly normal) [14, 15], γ-normal [6], π-normal [7], Δ-normal [4] (semi-nearly normal [11]) etc. exist in the literature. These spaces were introduced in different situations to study normality. Some of these variants of normality were utilized to obtain factorizations of normality (see [4, 5, 8]). In [1], A. V. Arhangel'skii and L. Ludwig introduced the concept of α-normal and β-normal spaces and Eva Murtinova in [12] provided an example of a β-normal Tychonoff space which is not normal. In this paper, we introduce the notion of almost β-normality which is a simultaneous generalization of almost normality and β-normality and obtain a decomposition of almost normality in terms of β-normality.

Let X be a topological space and let $A \subset X$. Throughout the present paper the closure of a set A will be denoted by $\overline{A}$ and the interior by $intA$. A set $U \subset X$ is said to be regularly open [10] if $U = int\,\overline{U}$. The complement of a regularly open set is called regularly closed. A space is κ-normal [15] (mildly normal [14]) if for every pair of disjoint regularly closed sets E, F of X there exist disjoint open subsets U and V of X such that $E \subseteq U$ and $F \subseteq V$. A topological space is said to be almost normal [13] if for every pair of disjoint closed sets A and B one of which is regularly closed, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. A topological space X is said to be almost regular if for every regularly closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $A \subseteq U$ and $x \in V$. A topological space X is said to be α-normal [1] if for any two disjoint closed subsets A and B of X there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap U$ is dense in B. A space X is β-normal [1] if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap U$ is

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dense in B, and $\overline{U} \cap \overline{V} = \emptyset$. A space X is said to be semi-normal if for every closed set A contained in an open set U, there exists a regularly open set V such that $A \subset V \subset U$.

2. Almost β-Normality

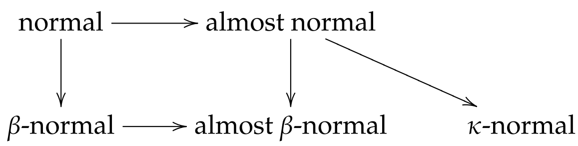
Definition 2.1. A topological space is called *almost β-normal* if for every pair of disjoint closed sets A and B, one of which is regularly closed, there exist disjoint open sets U and V such that $\overline{U \cap A} = A$, $\overline{V \cap B} = B$, and $\overline{U} \cap \overline{V} = \phi$.

From the definitions it is obvious that every normal space is β-normal and every β-normal space is almost β-normal.

Theorem 2.2. *Every almost normal space is almost β-normal.*

Proof. Let X be an almost normal space. Let A and B be two disjoint closed sets in X one of which (say A) is regularly closed. Since X is almost normal there exist disjoint open sets W and V containing A and B respectively. Since $W \cap V = \phi$, $W \cap \overline{V} = \phi$. Let $U = intA$. Then $\overline{U} \cap \overline{V} = \phi$, $\overline{U \cap A} = A$, and $\overline{V \cap B} = B$. So, the space is almost β-normal. □

The following implications hold but none are reversible.



Example 2.3. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{b\}, \{c\}, \{a, b, c\}, \{c, d\}, \{b, c\}, \{b, c, d\}\}$. Then the space (X, τ) is not almost β-normal since for regularly closed $A = \{a, b\}$ and disjoint closed set B = {d} there does not exist two open sets U and V such that $\overline{U \cap A} = A$, $\overline{B \cap V} = B$, and $\overline{U} \cap \overline{V} = \phi$.

Example 2.4. Let X be the union of any infinite set Υ and two distinct one point sets p and q. The modified Fort space on X as defined in [17] is almost β-normal but not β-normal. In X any subset of Υ is open and any set containing p or q open if and only if it contains all but a finite number of points in Υ. This space is not β-normal even not α-normal [1] because for disjoint closed sets {p} and {q} there does not exist two disjoint open sets separating them. The regularly closed sets of this space are finite subsets of Υ and sets of the form $A \cup \{p, q\}$, where $A \subseteq \Upsilon$ is infinite. Thus the space is almost β-normal.

Remark 2.5. If a β-normal space X satisfies the T_1 separation axiom, then the space X is regular is observed by A. V. Arhangel'skii and L. Ludwig in [1]. But if β-normality is replaced by almost β-normality then the proposition is not valid as Example 2.4 is an example of a T_1 almost β-normal space which is not regular, even not Hausdorff. Thus it is obvious to ask the question: which almost β-normal, T_1-spaces are regular? In the sequel, Theorem 2.17 provides a partial answer to this question.

Arhangel'skii and Ludwig in [1] have shown that a space is normal if and only if it is κ-normal and β-normal. Therefore, every non-normal space which is almost normal is an example of a κ-normal, almost β-normal space which is not β-normal.

Remark 2.6. In [12], Eva Murtinova provided an example of a β-normal Tyconoff space which is not normal. Such a space must also be almost β-normal and cannot be κ-normal.

It is very natural to ask under which additional conditions almost β-normality coincides with either almost normality or β-normality. The following results (Theorem 2.8, Theorem 2.21, Corollary 2.11, Corollary 2.12) provide answers to this question. Recall that a Hausdorff space X is said to be extremally disconnected if the closure of every open set in X is open. Further, the following generalized notions of normality defined by Kohli and Das are useful in the sequel to establish that almost β-normality coincides with almost normality under certain conditions.

A point $x \in X$ is called a *θ-limit point* [18] of A if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denotes the set of all θ-limit points of A. The set A is called θ-closed if $A = cl_\theta A$.

Definition 2.7. A topological space X is said to be

(*i*) *θ-normal* [8] if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;

(*ii*) *Weakly θ-normal* (wθ-normal)[8] if every pair of disjoint θ-closed sets are contained in disjoint open sets.

Theorem 2.8. *Every extremally disconnected almost β-normal space is almost normal.*

Proof. Let X be an extremally disconnected almost β-normal space and let A be a regularly closed set disjoint from the closed set B. By almost β-normality of X, there exist disjoint open sets U and V such that $\overline{U \cap A} = A$, $\overline{V \cap B} = B$ and $\overline{U} \cap \overline{V} = \phi$. Thus $A \subset \overline{U}$ and $B \subset \overline{V}$. By the extremally disconnectedness of X, $\overline{U}$ and $\overline{V}$ are disjoint open sets containing A and B respectively. □

Theorem 2.9. *Every T_1 almost β-normal space is almost regular.*

Proof. Let A be a regularly closed set in X and x be a point outside A. Since X is a T_1-space and every singleton is closed in a T_1-space, by almost β-normality there exist disjoint open sets U and V such that $x \in U$, $\overline{V \cap A} = A$, $\overline{U} \cap \overline{V} = \phi$. Since $A \subset \overline{V}$, U and $X - \overline{U}$ are disjoint open sets containing $\{x\}$ and A, respectively. Thus, the space is almost regular. □

Corollary 2.10. *In a T_1-space, weak θ-normality and almost β-normality implies κ-normality.*

Proof. Let X be a T_1 weakly θ-normal, almost β-normal space. Then by Theorem 2.9, X is almost regular. By a result of Kohli and Das [9] that every almost regular weakly θ-normal space is κ-normal, X is κ-normal. □

Corollary 2.11. *In the class of T_1, θ-normal spaces, every almost β-normal space is almost normal.*

Proof. Let X be a T_1 space which is θ-normal as well as almost β-normal. By Theorem 5.16 of Kohli and Das[9], X is almost normal. □

Corollary 2.12. *In the class of T_1, paracompact spaces, every almost β-normal space is almost normal.*

Proof. Since every paracompact space is θ-normal [8], the result holds by Corollary 2.11. □

Recall that a space X is said to be *almost compact* [3] if every open cover of X has a finite subcollection, the closure of whose members covers X.

Corollary 2.13. *An almost compact, almost β-normal, T_1-space is κ-normal.*

Proof. The proof is immediate from the result of Singal and Singal [14] that an almost regular almost compact space is κ-normal. □

Corollary 2.14. *A Lindelöf, almost β-normal, T_1-space is κ-normal.*

Proof. Since an almost regular Lindelöf space is κ-normal [14], the proof is immediate. □

Remark 2.15. The T_1 axiom in the above theorem cannot be relaxed since there exist almost β-normal spaces which are not almost regular.

Example 2.16. Let $X = \{a, b, c\}$ and $\tau = \{\{a\}, \{c\}, \{a, c\}, \phi, X\}$. Then X is vacuously normal, thus almost β-normal but not almost regular as the regularly closed set {a, b} and any point outside it cannot be separated by disjoint open sets.

Theorem 2.17. *In the class of T_1, semi-normal spaces, every almost β-normal space is regular.*

Proof. Let X be a T_1, semi-normal, and almost β-normal space. Let A be a closed subset of X and $x \notin A$. Since X is a T_1-space, the singlton set $\{x\}$ is closed. So by semi-normality of X, there exists a regularly open set U such that $\{x\} \subset U \subset X - A$. Here $F = X - U$ is a regularly closed set containing A with $x \notin F$. As X is an almost β-normal T_1-space, X is almost regular by Theorem 2.9. Thus there exist disjoint open sets V and W such that $x \in V$ and $A \subset F \subset W$. Hence X is regular. □

The following theorem provides a characterization of almost β-normality.

Theorem 2.18. *For any topological space X, the following are equivalent:*

1. *X is almost β-normal;*
2. *whenever $E, F \subseteq X$ are disjoint closed sets and E is regularly closed, there is an open set V such that $F = \overline{V \cap F}$ and $E \cap \overline{V} = \emptyset$;*
3. *whenever $E \subseteq X$ is closed, $U \subseteq X$ is regularly open, and $E \subseteq U$, there is an open set V such that $E = \overline{E \cap V} \subseteq \overline{V} \subseteq U$.*

Proof. [(1) ⇒ (2)] Suppose that $E, F \subseteq X$ are disjoint closed sets and E is regularly closed. Since X is almost β-normal, there exist open sets U and V such that $E = \overline{U \cap E} \subseteq \overline{U}$, $F = \overline{V \cap F} \subseteq \overline{V}$, and $\overline{U} \cap \overline{V} = \emptyset$. Then $E \cap \overline{V} = \emptyset$.

[(2) ⇒ (1)] Suppose that $E, F \subseteq X$ are disjoint closed sets and E is regularly closed. By the assumption, there exists an open set V such that $F = \overline{V \cap F}$ and $E \cap \overline{V} = \emptyset$. Let $U = \text{int}(E)$. Then $E = \overline{U \cap E}$ and $\overline{U} \cap \overline{V} = E \cap \overline{V} = \emptyset$.

[(1) ⇒ (3)] Suppose that E is closed, U is regularly open, and $E \subseteq U$. Since U is regularly open, $X \setminus U$ is regularly closed. Since X is almost β-normal, there are open sets O and V such that $X \setminus U = \overline{O \cap (X \setminus U)} \subseteq \overline{O}$, $E = \overline{V \cap E} \subseteq \overline{V}$, and $\overline{O} \cap \overline{V} = \emptyset$. Then $(X \setminus U) \cap \overline{V} = \emptyset$ which means that $\overline{V} \subseteq U$.

[(3) ⇒ (2)] Suppose that $E, F \subseteq X$ are disjoint closed sets and E is regularly closed. Then $F \subseteq X \setminus E$ and $X \setminus E$ is regularly open. By the hypothesis, there is an open set V such that $F = \overline{V \cap F} \subseteq \overline{V} \subseteq X \setminus E$. Then $\overline{V} \cap E = \emptyset$, as desired. □

The following result gives a decomposition of almost normality.

Theorem 2.19. *A space is almost normal if and only if it is almost β-normal and κ-normal.*

Proof. Let X be an almost β-normal and κ-normal space. Let A and B be two disjoint closed sets in X in which A is regularly closed. By almost β-normality of X, there exist disjoint open sets U and V such that $\overline{U} \cap \overline{V} = \phi$, $\overline{A \cap U} = A$ and $\overline{B \cap V} = B$. Thus $A \subset \overline{U}$ and $B \subset \overline{V}$. Here $\overline{U}$ and $\overline{V}$ are disjoint regularly closed sets . So by κ-normality, there exist disjoint open sets W_1 and W_2 such that $\overline{U} \subseteq W_1$ and $\overline{V} \subseteq W_2$. Hence X is almost normal. □

Corollary 2.20. *In a semi-normal and κ-normal space the following statements are equivalent :*

1. *X is normal;*
2. *X is almost normal;*
3. *X is β-normal;*

4. *X is almost β-normal.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (2) $\Rightarrow$ (4) are obvious. [(4) $\Rightarrow$ (1)] Let X be semi-normal, κ-normal and almost β-normal space. We have to show X is normal. By Theorem 2.19, X is almost normal. Since every almost normal, semi-normal space is normal [13], X is normal. □

Theorem 2.21. *Let X be a dense subspace of a product of metrizable spaces. Then X is almost normal if and only if X is almost β-normal.*

Proof. Since every dense subspace of any product of metrizable spaces is κ-normal ([2], [16]), by Theorem 2.19, the proof is immediate. □

It is known that, every β-normal space is α-normal, but in contrast almost β-normal spaces need not be α-normal which is evident from the Example 2.4 which is almost β-normal but not α-normal. The following Theorem provides a partial answer to the question: which almost β-normal spaces are α-normal ?

Theorem 2.22. *Every semi-normal, almost β-normal space is α-normal.*

Proof. Let X be a semi-normal, almost β-normal space. Let A and B be two disjoint closed sets in X. Thus $A \subset (X - B)$. By semi-normality, there exists a regularly open set F such that $A \subset F \subset (X - B)$. Now A and $(X - F)$ are two disjoint closed sets in X in which $X - F$ is a regularly closed set containing B. Thus by almost β-normality, there exist disjoint open sets U and V such that $\overline{U \cap A} = A$, $\overline{(X - F) \cap V} = X - F$, and $\overline{U} \cap \overline{V} = \phi$. Here $A = \overline{U \cap A} \subset \overline{U}$ and $(X - F) = \overline{(X - F) \cap V} \subset \overline{V}$. Thus U and $W = X - \overline{U}$ are two disjoint open sets such that $\overline{U \cap A} = A$ and $B \subset W$. Therefore, $\overline{W \cap B} = B$ and X is α-normal. □

The following examples show that a continuous image of an almost β-normal space need not be almost β-normal.

Example 2.23. Let X be the union of the set of integers $\mathbb{Z}$ and two distinct one point sets p and q with the modified Fort topology as defined in Example 2.4 and let $Y = \{a, b, c, d\}$ with the topology defined in Example 2.3. Define a function $f : X \to Y$ by

$$f(x) = \begin{cases} c \;;\; if \quad x \in \mathbb{Z} - \{0, 1\} \\ a \;;\; if \quad x = 0 \\ b \;;\; if \quad x = 1 \\ d \;;\; otherwise \end{cases}$$

Then f is a continuous function from X into Y. It is clear that X is almost β-normal but $f(X)$ is not.

Example 2.24. Let (X, τ) be a topological space which is not almost β-normal and let τ_D be the discrete topology on X. Define $f : (X, \tau_D) \to (X, \tau)$ by $f(x) = x$. Clearly, (X, τ_D) is almost β-normal and f is continuous, one-to-one, and onto.

Theorem 2.25. *Suppose that X and Y are topological spaces, X is almost β-normal, and $f : X \to Y$ is onto, continuous, open, and closed. Then Y is almost β-normal.*

Proof. Suppose that $E, F \subseteq Y$ are disjoint closed sets and E is regularly closed. Since f is continuous, $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint closed sets. To see that $f^{-1}(E) = \overline{f^{-1}(\text{int}(E))}$, suppose that $W \subseteq X$ is open such that $W \cap f^{-1}(E) \neq \emptyset$. Then $f(W)$ is open in Y and $f(W) \cap E = f(W) \cap \overline{\text{int}(E)} \neq \emptyset$ which implies that $f(W) \cap \text{int}(E) \neq \emptyset$. Hence, $W \cap f^{-1}(\text{int}(E)) \neq \emptyset$ and so $f^{-1}(E) = \overline{f^{-1}(\text{int}(E))}$. Since $f^{-1}(E) = \overline{f^{-1}(\text{int}(E))}$, $f^{-1}(E)$ is a regularly closed set. So there exists an open set $U \subseteq X$ such that $f^{-1}(F) = \overline{f^{-1}(F) \cap U}$ and $\overline{U} \cap f^{-1}(E) = \emptyset$. Since $\overline{U} \cap f^{-1}(E) = \emptyset$, $f(\overline{U}) \cap E = \emptyset$. Also, note that $f(U)$ is open and $f(\overline{U})$ is closed. Since $f(\overline{U})$ is a closed set containing $f(U)$, $\overline{f(U)} \subseteq f(\overline{U})$. So $\overline{f(U)} \cap E = \emptyset$. It remains to show that $F = \overline{F \cap f(U)}$. To see this, let $y \in F$ and O be an open set containing y. Then $f^{-1}(y) \subseteq [f^{-1}(F) \cap f^{-1}(O)]$. Since $f^{-1}(F) = \overline{f^{-1}(F) \cap U}$, $f^{-1}(F) \cap U \cap f^{-1}(O) \neq \emptyset$. Hence, $F \cap f(U) \cap O = f(f^{-1}(F)) \cap f(U) \cap f(f^{-1}(O)) \supseteq f[f^{-1}(F) \cap U \cap f^{-1}(O)] \neq \emptyset$, as desired. □

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