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On Fractional Differential Equations with State-Dependent Delay via Kuratowski Measure of Noncompactness

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Abstract. This paper is devoted to study the existence of mild solutions for semilinear functional differential equations with state-dependent delay involving the Riemann-Liouville fractional derivative in a Banach space and resolvent operator. The arguments are based upon Mönch's fixed point theorem and the technique of measure of noncompactness.

1. Introduction

This paper is concerned with existence of mild solutions defined on a compact real interval for fractional order semilinear functional differential equations with state-dependent delay of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y(t - \rho(y(t)))), \quad t \in J = [0, b], \quad 0 < \alpha < 1 \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0] \tag{2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E. $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |.|)$ a real Banach space. ρ is a positive bounded continuous function on $C([-r, 0], E)$. r is the maximal delay defined by

$$r = \sup_{y \in C} \rho(y).$$

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. For the theory of differential equations with state dependent delay and their applications, we reefer the reader to the papers [5, 9].

The fractional differential equations are valuable tools in the modeling of many phenomena in various fields of science and engineering [6, 7]. On the other hand, the integrodifferential equations arise in various

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applications such as viscoelasticity, heat equations, and many other physical phenomena for details, see [13, 14, 16, 17]. Moreover, the Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decades see for instance [2, 3, 11] and references cited therein.

The principal goal of this paper is to extend such results to the case of state dependent delay by virtue of resolvent operator and to initiate the application of the technique of measures of noncompactness to investigate the problem of the existence of mild solutions for (1)-(2). Especially that technique combined with an appropriate fixed point theorem has proved to be a very useful tool in the study of the existence of solutions for several types of integral and differential equations; see for example [4, 8, 12, 15, 19]. In Section 2 we recall some definitions and preliminary facts which will be used in the sequel. In Section 3, we give our main existence results. An example will be presented in the last section illustrating the abstract theory.

2. Preliminaries

In this section, we recall some definitions and propositions of fractional calculus and resolvent operators. Let E be a Banach space. By $C(J,E)$ we denote the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

$C([-r,0],E)$ is endowed with norm defined by

$$\|\psi\|_C = \sup\{|\psi(\theta)| : \theta \in [-r,0]\}.$$

$B(E)$ denotes the Banach space of all bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J,E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

$L^\infty(J,E)$ denotes the Banach space of measurable functions $y : J \to E$ which are bounded, equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : \|y(t)\| < c,\ a.e.\ t \in J\}.$$

For a given set V of functions $v : [-r,b] \longrightarrow E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, t \in [-r,b]$$

and

$$V(J) = \{v(t) : v \in V, t \in [-r,b]\}$$

Definition 2.1. *[13, 17] The Riemann-Liouville fractional primitive of order $\alpha \in \mathbb{R}^+$ of a function $h : (0,b] \to E$ is defined by*

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

provided the right hand side exists pointwise on $(0,b]$, where Γ is the gamma function.

Definition 2.2. *[13, 17] The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a continuous function $h : (0,b] \to E$ is defined by*

$$\begin{aligned} \frac{d^\alpha h(t)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds \\ &= \frac{d}{dt} I_0^{1-\alpha} h(t). \end{aligned}$$

Definition 2.3. *A map $f : J \times C([-r, 0], E) \longrightarrow E$ is said to be Carathéodory if*

i) *$t \longmapsto f(t, u)$ is measurable for each $u \in C([-r, 0], E)$;*

ii) *$u \longmapsto F(t, u)$ is continuous for almost each $t \in J$.*

Consider the fractional differential equation

$$D^{\alpha} y(t) = Ay(t) + f(t), \quad t \in J,\ 0 < \alpha < 1,\ \ y(0) = 0, \tag{3}$$

where A is a closed linear unbounded operator in E and $f \in C(J, E)$. Equation (3) is equivalent to the following integral equation [13]

$$y(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ \ t \in J. \tag{4}$$

This equation can be written in the following form of integral equation

$$y(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ay(s) ds, \ \ t \geq 0, \tag{5}$$

where

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{6}$$

Examples where the exact solution of (3) and the integral equation (4) are the same, are given in [3]. Let us assume that the integral equation (5) has an associated resolvent operator $(S(t))_{t\geq 0}$ on E.

Next we define the resolvent operator of the integral equation (5).

Definition 2.4. *[18, Definition 1.1.3] A one parameter family of bounded linear operators $(S(t))_{t\geq 0}$ on E is called a resolvent operator for* (4) *if the following conditions hold:*

(a) $S(\cdot)x \in C([0, \infty), E)$ and $S(0)x = x$ for all $x \in E$;

(b) $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every $t \geq 0$;

(c) for every $x \in D(A)$ and $t \geq 0$,

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS(s)x ds. \tag{7}$$

Here and hereafter we assume that the resolvent operator $(S(t))_{t\geq 0}$ is analytic [18, Chapter 2], and there exist a function $\varphi_A \in L^1_{loc}([0, \infty), \mathbb{R}^+)$ such that $\|S'(t)x\| \leq \varphi_A(t)\|x\|_{[D(A)]}$ for all $t > 0$ and each $x \in D(A)$.

We have the following concept of solution using Definition 1.1.1 in [18].

Definition 2.5. *A function $u \in C(J, E)$ is called a mild solution of the integral equation (5) on J if $\int_0^t (t-s)^{\alpha-1} u(s) ds \in D(A)$ for all $t \in J$, $h(t) \in C(J, E)$ and*

$$u(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + h(t), \quad \forall t \in J.$$

The next result follows from [18, Proposition I.1.2, Theorem II.2.4, Corollary II.2.6].

Lemma 2.6. *Under the above conditions the following properties are valid.*

(i) *If $u(\cdot)$ is a mild solution of (5) on J, then the function $t \to \int_0^t S(t-s)h(s)ds$ is continuously differentiable on J, and*

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)h(s)ds, \quad \forall t \in J.$$

(ii) *If $h \in C^\beta(J,E)$ for some $\beta \in (0,1)$, then the function defined by*

$$u(t) = S(t)(h(t)-h(0)) + \int_0^t S'(t-s)[h(s)-h(t)]ds + S(t)h(0), \quad t \in J,$$

is a mild solution of (5) on J.

(iii) *If $h \in C(J,[D(A)])$ then the function $u : J \to E$ defined by*

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \quad t \in J,$$

is a mild solution of (5) on J.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.7. *[4] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \longrightarrow [0,\infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties (for more details see [4]).

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$

(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$

(d) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$

(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$

(f) $\alpha(convB) = \alpha(B)$

Theorem 2.8. *[1, 15] Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.9. *[19] Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J \times J$ and f a function from $J \times C([-r,0],E) \longrightarrow E$ which satisfies the Carathéodory conditions and there exists $p \in L^1(J,\mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset C([-r,0],E)$ we have*

$$\lim_{k\to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B); \text{ here } J_{t,k} = [t-k,t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\Big(\Big\{\int_J G(s,t)f(s,y_s)ds : y \in V\Big\}\Big) \leq \int_J \|G(t,s)\|p(s)\alpha(V(s))ds.$$

3. Main Result

In this section we give our main existence results for problem (1)-(2). This problem is equivalent to the following integral equation

$$y(t) = \begin{cases} \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Motivated by Lemma 2.6 and the above representation, we introduce the concept of mild solution.

Definition 3.1. *We say that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (1)-(2) if:*

1. $\displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds \in D(A)$ *for* $t \in J$,
2. $y(t) = \phi(t)$, $t \in [-r, 0]$, *and*
3. $y(t) = \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds$, $t \in J$.

Suppose that there exists a resolvent $(S(t))_{t\geq 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.6 (iii), if $y : [-r, b] \to E$ is a mild solution of (1)-(2), then

$$y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds & \\ + \displaystyle\int_0^t S'(t-s)\Big(\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s (s-\tau)^{\alpha-1} f(\tau, y(\tau-\rho(y(\tau))))d\tau\Big)ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

To prove the main results, we assume the following conditions:

(H1) The operator $S'(t)$ is compact for all $t > 0$; and

$$\|S'(t)x\| \leq \varphi_A(t)\|x\|_{[D(A)]} \ \textit{for all } t > 0 \textit{ and each } x \in D(A)$$

(H2) $f : J \times C([-r, 0], E) \longrightarrow E$ is of Carathéodory.

(H3) There exist functions $p \in L^\infty(J, \mathbb{R}_+)$ such that

$$|f(t, u)| \leq p(t)(\|u\|_C + 1), \ \textit{for a.e. } t \in J \textit{ and } u \in C([-r, 0], E).$$

(H4) For almost each $t \in J$ and each bounded set $B \subset C([-r, 0], E)$ we have

$$\lim_{k\to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B); \ \textit{here } J_{t,k} = [t-k, t] \cap J.$$

Our main result reads as follows:

Theorem 3.2. *Assume that the conditions $(H1) - (H4)$ are satisfied. Then the problem (1)-(2) has at least one mild solution on $[-r, b]$, provident that*

$$\frac{b^\alpha \|p\|_{L^\infty}(1 + \|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)} < 1. \tag{8}$$

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $N : C([-r,b],E) \to C([-r,b],E)$ defined by,

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds & \\ + \displaystyle\int_0^t S'(t-s)\Big(\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s (s-\tau)^{\alpha-1} f(\tau, y(\tau-\rho(y(\tau))))d\tau\Big)ds, & t \in [0,b]. \end{cases}$$

Let $\gamma > 0$ be such that

$$\gamma \geq \frac{b^\alpha \|p\|_{L^\infty}}{\Gamma(\alpha+1) - b^\alpha \|p\|_{L^\infty}} \tag{9}$$

and consider the set

$$D_\gamma = \{y \in C([-r,b],E) : \|y\|_\infty \leq \gamma\}$$

Clearly, the subset D_γ is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 3.2.

In order to prove that N is completely continuous, we divide the operator N into two operators:

$$N_1(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds,$$

and

$$N_2(y)(t) = \int_0^t S'(t-s) N_1(y)(s)ds.$$

We prove that N_1 and N_2 are completely continuous.

Step 1: N_1 is completely continuous.
At first, we prove that N_1 is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C([-r,b],E)$, then for $t \in [0,b]$. Note that $-r \leq s - \rho(y(s)) \leq s$ for each $s \in J$ we have,

$$|N_1(y_n)(t) - N_1(y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, y_n(s-\rho(y_n(s)))) - f(s, y(s-\rho(y(s))))| ds$$

Since f is a Carathéodory function for $t \in J$, and from the continuity of ρ, we have by the dominated convergence theorem of Lebesgue, the right member of the above inequality tends to zero as $n \to \infty$.

$$\|N_1(y_n) - N_1(y)\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Thus N_1 is continuous.
Next, we will prove that $N_1(D_\gamma) \subset D_\gamma$ is bounded. For each $y \in D_\gamma$ by $(H3)$ and (8) we have for each $t \in [0,b]$

$$\begin{aligned} |N_1(y)(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, y(s-\rho(y(s)))) \Big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)(\|y(s)\| + 1)ds \\ &\leq \frac{(\gamma+1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds \\ &\leq \frac{b^\alpha(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha+1)} \\ &\leq \gamma \end{aligned}$$

Then $N_1(D_\gamma) \subset D_\gamma$.
Now, we show prove that $N_1(D_\gamma)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for any $y \in D_\gamma$;

$$\begin{aligned}
& |N_1(y)(\tau_2) - N_1(y)(\tau_1)| \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, y(s-\rho(y(s))))ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1 - s)^{\alpha-1}f(s, y(s-\rho(y(s))))ds\right| \\
\leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, y(s-\rho(y(s))))ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, y(s-\rho(y(s))))ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, y(s-\rho(y(s))))ds\right| \\
\leq & \frac{(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha)}\Big(\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds \\
& + \int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}ds\Big).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero.
Then $N_1(D_\gamma)$ is continuous and completely continuous

Step 2: N_2 is completely continuous.
The operator N_2 is continuous, since $S'(\cdot) \in C([0, b], B(E))$ and N_1 is continuous as proved in Step 1.
For $y \in D_\gamma$ we have

$$\begin{aligned}
|N_2(y)(t)| & \leq \int_0^t |S'(t-s)||N_1(y)(s)|ds \\
& \leq \int_0^t \varphi_A(t-s)\|N_1(y)(s)\|_{[D(A)]}ds \\
& \leq \frac{\|\varphi_A\|_{L^1} b^\alpha(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha+1)} \\
& \leq \gamma
\end{aligned}$$

Then $N_2(D_\gamma) \subset D_\gamma$.
Next, we shall show that $N_2(D_\gamma)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for any $y \in D_\gamma$;

$$\begin{aligned}
|N_2(y)(\tau_2) - N_2(y)(\tau_1)| & = \left|\int_0^{\tau_2} S'(\tau_2 - s)N_1(y)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1 - s)N_1(y)(\tau_1)ds\right| \\
& \leq \frac{b^\alpha(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha+1)}\Big(\int_0^{\tau_1-\epsilon}|S'(\tau_2 - s) - S'(\tau_1 - s)|\,ds \\
& + \int_{\tau_1-\epsilon}^{\tau_1}|S'(\tau_2 - s) - S'(\tau_1 - s)|\,ds + \int_{\tau_1}^{\tau_2}|S'(\tau_2 - s)|ds\Big).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero.
Then $N_2(D_\gamma)$ is continuous and completely continuous

Now let V be a subset of D_γ such that $V \subset \overline{conv}(N(V) \cup \{0\})$.
V is bounded and equicontinuous and therefore the function $v \longrightarrow v(t) = \alpha(V(t))$ is continuous on $[-r, b]$. By (H4), Lemma 2.9 and the properties of the measure α we have for each $t \in [-r, b]$

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\alpha(V(s))ds \\
&\quad + \int_0^t S'(t-s)\Big(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} p(s)\alpha(V(\tau))d\tau\Big)ds \\
&\leq \frac{\|p\|_{L^\infty}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds + \frac{\|p\|_{L^\infty}\|\varphi_A\|_{L^1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds \\
&\leq \|v\|_\infty \frac{b^\alpha \|p\|_{L^\infty}}{\Gamma(\alpha+1)} + \|v\|_\infty \frac{b^\alpha \|p\|_{L^\infty}\|\varphi_A\|_{L^1}}{\Gamma(\alpha+1)} \\
&\leq \|v\|_\infty \frac{b^\alpha \|p\|_{L^\infty}(1 + \|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)}
\end{aligned}$$

This means that

$$\|v\|_\infty \Big(1 - \frac{b^\alpha \|p\|_{L^\infty}(1 + \|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)}\Big) \leq 0$$

By (8) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in [-r, b]$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_γ. Applying now Theorem 3.2 we conclude that N has a fixed point which is a mild solution for the problem (1)-(2).

4. An Example

To apply our pervious result, we consider the following partial functional differential equation with fractional order for some $p > 1$

$$\begin{cases}
\frac{\partial^\alpha}{\partial t^\alpha} u(t,y) = \Delta u(t,y) + \theta(t)|u(t - \tau(u(t,y)), y)|^p, \quad for\ y \in \Omega, t \in [0,T]\ and\ 0 < \alpha < 1; \\
u(t,y) = 0, \quad for\ \ y \in \partial\Omega\ and\ t \in [0,T]; \\
u(t,y) = u_0(t,y), \quad for\ \ y \in \Omega\ and\ -\tau_{max} \leq t \leq 0.
\end{cases} \tag{10}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. $u_0 \in C^2([-\tau_{max}, 0]\times\Omega, \mathbb{R}^n)$, θ is a continuous function from $[0,T]$ to $\mathbb{R}$ and $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$. The delay function τ is bounded positive continuous function in $\mathbb{R}^n$, let τ_{max} be the maximal delay which is defined by

$$\tau_{max} = \sup_{y\in\mathbb{R}} \tau(y).$$

Let $E = L^2[0,\pi]$ and let A be the operator given by $Aw = w''$ with domain $D(A) = \{w \in E, w, w'$ are absolutely continuous, $w'' \in E, w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^\infty n^2 (w, w_n) w_n, \qquad w \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \sin(nx), n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq0}$ on E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)w_n, \qquad w \in E.$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

From [18, Example 2.2.1] we know that the integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)ds, \ s \geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda^{\alpha} - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [2]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A),\ t > 0$.
Let f be the function defined from $[0, T] \times E$ to E by

$$f(t, \varphi)(y) \quad = \quad \theta(t)|\varphi(y)|^p \quad for \ \ \varphi \in E \ and \ \ y \in \Omega.$$

Let u be a solution of Equation (10). Then $y(t) = u(t, .)$ is a solution of the following equation

$$\begin{cases} D^{\alpha} y(t) & = Ay(t) + f(t, y(t - \tau(y(t)))) \quad for \ \ t \in [0, T], 0 < \alpha < 1; \\ y(t) & = \phi(t) \quad , t \in [-\tau_{max}, 0], \end{cases}$$

where the initial value function ϕ is given by

$$\phi(t)(y) \quad = \quad u_0(t, y) \quad for \ \ t \in [-\tau_{max}, 0] \ and \ \ y \in \Omega.$$

We can show that problem (1.1) − (1.2) is an abstract formulation of problem (10). Under suitable conditions, Theorem 3.2 implies that problem (10) has a unique solution y on $[-\tau_{max}, T] \times \Omega$.

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