Filomat 31:2 (2017), 197–206
DOI 10.2298/FIL1702197J

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Norm Inequalities for Elementary Operators and Other Inner Product Type Integral Transformers with the Spectra Contained in the Unit Disc

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Abstract. If $\{\mathscr{A}_t\}_{t\in\Omega}$ and $\{\mathscr{B}_t\}_{t\in\Omega}$ are weakly*-measurable families of bounded Hilbert space operators such that transformers $X \mapsto \int_\Omega \mathscr{A}_t^* X \mathscr{A}_t \, d\mu(t)$ and $X \mapsto \int_\Omega \mathscr{B}_t^* X \mathscr{B}_t \, d\mu(t)$ on $\mathcal{B}(\mathcal{H})$ have their spectra contained in the unit disc, then for all bounded operators X

$$\left\|\Delta_{\mathscr{A}} X \Delta_{\mathscr{B}}\right\| \leqslant \left\| X - \int_\Omega \mathscr{A}_t^* X \mathscr{B}_t \, d\mu(t)\right\|, \tag{1}$$

where $\Delta_{\mathscr{A}} \overset{def}{=} s\text{-}\lim_{r\nearrow 1}\Big(I + \sum_{n=1}^\infty r^{2n} \int_\Omega \cdots \int_\Omega \left|\mathscr{A}_{t_1}\cdots\mathscr{A}_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n)\Big)^{-1/2}$ and $\Delta_{\mathscr{B}}$ by analogy.

If additionally $\sum_{n=1}^\infty \int_{\Omega^n} \left|\mathscr{A}_{t_1}^*\cdots\mathscr{A}_{t_n}^*\right|^2 d\mu^n(t_1,\cdots,t_n)$ and $\sum_{n=1}^\infty \int_{\Omega^n} \left|\mathscr{B}_{t_1}^*\cdots\mathscr{B}_{t_n}^*\right|^2 d\mu^n(t_1,\cdots,t_n)$ both represent bounded operators, then for all $p,q,s \geqslant 1$ such that $\frac{1}{q}+\frac{1}{s}=\frac{2}{p}$ and for all Schatten p trace class operators X

$$\left\|\Delta_{\mathscr{A}}^{1-\frac{1}{q}} X \Delta_{\mathscr{B}}^{1-\frac{1}{s}}\right\|_p \leqslant \left\|\Delta_{\mathscr{A}^*}^{-\frac{1}{q}}\Big(X - \int_\Omega \mathscr{A}_t^* X \mathscr{B}_t d\mu(t)\Big)\Delta_{\mathscr{B}^*}^{-\frac{1}{s}}\right\|_p. \tag{2}$$

If at least one of those families consists of bounded commuting normal operators, then (1) holds for all unitarily invariant Q-norms. Applications to the shift operators are also given.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ and $\mathfrak{C}_\infty(\mathcal{H})$ denote respectively spaces of all bounded and all compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space $\mathcal{H}$. Each "symmetric gauge (s.g.) function" (also known as symmetric norming functions) Φ on sequences gives rise to a symmetric norm or a unitarily invariant (u.i.) norm on operators defined by $\|X\|_\Phi \overset{def}{=} \Phi(\{s_n(X)\}_{n=1}^\infty)$, with $s_1(X) \geqslant s_2(X) \geqslant \cdots$ being the singular values of X. We will denote by the symbol $|||\cdot|||$ any such norm, which is therefore defined on a naturally associated norm ideal $\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ of $\mathfrak{C}_\infty(\mathcal{H})$ and satisfies the invariance property $|||UXV||| = |||X|||$ for all $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ and for all unitary operators U, V. Even more, $|||AXB||| \leqslant |||CXD|||$ whenever $A^*A \leqslant C^*C$ and $BB^* \leqslant DD^*$. This is the consequence of Ky-Fan dominance property, which says that $|||X||| \leqslant |||Y|||$ iff

2010 *Mathematics Subject Classification*. Primary 47B49; Secondary 47B47, 47A56, 47A30, 47A63, 47B10, 47B15, 47B37, 47A10
Keywords. Norm inequalities, elementary operators, inner product type integral transformers, spectral radius defect operator, Q-norms, shift operator
Received: 27 August 2015; Accepted: 15 September 2015
Communicated by Dragan S. Djordjević
Research supported by MPNTR grant No. 174017, Serbia
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$\sum_{k=1}^{n} s_k(X) \leqslant \sum_{k=1}^{n} s_k(Y)$ for all $n \in \mathbb{N}$, and the monotonicity of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of compact self-adjoint operators, which gives that

$$s_n(AXB) = \lambda_n^{\frac{1}{2}}(B^*X^*A^*AXB) \leqslant \lambda_n^{\frac{1}{2}}(B^*X^*C^*CXB) = \lambda_n^{\frac{1}{2}}(CXBB^*X^*C^*) \leqslant \lambda_n^{\frac{1}{2}}(CXDD^*X^*C^*) = s_n(CXD) \tag{3}$$

for all $n \in \mathbb{N}$, because $B^*X^*A^*AXB \leqslant B^*X^*C^*CXB$ implies $\lambda_n(B^*X^*A^*AXB) \leqslant \lambda_n(B^*X^*C^*CXB)$ and similarly $CXBB^*X^*C^* \leqslant CXDD^*X^*C^*$ implies $\lambda_n(CXBB^*X^*C^*) \leqslant \lambda_n(CXDD^*X^*C^*)$.

Each norm $|||\cdot|||$ is lower semi-continuous, i.e., $|||w-\lim_{n\to\infty} X_n||| \leqslant \liminf_{n\to\infty} |||X_n|||$. This follows from the well known representation formula $|||X||| = \sup\left\{\frac{|\mathrm{tr}(XY)|}{|||Y|||_*} : Y \text{ is finite dimensional}\right\}$, where $|||\cdot|||_*$ stands for the dual norm of $|||\cdot|||$ (see Th. 2.7 (d) in [18]).

One way to modify a s.g. function Φ is to introduce for $p \geqslant 1$ its (degree) p modification $\Phi^{(p)}$ as a new s.g. function by

$$\Phi^{(p)}\left((z_n)_{n=1}^{\infty}\right) \stackrel{def}{=} \sqrt[p]{\Phi\left((|z_n|^p)_{n=1}^{\infty}\right)},$$

which will be defined on its natural domain consisting of all complex sequences $z = (z_n)_{n=1}^{\infty}$ complying with $(|z_n|^p)_{n=1}^{\infty} \in \ell_\Phi$. A simple proof that $\Phi^{(p)}$ is a s.g. function can be found in [11].

For example, if we denote by ℓ a s.g. function determining the norm in ℓ^1, then we see that $\ell^{(p)}$ is exactly the s.g. function determining the norm in ℓ^p. More generally, this gives the way for p modification $\|\cdot\|_{\Phi^{(p)}}$ of any u.i. $\mathfrak{C}_\Phi(\mathcal{H})$ norm $\|\cdot\|_\Phi$ trough the formula

$$\|X\|_{\Phi^{(p)}} \stackrel{def}{=} \||X|^p\|_\Phi^{\frac{1}{p}} \qquad \text{for all } X \in \mathcal{B}(\mathcal{H}) \text{ such that } |X|^p \in \mathfrak{C}_\Phi(\mathcal{H}). \tag{4}$$

Schatten tracial p-norms defined as $\|X\|_p \stackrel{def}{=} \sqrt[p]{\sum_{n=1}^{\infty} s_n^p(X)}$ for $1 \leqslant p < \infty$, are exactly p-modification of the trace norm $\|\cdot\|_1$. Another widely known class of such norms are so called Q-norms, which represent a (degree) 2 modifications of some other u.i. norms. Given $f, g \in \mathcal{H}$, we will use the notation $f \otimes g^\star$ for one dimensional operators $f \otimes g^\star : \mathcal{H} \to \mathcal{H} : h \mapsto \langle h, g\rangle f$, known to have their linear span dense in each of $\mathfrak{C}_p(\mathcal{H})$ for $1 \leq p \leq \infty$. For a more complete account of the theory of norm ideals, the interested reader is referred to [5], [18] and [17].

For an operator valued (o.v.) function $\mathscr{A} : \Omega \to \mathcal{B}(\mathcal{H}) : t \mapsto \mathscr{A}_t$ we say to be weak*-measurable if $t \mapsto \langle \mathscr{A}_t f, g\rangle$ is measurable for all $f, g \in \mathcal{H}$. If $t \mapsto \langle \mathscr{A}_t f, g\rangle$ is in $L^1(\Omega, \mu)$ for all $f \in \mathcal{H}$, then $t \mapsto \mathrm{tr}(\mathscr{A}_t Y)$ is also in $L^1(\Omega, \mu)$ for all $Y \in \mathfrak{C}_1(\mathcal{H})$ and there exist Gel'fand or weak*-integral $\int_\Omega \mathscr{A}\, d\mu \in \mathcal{B}(\mathcal{H})$ such that

$$\mathrm{tr}\left(\int_\Omega \mathscr{A}\, d\mu\, Y\right) = \int_\Omega \mathrm{tr}(\mathscr{A}_t Y)\, d\mu(t) \qquad \text{for all } Y \in \mathfrak{C}_1(\mathcal{H}).$$

Specially, $\left\langle \int_\Omega \mathscr{A}\, d\mu\, f, g\right\rangle = \int_\Omega \langle \mathscr{A}_t f, g\rangle\, d\mu(t)$ for all $f, g \in \mathcal{H}$, and this is exactly the relation that entirely defines $\int_\Omega \mathscr{A}\, d\mu$. An example of the very useful weak*-integral is $\int_\Omega \mathscr{A}^*\mathscr{A}\, d\mu = \int_\Omega |\mathscr{A}|^2\, d\mu$, with the associated quadratic form $\left\langle \int_\Omega \mathscr{A}^*\mathscr{A}\, d\mu f, f\right\rangle = \int_\Omega \left\|\mathscr{A} f\right\|^2 d\mu$ for all $f \in \mathcal{H}$, provided by the finiteness of the last term in expression.

For weakly*-measurable o.v. functions $\mathscr{A}, \mathscr{B} : \Omega \to \mathcal{B}(\mathcal{H})$ and for all $X \in \mathcal{B}(\mathcal{H})$ a function $t \mapsto \mathscr{A}_t X \mathscr{B}_t$ is also weakly*-measurable one. If this function is weakly*-integrable for all $X \in \mathcal{B}(\mathcal{H})$, then this inner product type linear transformation $X \mapsto \int_\Omega \mathscr{A}_t X \mathscr{B}_t\, d\mu(t)$ will be called inner product type integral (i.p.t.i.) transformer on $\mathcal{B}(\mathcal{H})$ and denoted by $\int_\Omega \mathscr{A}_t \otimes \mathscr{B}_t\, d\mu(t)$. A special case when μ is a counting measure on $\mathbb{N}$ is mostly known and widely investigated, and such transformers are known as elementary mappings or elementary operators.

As shown in Lemma 3.1 (a) in [9], a sufficient condition for $\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu$ to be bounded on $\mathcal{B}(\mathcal{H})$ is provided when both $\mathscr{A}^*\mathscr{A}$ and $\mathscr{B}^*\mathscr{B}$ are weak*-integrable. If each of families $(\mathscr{A}_t)_{t\in\Omega}$ and $(\mathscr{B}_t)_{t\in\Omega}$ consists of commuting normal operators, then by Th. 3.2 in [9] the i.p.t.i. transformer $\int_\Omega \mathscr{A}_t \otimes \mathscr{B}_t\, d\mu(t)$ leaves every u.i. norm ideal $\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ invariant and the following Cauchy-Schwarz inequality holds:

$$\left|\left|\left|\int_\Omega \mathscr{A}_t X \mathscr{B}_t \, d\mu(t)\right|\right|\right| \leqslant \left|\left|\left|\sqrt{\int_\Omega \mathscr{A}_t^* \mathscr{A}_t \, d\mu(t)}\, X \sqrt{\int_\Omega \mathscr{B}_t^* \mathscr{B}_t \, d\mu(t)}\right|\right|\right|. \tag{5}$$

As noted in [7] p. 8–9, double operator integrals (d.o.i.) defined by the apparatus developed by Birman and Solomyak (see review articles [3] and [4]) can be seen as an example of weak*-integrals and they have found various application, including operators means and related topics (see ([7], [14] and references therein). Moreover, with given self-adjoint operators H and K, for an d.o.i. induced transformer to be bounded on $\mathcal{C}_1(\mathcal{H})$ it is necessary and sufficient to be of the form $\int_\Omega \alpha(H,t)\otimes\beta(K,t)\,d\mu(t)$, such that $\left\|\int_\Omega |\alpha(H,t)|^2\,d\mu(t)\right\|\cdot\left\|\int_\Omega |\alpha(K,t)|^2\,d\mu(t)\right\| < +\infty$, as established by the celebrated result of Peller in [15] and [16]. This shows that in this case d.o.i. induced transformers can be seen as a special case of i.p.t.i. transformers when $\mathscr{A}_t = \alpha(H,t)$ and $\mathscr{B}_t = \beta(H,t)$ for some self-adjoint (or unitary) operators H and K and for all $t \in \Omega$. Anyway, any successfulness of the application of d.o.i. or weak*-integrals in practice relies of the optimality of the chosen integral representation for the considered transformer, with means inequalities in [6], [7] and [14] as examples of such good practice.

Normality and commutativity condition in (5) can be dropped for Schatten tracial p norms as shown in Th. 3.3 in [9] and this represents a type of noncommutative (extension of) theory beyond d.o.i. transformers. For some applications of this theorem see [12], as well as [9] for the improved estimate for the solution of the Lyapunov equation given in [2].

In Th. 3.1 in [10] a formula for the exact norm of i.p.t.i. transformer $\int_\Omega \mathscr{A}_t\otimes\mathscr{B}_t\,d\mu(t)$ acting on $\mathcal{C}_2(\mathcal{H})$ is found. In Th. 2.1 in [10] the exact norm of i.p.t.i. transformer $\int_\Omega \mathscr{A}_t^*\otimes\mathscr{A}_t\,d\mu(t)$ is given for two specific cases:

$$\left\|\int_\Omega \mathscr{A}_t^*\otimes\mathscr{A}_t\,d\mu(t)\right\|_{\mathcal{B}(\mathcal{H})\to\mathcal{C}_\Phi(\mathcal{H})} = \left\|\int_\Omega \mathscr{A}_t^*\mathscr{A}_t\,d\mu(t)\right\|_\Phi = \left\|\int_\Omega \mathscr{A}_t^*\otimes\mathscr{A}_t\,d\mu(t)(I)\right\|_\Phi, \tag{6}$$

$$\left\|\int_\Omega \mathscr{A}_t^*\otimes\mathscr{A}_t\,d\mu(t)\right\|_{\mathcal{C}_\Phi(\mathcal{H})\to\mathcal{C}_1(\mathcal{H})} = \left\|\int_\Omega \mathscr{A}_t\mathscr{A}_t^*\,d\mu(t)\right\|_{\Phi_*} = \left\|\int_\Omega \mathscr{A}_t\otimes\mathscr{A}_t^*\,d\mu(t)(I)\right\|_{\Phi_*}, \tag{7}$$

where Φ_* stands for a s.g. function related to the norm in the dual space $\mathcal{C}_\Phi(\mathcal{H})^*$.

If both families $\{\mathscr{A}_t\}_{t\in\Omega}$ and $\{\mathscr{B}_t\}_{t\in\Omega}$ consist of commuting normal operators, such that $\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu \leqslant I$ and $\int_\Omega \mathscr{B}^*\mathscr{B}\,d\mu \leqslant I$, then for all $X\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$

$$\left|\left|\left|\sqrt{I-\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu}\,X\sqrt{I-\int_\Omega \mathscr{B}^*\mathscr{B}\,d\mu}\right|\right|\right| \leqslant \left|\left|\left|X-\int_\Omega \mathscr{A}X\mathscr{B}\,d\mu\right|\right|\right|. \tag{8}$$

A central result of this paper will be the extension of this inequality to the noncommutative settings, to the families consisting of not necessarily normal, nor commuting operators.

2. Preliminaries

First we will consider a spectral radius formula for i.p.t.i. transformers.

Lemma 2.1. *Let $\int_\Omega \mathscr{A}^*\otimes\mathscr{B}\,d\mu : \mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H}) : X\mapsto \int_\Omega \mathscr{A}_t^* X\mathscr{B}_t\,d\mu(t)$, then for its spectral radius we have*

$$\begin{aligned} r\Big(\int_\Omega \mathscr{A}^*\otimes\mathscr{B}\,d\mu\Big) &\leqslant \inf_{n\in\mathbb{N}}\left\|\int_{\Omega^n}\left|\mathscr{A}_{t_1}\cdots\mathscr{A}_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n)\right\|^{\frac{1}{2n}} \inf_{n\in\mathbb{N}}\left\|\int_{\Omega^n}\left|\mathscr{B}_{t_1}\cdots\mathscr{B}_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n)\right\|^{\frac{1}{2n}} \\ &= \lim_{n\to\infty}\left\|\int_{\Omega^n}\left|\mathscr{A}_{t_1}\cdots\mathscr{A}_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n)\right\|^{\frac{1}{2n}} \lim_{n\to\infty}\left\|\int_{\Omega^n}\left|\mathscr{B}_{t_1}\cdots\mathscr{B}_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n)\right\|^{\frac{1}{2n}}. \end{aligned} \tag{9}$$

If $\mathscr{A}_t = \mathscr{B}_t$ for all $t\in\Omega$, then inequality in (9) turns into equality.

Proof. First we prove that we have equality in (9) when $\mathscr{A}_t = \mathscr{B}_t$ for all $t \in \Omega$. Next, $\mathcal{B}(\mathcal{H})$ case of formula (6) gives us the norm of the $\mathcal{B}(\mathcal{H})$ transformer $\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu$:

$$\left\|\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right\|_{\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})} = \left\|\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\,(I)\right\| = \left\|\int_\Omega \mathscr{A}_t^* \mathscr{A}_t\, d\mu(t)\right\| = \left\|\int_\Omega |\mathscr{A}_t|^2\, d\mu(t)\right\|. \tag{10}$$

With $\underbrace{\mu \times \cdots \times \mu}_{\text{n - times}}$ already denoted by μ^n, let also $\mathscr{A}^{|n\rangle}_{(t_1,\cdots,t_n)} \stackrel{def}{=} \mathscr{A}_{t_1} \cdots \mathscr{A}_{t_n}$ and $\mathscr{B}^{|n\rangle}_{(t_1,\cdots,t_n)} \stackrel{def}{=} \mathscr{B}_{t_1} \cdots \mathscr{B}_{t_n}$. As formula (10) holds for $\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right)^n = \int_\Omega \cdots \int_\Omega \mathscr{A}_{t_n}^* \cdots \mathscr{A}_{t_1}^* \otimes \mathscr{A}_{t_1} \cdots \mathscr{A}_{t_n}\, d\mu^n(t_1, \cdot\cdot, t_n) = \int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{A}^{|n\rangle}\, d\mu^n$ as well, therefore

$$\left\|\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right)^n\right\| = \left\|\int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{A}^{|n\rangle}\, d\mu^n\right\| = \left\|\left(\int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{A}^{|n\rangle}\, d\mu^n\right)(I)\right\| = \left\|\int_{\Omega^n} \left|\mathscr{A}^{|n\rangle}\right|^2 d\mu^n\right\|. \tag{11}$$

Now, the equality in (9) follows from (11) by the very definition of the spectral radius

$$r\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right) \stackrel{def}{=} \inf_{n\in\mathbb{N}} \left\|\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right)^n\right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right)^n\right\|^{\frac{1}{n}}$$
$$= \inf_{n\in\mathbb{N}} \left\|\int_{\Omega^n} \left|\mathscr{A}^{|n\rangle}\right|^2 d\mu\right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|\int_{\Omega^n} \left|\mathscr{A}^{|n\rangle}\right|^2 d\mu\right\|^{\frac{1}{n}},$$

which proves the equality case in (9).

To treat the general case, note that $\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\right)^n = \int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{B}^{|n\rangle}\, d\mu^n$, which by Lemma 3.1. (a1) of [9] applied to $\int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{B}^{|n\rangle}\, d\mu^n$ gives

$$\left\|\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\right)^n\right\|^{\frac{1}{n}} = \left\|\int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{B}^{|n\rangle}\, d\mu^n\right\|^{\frac{1}{n}}$$
$$\leqslant \left\|\int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{A}^{|n\rangle}\, d\mu^n\right\|^{\frac{1}{2n}} \left\|\int_{\Omega^n} \mathscr{B}^{|n\rangle *} \otimes \mathscr{B}^{|n\rangle}\, d\mu^n\right\|^{\frac{1}{2n}} = \left\|\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right)^n\right\|^{\frac{1}{2n}} \left\|\left(\int_\Omega \mathscr{B}^* \otimes \mathscr{B}\, d\mu\right)^n\right\|^{\frac{1}{2n}}. \tag{12}$$

Finally, by letting $n \to \infty$ in (12) we get the spectral radius formula

$$r\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\right) \leqslant \sqrt{r\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right) r\left(\int_\Omega \mathscr{B}^* \otimes \mathscr{B}\, d\mu\right)},$$

according to the already proven part of the proposition. But this is nothing else than (9), as proclaimed. □

In the situation that we will consider bellow, it says that the spectrum of the transformer $\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu$ is contained in the unit disc iff $\inf_{n\in\mathbb{N}} \left\|\int_\Omega \cdots \int_\Omega \left|\mathscr{A}_{t_1} \cdots \mathscr{A}_{t_n}\right|^2 d\mu^n(t_1, \cdot\cdot, t_n)\right\|^{\frac{1}{n}} \leqslant 1$. Also, if additionally $r\left(\int_\Omega \mathscr{B}^* \otimes \mathscr{B}\, d\mu\right) \leqslant 1$, then $r\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\right) \leqslant 1$ as well.

Definition 2.1. *Let $\mathscr{A} : \Omega \to \mathcal{B}(\mathcal{H})$ be weakly* - measurable family, such that $r\left(\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu\right) \leqslant 1$. For the transformer $\int_\Omega \mathscr{A}^* \otimes \mathscr{A}\, d\mu$ we define its associated spectral (radius) defect operator:*

$$\Delta_{\mathscr{A}} \stackrel{def}{=} s\text{-}\lim_{r\nearrow 1}\left(I + \sum_{n=1}^{\infty} r^{2n} \int_{\Omega^n} \left|\mathscr{A}_{t_1} \cdots \mathscr{A}_{t_n}\right|^2 d\mu^n(t_1, \cdot\cdot, t_n)\right)^{-1/2} = \sqrt{s\text{-}\lim_{r\nearrow 1}\left(I + \sum_{n=1}^{\infty} r^{2n} \int_{\Omega^n} \left|\mathscr{A}_{t_1} \cdots \mathscr{A}_{t_n}\right|^2 d\mu^n(t_1, \cdot\cdot, t_n)\right)^{-1}}. \tag{13}$$

Correctness of this definition is based on the fact that family of operators appearing in (13) represents a family of strongly decreasing (by r) positive contractions, due to the operator monotonicity of the function $t \mapsto \sqrt{t}$ on $[0, +\infty)$. Consequently, it strongly converges and $\Delta_{\mathscr{A}}$ itself is therefore a positive contraction.

Last equality in (13) is a consequence of the fact that $\Delta^2_{\mathscr{A}} = s-\lim_{r\nearrow 1}\Big(I + \sum_{n=1}^{\infty} r^{2n}\int_{\Omega^n} \big|\mathscr{A}_{t_1}\cdots\mathscr{A}_{t_n}\big|^2 d\mu^n(t_1,\cdot\cdot\cdot,t_n)\Big)^{-1}$, due to the continuity of multiplication of operators in the strong operator topology.

Remark 1: If $I + \sum_{n=1}^{\infty}\int_{\Omega^n}\big|\mathscr{A}^{|n\rangle}\big|^2 d\mu^n$ represents a bounded Hilbert space operator (which is by the Banach-Steinhaus theorem equivalent to the property that $\sum_{n=1}^{\infty}\int_{\Omega^n}\big\|\mathscr{A}^{|n\rangle}f\big\|^2 d\mu^n < +\infty$ for every $f \in \mathcal{H}$), then it is invertible and its inverse is exactly $\Delta^2_{\mathscr{A}}$. When this argument is applied to $r\mathscr{A}$ instead of $\mathscr{A}$, then we realize that in fact
$\Delta^{-2}_{r\mathscr{A}} = I + \sum_{n=1}^{\infty} r^{2n}\int_{\Omega}\cdots\int_{\Omega}\big|\mathscr{A}_{t_1}\cdots\mathscr{A}_{t_n}\big|^2 d\mu^n(t_1,\cdot\cdot\cdot,t_n)$, and so (13) actually says that $\Delta_{\mathscr{A}} = s-\lim_{r\nearrow 1}\Delta_{r\mathscr{A}}$. Moreover, as $\Delta^{-2}_{r\mathscr{A}} = I + r^2\int_{\Omega}\mathscr{A}_t^*\Delta^{-2}_{r\mathscr{A}}\mathscr{A}_t\,d\mu(t)$, it follows $I = \Delta^2_{r\mathscr{A}} + r^2\int_{\Omega}\Delta_{r\mathscr{A}}\mathscr{A}_t^*\Delta^{-2}_{r\mathscr{A}}\mathscr{A}_t\Delta_{r\mathscr{A}}\,d\mu(t)$, from which we derive by the limiting process $I - \Delta^2_{\mathscr{A}} = s-\lim_{r\nearrow 1}\int_{\Omega}\big|\Delta^{-1}_{r\mathscr{A}}\mathscr{A}_t\Delta_{r\mathscr{A}}\big|^2 d\mu(t) \geqslant 0$. This also shows that $\Delta_{\mathscr{A}}$ is a positive contraction, as well as how much $\Delta^2_{\mathscr{A}}$ declines from I.

An appropriate use of (6) can also show us that $\big\|\sum_{n=0}^{\infty} r^{2n}\int_{\Omega^n}\mathscr{A}^{|n\rangle *}\otimes\mathscr{A}^{|n\rangle}\,d\mu^n\big\| = \big\|\Delta^{-2}_{r\mathscr{A}}\big\|$ (where, in the sense of definition (24) applied to $\mathscr{A} = \mathscr{B}$, the summand for $n = 0$ is understood as the identity transformer on $\mathcal{B}(\mathcal{H})$).

Example 1. For the right unilateral shift $S : \ell^2_{\mathbb{N}} \to \ell^2_{\mathbb{N}} : (x_1, \cdots, x_n, \cdots) \mapsto (0, x_1, \cdots, x_n, \cdots)$ its adjoint operator is the left unilateral shift $S^* : \ell^2_{\mathbb{N}} \to \ell^2_{\mathbb{N}} : (x_1, \cdots, x_n, \cdots) \mapsto (x_2, \cdots, x_n, \cdots)$. For any $n \in \mathbb{N}$ we have $S^{*n}S^n = I$ and $S^nS^{*n} = I - \sum_{j=1}^{n} e_j \otimes e_j^\star$, where $\{e_n\}_{n=1}^{\infty}$ stands for the standard basis of $\ell^2_{\mathbb{N}}$. Thus

$$\Delta_{S^n} = s-\lim_{r\nearrow 1}\Delta_{rS^n} = s-\lim_{r\nearrow 1}\Big(\sum_{k=0}^{\infty} r^{2k}S^{*kn}S^{kn}\Big)^{-1/2} = s-\lim_{r\nearrow 1}\sqrt{1-r^2}I = 0.$$

Let us denote $P_m \overset{def}{=} \sum_{j=1}^{m} e_j \otimes e_j^\star$ for all $m \in \mathbb{N}$. Then, a direct computation reveals that

$$\begin{aligned}\Delta_{S^{*n}} &= s-\lim_{r\nearrow 1}\Delta_{rS^{*n}} = s-\lim_{r\nearrow 1}\Big(\sum_{k=0}^{\infty} r^{2k}S^{kn}S^{*kn}\Big)^{-1/2} = s-\lim_{r\nearrow 1}\Big(\sum_{k=0}^{\infty} r^{2k}\big(I - P_{kn}\big)\Big)^{-1/2}\\ &= s-\lim_{r\nearrow 1}\Big(\sum_{k=0}^{\infty} r^{2k}\sum_{l=k+1}^{\infty}\big(P_{ln} - P_{(l-1)n}\big)\Big)^{-1/2} = s-\lim_{r\nearrow 1}\Big(\sum_{l=1}^{\infty}\sum_{k=0}^{l-1} r^{2k}\sum_{j=n(l-1)+1}^{nl} e_j\otimes e_j^\star\Big)^{-1/2}\\ &= s-\lim_{r\nearrow 1}\sum_{l=1}^{\infty}\frac{1}{\sqrt{\sum_{k=0}^{l-1} r^{2k}}}\sum_{j=n(l-1)+1}^{nl} e_j\otimes e_j^\star = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\sum_{j=n(k-1)+1}^{nk} e_j\otimes e_j^\star. \end{aligned} \tag{14}$$

In other words, $\Delta_{S^{*n}}$ is in $\mathfrak{C}_p(\mathcal{H})$ for all $p > 2$, with its eigenvalue sequence $\big\{\frac{1}{\sqrt{k}}\big\}_{k=1}^{\infty}$ and each of its eigenvalues has the multiplicity n.

Another situation when an explicit formula for $\Delta_{\mathscr{A}}$ can be given is in the case when this family consists of commuting normal operators.

Lemma 2.2. *If $\{\mathscr{A}_t\}_{t\in\Omega}$ consists of commuting normal operators and $\int_{\Omega}\mathscr{A}^*\mathscr{A}\,d\mu \leqslant I$, then*

$$\Delta_{\mathscr{A}} = \sqrt{I - \int_{\Omega}\mathscr{A}^*\mathscr{A}\,d\mu}. \tag{15}$$

Proof. Since $\{\mathscr{A}_t\}_{t\in\Omega}$ are commuting normal operators, then $\mathscr{A}_t$ commute with $\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu$ for every $t\in\Omega$, so that consequently we have

$$\int_{\Omega^n} \left|\mathscr{A}^{|n\rangle}\right|^2 d\mu^n = \int_{\Omega^n} \left|\mathscr{A}_{t_1}\cdots\mathscr{A}_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n) = \left(\int_\Omega \mathscr{A}_t^*\mathscr{A}_t\,d\mu(t)\right)^n = \left(\int_\Omega |\mathscr{A}|^2\,d\mu\right)^n.$$

Therefore $r\left(\int_\Omega \mathscr{A}^*\otimes\mathscr{A}\,d\mu\right) = \left\|\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu\right\| \leqslant 1$, and we also have

$$\begin{aligned}\Delta_{\mathscr{A}} &= s-\lim_{r\nearrow 1}\left(I+\sum_{n=1}^\infty r^{2n}\int_{\Omega^n}\left|\mathscr{A}^{|n\rangle}\right|^2 d\mu^n\right)^{-\frac12} = s-\lim_{r\nearrow 1}\left(I+\sum_{n=1}^\infty r^{2n}\left(\int_\Omega |\mathscr{A}|^2\,d\mu\right)^n\right)^{-\frac12}\\ &= s-\lim_{r\nearrow 1}\sqrt{I-r^2\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu} = \sqrt{I-\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu}. \qquad (16)\end{aligned}$$

Equality (16) can easily be checked by the use of the spectral theorem for positive contraction $\int_\Omega \mathscr{A}^*\mathscr{A}\,d\mu$. □

3. Main Results and Applications

We start with the norm inequalities for i.p.t.i. transformers acting on $\mathfrak{C}_p(\mathcal{H})$.

Theorem 3.1. *Let $\{\mathscr{A}_t\}_{t\in\Omega}$ and $\{\mathscr{B}_t\}_{t\in\Omega}$ be weakly*-measurable families of bounded operators such that $r\left(\int_\Omega \mathscr{A}^*\otimes\mathscr{A}\,d\mu\right)\leqslant 1$ and $r\left(\int_\Omega \mathscr{B}^*\otimes\mathscr{B}\,d\mu\right)\leqslant 1$. Then for all $X\in\mathfrak{B}(\mathcal{H})$*

$$\|\Delta_{\mathscr{A}}X\Delta_{\mathscr{B}}\| \leqslant \left\|X-\int_\Omega \mathscr{A}_t^*X\mathscr{B}_t\,d\mu(t)\right\|. \qquad (17)$$

If additionally $p\geqslant 2$ and

$$\sum_{n=1}^\infty \int_{\Omega^n}\left\|\mathscr{A}_{t_1}^*\cdots\mathscr{A}_{t_n}^*f\right\|^2 d\mu^n(t_1,\cdots,t_n) < +\infty \qquad \text{for all } f\in\mathcal{H}, \qquad (18)$$

then $r\left(\int_\Omega \mathscr{A}\otimes\mathscr{A}^*\,d\mu\right)\leqslant 1$ and for all $X\in\mathfrak{C}_p(\mathcal{H})$

$$\left\|\Delta_{\mathscr{A}}^{1-\frac2p}X\Delta_{\mathscr{B}}\right\|_p \leqslant \left\|\Delta_{\mathscr{A}^*}^{-\frac2p}\left(X-\int_\Omega \mathscr{A}_t^*X\mathscr{B}_t\,d\mu(t)\right)\right\|_p. \qquad (19)$$

Similarly, when $p\geqslant 2$ and

$$\sum_{n=1}^\infty \int_{\Omega^n}\left\|\mathscr{B}_{t_1}^*\cdots\mathscr{B}_{t_n}^*f\right\|^2 d\mu^n(t_1,\cdots,t_n) < +\infty \qquad \text{for all } f\in\mathcal{H}, \qquad (20)$$

then $r\left(\int_\Omega \mathscr{B}\otimes\mathscr{B}^*\,d\mu\right)\leqslant 1$ and for all $X\in\mathfrak{C}_p(\mathcal{H})$

$$\left\|\Delta_{\mathscr{A}}X\Delta_{\mathscr{B}}^{1-\frac2p}\right\|_p \leqslant \left\|\left(X-\int_\Omega \mathscr{A}_t^*X\mathscr{B}_t\,d\mu(t)\right)\Delta_{\mathscr{B}^*}^{-\frac2p}\right\|_p. \qquad (21)$$

If $p,q,s\geqslant 1$ are such that $\frac1q+\frac1s=\frac2p$ and if both conditions (18) and (20) are fulfilled, then

$$r\left(\int_\Omega \mathscr{A}\otimes\mathscr{A}^*\,d\mu\right)\leqslant 1 \quad \text{and} \quad r\left(\int_\Omega \mathscr{B}\otimes\mathscr{B}^*\,d\mu\right)\leqslant 1. \qquad (22)$$

and for all $X\in\mathfrak{C}_p(\mathcal{H})$

$$\left\|\Delta_{\mathscr{A}}^{1-\frac1q}X\Delta_{\mathscr{B}}^{1-\frac1s}\right\|_p \leqslant \left\|\Delta_{\mathscr{A}^*}^{-\frac1q}\left(X-\int_\Omega \mathscr{A}_t^*X\mathscr{B}_t\,d\mu(t)\right)\Delta_{\mathscr{B}^*}^{-\frac1s}\right\|_p. \qquad (23)$$

Proof. We restrict ourselves to the proof of (23), as it contains all essential steps for the proof of (17), (19) and (21). In accordance with the already used notation, let $\mathscr{A}^{*|n\rangle}(t_1,\cdots,t_n) \overset{def}{=} \mathscr{A}^*_{t_1}\cdots\mathscr{A}^*_{t_n}$ and $\mathscr{B}^{*|n\rangle}(t_1,\cdots,t_n) \overset{def}{=} \mathscr{B}^*_{t_1}\cdots\mathscr{B}^*_{t_n}$ for all $(t_1,\cdots,t_n) \in \Omega^n$. Adding to the previous notation, for $C, D \in \mathcal{B}(\mathcal{H})$ let

$$\int_{\Omega^0} C\mathscr{A}^{|0\rangle *} \otimes \mathscr{B}^{|0\rangle} D\, d\mu^0 \overset{def}{=} C \otimes D : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) : X \mapsto CXD \tag{24}$$

and let $\int_{\Omega^0} \left|\mathscr{A}^{|0\rangle}\right|^2 d\mu^0 \overset{def}{=} \int_{\Omega^0} \left|\mathscr{A}^{*|0\rangle}\right|^2 d\mu^0 \overset{def}{=} \int_{\Omega^0} \left|\mathscr{B}^{|0\rangle}\right|^2 d\mu^0 \overset{def}{=} \int_{\Omega^0} \left|\mathscr{B}^{*|0\rangle}\right|^2 d\mu^0 \overset{def}{=} I$, the identity operator on $\mathcal{H}$. First, we will prove (22). Condition (18) provides that $\sum_{n=0}^{\infty} \int_{\Omega^n} \left|\mathscr{A}^*_{t_1}\cdots\mathscr{A}^*_{t_n}\right|^2 d\mu^n(t_1,\cdots,t_n)$ is a bounded Hilbert space operator, and it actually equals to $\Delta^{-2}_{\mathscr{A}^*}$. Obviously, it is bounded from bellow by I. As

$$\left\|\left(\int_\Omega \mathscr{A} \otimes \mathscr{A}^* d\mu\right)^n\right\| = \left\|\int_{\Omega^n} \mathscr{A}^{|n\rangle} \otimes \mathscr{A}^{|n\rangle *}\, d\mu^n\right\| = \left\|\int_{\Omega^n} \left|\mathscr{A}^{*|n\rangle}\right|^2 d\mu^n\right\| \leqslant \left\|\sum_{n=0}^{\infty} \int_{\Omega^n} \left|\mathscr{A}^{*|n\rangle}\right|^2 d\mu^n\right\| = \left\|\Delta^{-2}_{\mathscr{A}^*}\right\|,$$

it follows that

$$r\left(\int_\Omega \mathscr{A} \otimes \mathscr{A}^* d\mu\right) = \left\|\left(\int_\Omega \mathscr{A} \otimes \mathscr{A}^* d\mu\right)^n\right\|^{\frac{1}{n}} \leqslant \inf_{n\in\mathbb{N}} \left\|\Delta^{-2}_{\mathscr{A}^*}\right\|^{\frac{1}{n}} = 1.$$

Similarly, $\Delta^{-2}_{\mathscr{B}^*} = \sum_{n=0}^{\infty} \int_{\Omega^n} \left|\mathscr{B}^{|n\rangle *}\right|^2 d\mu^n$ is a bounded operator, and also $r\left(\int_\Omega \mathscr{B} \otimes \mathscr{B}^* d\mu\right) \leqslant 1$. For every $r \in [0,1)$ we have

$$\left(I - r^2\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\right)^{-1} = \sum_{n=0}^{\infty} r^{2n} \left(\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\right)^n = \sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{A}^{|n\rangle *} \otimes \mathscr{B}^{|n\rangle}\, d\mu^n, \tag{25}$$

and therefore

$$\left\|\Delta^{1-\frac{1}{q}}_{r\mathscr{A}} X \Delta^{1-\frac{1}{s}}_{r\mathscr{B}}\right\|_p = \left\|\sum_{n=0}^{\infty} r^{2n} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \int_{\Omega^n} \mathscr{A}^{|n\rangle *}\left(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\right)\mathscr{B}^{|n\rangle}\, d\mu^n \Delta^{1-\frac{1}{s}}_{r\mathscr{B}}\right\|_p \leqslant \left\|C_r\left(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B} d\mu\right)D_r\right\|_p, \tag{26}$$

by virtue of Th. 3.3 in [9], where

$$\begin{aligned} C_r &\overset{def}{=} \left(\sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{A}^{|n\rangle} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \left(\sum_{n=0}^{\infty} r^{2n} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \int_{\Omega^n} \left|\mathscr{A}^{|n\rangle}\right|^2 d\mu^n \Delta^{1-\frac{1}{q}}_{r\mathscr{A}}\right)^{q-1} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \mathscr{A}^{|n\rangle *}\, d\mu^n\right)^{\frac{1}{2q}} \\ &= \left(\sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{A}^{|n\rangle} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \left(\Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \Delta^{-2}_{r\mathscr{A}} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}}\right)^{q-1} \Delta^{1-\frac{1}{q}}_{r\mathscr{A}} \mathscr{A}^{|n\rangle *}\, d\mu^n\right)^{\frac{1}{2q}} = \left(\sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{A}^{|n\rangle} \mathscr{A}^{|n\rangle *} d\mu^n\right)^{\frac{1}{2q}} = \Delta^{-\frac{1}{q}}_{r\mathscr{A}^*} \end{aligned} \tag{27}$$

and, by analogy,

$$D_r \overset{def}{=} \left(\sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{B}^{|n\rangle} \Delta^{1-\frac{1}{s}}_{r\mathscr{B}} \left(\sum_{n=0}^{\infty} r^{2n} \Delta^{1-\frac{1}{s}}_{r\mathscr{B}} \int_{\Omega^n} \left|\mathscr{B}^{|n\rangle}\right|^2 d\mu^n \Delta^{1-\frac{1}{s}}_{r\mathscr{B}}\right)^{s-1} \Delta^{1-\frac{1}{s}}_{r\mathscr{B}} \mathscr{B}^{|n\rangle *}\, d\mu^n\right)^{\frac{1}{2s}} = \Delta^{-\frac{1}{s}}_{r\mathscr{B}^*}. \tag{28}$$

Thus we have proved (23) for $r\mathscr{A}$ and $r\mathscr{B}$ instead of $\mathscr{A}$ and $\mathscr{B}$ respectively. First note that $\Delta^{-2}_{r\mathscr{A}^*} \leqslant \Delta^{-2}_{\mathscr{A}^*}$ implies $\Delta^{-\frac{2}{q}}_{r\mathscr{A}^*} \leqslant \Delta^{-\frac{2}{q}}_{\mathscr{A}^*}$ due to the operator monotonicity of the function $t \mapsto t^{\frac{1}{q}}$ on $[0,+\infty)$. Similarly $\Delta^{-\frac{2}{s}}_{r\mathscr{B}^*} \leqslant \Delta^{-\frac{2}{s}}_{\mathscr{B}^*}$, which due to monotonicity property (3) gives

$$\left\|\Delta^{-\frac{1}{q}}_{r\mathscr{A}^*}\left(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\right)\Delta^{-\frac{1}{s}}_{r\mathscr{B}^*}\right\|_p \leqslant \left\|\Delta^{-\frac{1}{q}}_{\mathscr{A}^*}\left(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B} d\mu\right)\Delta^{-\frac{1}{s}}_{\mathscr{B}^*}\right\|_p.$$

All we have to do now is to invoke the lower semicontinuity of Schatten tracial p norms to see that

$$\left\|\Delta_{\mathscr{A}}^{1-\frac{1}{q}} X \Delta_{\mathscr{B}}^{1-\frac{1}{s}}\right\|_p = \left\| w\!-\!\lim_{r\nearrow 1} \Delta_{r\mathscr{A}}^{1-\frac{1}{q}} X \Delta_{r\mathscr{B}}^{1-\frac{1}{s}}\right\|_p \leqslant \liminf_{r\nearrow 1} \left\|\Delta_{r\mathscr{A}^*}^{-\frac{1}{q}}\Big(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\Big)\Delta_{r\mathscr{B}^*}^{-\frac{1}{s}}\right\|_p$$
$$\leqslant \liminf_{r\nearrow 1} \left\|\Delta_{\mathscr{A}^*}^{-\frac{1}{q}}\Big(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\Big)\Delta_{\mathscr{B}^*}^{-\frac{1}{s}}\right\|_p = \left\|\Delta_{\mathscr{A}^*}^{-\frac{1}{q}}\Big(X - \int_\Omega \mathscr{A}^* X \mathscr{B} d\mu\Big)\Delta_{\mathscr{B}^*}^{-\frac{1}{s}}\right\|_p,$$

which concludes the proof of (23).

(17) could essentially be seen as the special case of (23) for $\frac{1}{q} = \frac{1}{s} = 0$, with almost identical proof which differs from the just presented one only by the use of $\mathcal{B}(\mathcal{H})$ norm $\|\cdot\|$ instead of Schatten tracial p norm $\|\cdot\|_p$, the use Lemma 3.1. (12) of [9] instead of Th. 3.3 in [9] and 0 instead of $\frac{1}{q}$ and $\frac{1}{s}$. Thus, requirements (18) and (20) are not needed in this occasion. Similarly, (18) and (20) are not needed for the proof of (21) and (19) respectively. □

For arbitrary $A, B \in \mathcal{B}(\mathcal{H})$ a bilateral multiplier transformer $A^* \otimes B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) : X \mapsto A^* X B$ gives the simplest example of i.p.t.i. transformer, with the measure space consisting of a single point. When $r(A) \leqslant 1$ and $r(B) \leqslant 1$, then

$$r(A^* \otimes A) = \inf_{n\in\mathbb{N}} \left\||A^n|^2\right\|^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} \|A^n\|^{\frac{1}{n}} = r(A) = r(A^*) = r(A \otimes A^*) \leqslant 1,$$

while $\Delta_{rA}^{-2} = \sum_{n=0}^{\infty} r^{2n} A^{*n} A^n$ and consequently $\Delta_A = s\text{-}\lim_{r\nearrow 1}\big(\sum_{n=0}^{\infty} r^{2n} A^{*n} A^n\big)^{-1/2} = \sqrt{s\text{-}\lim_{r\nearrow 1}\big(\sum_{n=0}^{\infty} r^{2n} A^{*n} A^n\big)^{-1}}$, based on (13). With a similar conclusions for Δ_B, Th. 3.1 gives

Corollary 3.2. *If $r(A) \leqslant 1$ and $r(B) \leqslant 1$ for some $A, B \in \mathcal{B}(\mathcal{H})$, then for all $X \in \mathcal{B}(\mathcal{H})$*

$$\left\|\sqrt{s\text{-}\lim_{r\nearrow 1}\Big(\sum_{n=0}^{\infty} r^{2n} A^{*n} A^n\Big)^{-1}}\, X \sqrt{s\text{-}\lim_{r\nearrow 1}\Big(\sum_{n=0}^{\infty} r^{2n} B^{*n} B^n\Big)^{-1}}\right\| \leqslant \left\|X - A^* X B\right\|. \tag{29}$$

If additionally $\sum_{n=1}^{\infty} \|A^{*n} f\|^2 < +\infty$ for all $f \in \mathcal{H}$, then for all $p \geqslant 2$ and for all $X \in \mathfrak{C}_p(\mathcal{H})$

$$\left\|\left(s\text{-}\lim_{r\nearrow 1}\Big(\sum_{n=0}^{\infty} r^{2n} A^{*n} A^n\Big)^{-1}\right)^{\frac{1}{2}-\frac{1}{p}} X\, s\text{-}\lim_{r\nearrow 1}\Big(\sum_{n=0}^{\infty} r^{2n} B^{*n} B^n\Big)^{-\frac{1}{2}}\right\|_p \leqslant \left\|\Big(\sum_{n=0}^{\infty} A^n A^{*n}\Big)^{\frac{1}{p}}\big(X - A^* X B\big)\right\|_p. \tag{30}$$

Alternatively, if $\sum_{n=1}^{\infty} \|B^{*n} f\|^2 < +\infty$ for all $f \in \mathcal{H}$, then for all $p \geqslant 2$ and for all $X \in \mathfrak{C}_p(\mathcal{H})$

$$\left\|s\text{-}\lim_{r\nearrow 1}\Big(\sum_{n=0}^{\infty} r^{2n} A^{*n} A^n\Big)^{-\frac{1}{2}} X \left(s\text{-}\lim_{r\nearrow 1}\Big(\sum_{n=0}^{\infty} r^{2n} B^{*n} B^n\Big)^{-1}\right)^{\frac{1}{2}-\frac{1}{p}}\right\|_p \leqslant \left\|\big(X - A^* X B\big)\Big(\sum_{n=0}^{\infty} B^n B^{*n}\Big)^{\frac{1}{p}}\right\|_p. \tag{31}$$

In the case when operator A is a normal contraction, formula (15) gives $\Delta_A = \sqrt{I - A^* A}$, and thus $\sqrt{I - A^* A}$ represents also the defect operator D_A for A, according to the notation in [13]. Hence (29) actually generalize Th. 2.3. from [8] to non-normal operators with their spectra in the unit disc, in the case of uniform norm.

In the case of shift operator, formula (14) gives

Corollary 3.3. *For the right unilateral shift S, for all $m, n \in \mathbb{N}$ and for all $X \in \mathcal{B}(\mathcal{H})$ we have*

$$\left\|\Big(\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sum_{l=m(k-1)+1}^{mk} e_l \otimes e_l^{\star}\Big) X \Big(\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sum_{l=n(k-1)+1}^{nk} e_l \otimes e_l^{\star}\Big)\right\| \leqslant \left\|X - S^m X S^{*n}\right\|. \tag{32}$$

Besides shifts and other contractions, Cor. 3.2 is applicable to operators of the form $A = TCT^{-1}$ and $B = WDW^{-1}$, for some contractions C and D and some invertible operators T and W. This cames from the simple fact that spectras of A and C coincide, as well as spectras of B and D. In fact, it is well known that operators similar to contractions are exactly those operators which spectrum is contained in the unit disc (see Cor. 8.2. in [13]).

To consider the validity of Th. 3.1 for an arbitrary u.i. Q-norm we need

Lemma 3.4. *Let $\mathscr{C}, \mathscr{D} : \Omega \to \mathfrak{B}(\mathcal{H})$ be weakly*-measurable families such that $\{\mathscr{C}_t\}_{t\in\Omega}$ consists of commuting normal operators and $\int_\Omega \|\mathscr{C}_t f\|^2 + \|\mathscr{D}_t f\|^2 \, d\mu(t) < +\infty$ for all $f \in \mathcal{H}$. Then for any Q-norm $|||\cdot|||_{(2)}$ and for any $X \in \mathfrak{C}_{|||\cdot|||_{(2)}}(\mathcal{H})$*

$$\left|\left|\left| \int_\Omega \mathscr{C}^* X \mathscr{D} \, d\mu \right|\right|\right|_{(2)} \leqslant \left|\left|\left| \sqrt{\int_\Omega \mathscr{C}^* \mathscr{C} \, d\mu} \, X \right|\right|\right|_{(2)} \left\| \sqrt{\int_\Omega \mathscr{D}^* \mathscr{D} \, d\mu} \right\|. \tag{33}$$

Similarly, when $\{\mathscr{D}_t\}_{t\in\Omega}$ consists of commuting normal operators, then

$$\left|\left|\left| \int_\Omega \mathscr{C}^* X \mathscr{D} \, d\mu \right|\right|\right|_{(2)} \leqslant \left\| \sqrt{\int_\Omega \mathscr{C}^* \mathscr{C} \, d\mu} \right\| \cdot \left|\left|\left| X \sqrt{\int_\Omega \mathscr{D}^* \mathscr{D} \, d\mu} \right|\right|\right|_{(2)}. \tag{34}$$

Proof. Let us remember (4) that $|||X|||_{(2)}$ denote a Q-norm $|||X^*X|||^{1/2}$ of any X such that $X^*X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$. Based on the Th. 3.1. (e) of [9] we have

$$\left|\left|\left| \left| \int_\Omega \mathscr{C}^* X \mathscr{D} \, d\mu \right|^2 \right|\right|\right| = \left|\left|\left| \int_\Omega \mathscr{C}^* X \mathscr{D} \, d\mu \right|\right|\right|_{(2)}^2 \leqslant \left|\left|\left| \int_\Omega \mathscr{C}^* X X^* \mathscr{C} \, d\mu \right|\right|\right| \cdot \left\| \int_\Omega \mathscr{D}^* \mathscr{D} \, d\mu \right\|, \tag{35}$$

where we took that $\alpha = 2$, $\theta = 0$, $\|\cdot\|_{\Phi_1} = \|\cdot\|_{\Phi_2} = |||\cdot|||$ and $\|\cdot\|_{\Phi_3} = \|\cdot\|$. Furthermore, if the family $\{\mathscr{C}_t\}_{t\in\Omega}$ consists of commuting normal operators, we have

$$\left|\left|\left| \int_\Omega \mathscr{C}^* X X^* \mathscr{C} \, d\mu \right|\right|\right| \leqslant \left|\left|\left| \sqrt{\int_\Omega \mathscr{C}^* \mathscr{C} \, d\mu} \, X X^* \sqrt{\int_\Omega \mathscr{C}^* \mathscr{C} \, d\mu} \right|\right|\right| = \left|\left|\left| \sqrt{\int_\Omega \mathscr{C}^* \mathscr{C} \, d\mu} \, X \right|\right|\right|_{(2)}^2, \tag{36}$$

where we used Th. 3.2. of [9]. Finally, we get (33) from (35) and (36). The proof for (34) goes by analogy. □

Theorem 3.5. *Let $\mathscr{A}, \mathscr{B} : \Omega \to \mathfrak{B}(\mathcal{H})$ be weakly*-measurable families such that*

$$\inf_{n\in\mathbb{N}} \left\| \int_{\Omega^n} \left| \mathscr{A}_{t_1} \cdots \mathscr{A}_{t_n} \right|^2 d\mu^n(t_1, \cdots, t_n) \right\|^{\frac{1}{n}} \leqslant 1 \quad \text{and} \quad \inf_{n\in\mathbb{N}} \left\| \int_{\Omega^n} \left| \mathscr{B}_{t_1} \cdots \mathscr{B}_{t_n} \right|^2 d\mu^n(t_1, \cdots, t_n) \right\|^{\frac{1}{n}} \leqslant 1,$$

let $|||\cdot|||$ be an arbitrary u.i. norm and let $X^*X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$. If $\{\mathscr{A}_t\}_{t\in\Omega}$ consists of commuting normal operators, then

$$\left|\left|\left| \sqrt{I - \int_\Omega \mathscr{A}_t^* \mathscr{A}_t \, d\mu(t)} \, X \Delta_{\mathscr{B}} \right|\right|\right|_{(2)} \leqslant \left|\left|\left| X - \int_\Omega \mathscr{A}_t^* X \mathscr{B}_t \, d\mu(t) \right|\right|\right|_{(2)}. \tag{37}$$

Similarly, if $\{\mathscr{B}_t\}_{t\in\Omega}$ consists of commuting normal operators, then

$$\left|\left|\left| \Delta_{\mathscr{A}} X \sqrt{I - \int_\Omega \mathscr{B}_t^* \mathscr{B}_t \, d\mu(t)} \right|\right|\right|_{(2)} \leqslant \left|\left|\left| X - \int_\Omega \mathscr{A}_t^* X \mathscr{B}_t \, d\mu(t) \right|\right|\right|_{(2)}. \tag{38}$$

Proof. Let $0 \leqslant r < 1$. Based on the expansion (25) and the previous Lemma 3.4 we have

$$\begin{aligned}
\left\|\left|\Delta_{r\mathscr{A}} X \Delta_{r\mathscr{B}}\right|\right\|_{(2)} &= \left\|\left|\Delta_{r\mathscr{A}}\Big(\sum_{n=0}^{\infty} r^{2n}\Big(\int_\Omega \mathscr{A}^* \otimes \mathscr{B}\, d\mu\Big)^n \Big(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\Big)\Big)\Delta_{r\mathscr{B}}\right|\right\|_{(2)} \\
&= \left\|\left|\sum_{n=0}^{\infty} r^{2n} \Delta_{r\mathscr{A}}^{1-\frac{1}{q}} \int_{\Omega^n} \mathscr{A}^{|n\rangle *}\Big(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\Big)\mathscr{B}^{|n\rangle}\, d\mu^n \Delta_{r\mathscr{B}}^{1-\frac{1}{s}}\right|\right\|_{(2)} \\
&\leqslant \left\|\left|\sqrt{\Delta_{r\mathscr{A}} \sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{A}^{|n\rangle *}\mathscr{A}^{|n\rangle}\, d\mu^n \Delta_{r\mathscr{A}}}\Big(X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\Big)\right|\right\|_{(2)} \left\|\sqrt{\Delta_{r\mathscr{B}} \sum_{n=0}^{\infty} r^{2n} \int_{\Omega^n} \mathscr{B}^{|n\rangle *}\mathscr{B}^{|n\rangle}\, d\mu^n \Delta_{r\mathscr{B}}}\right\| \\
&= \left\|\left|X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\right|\right\|_{(2)}.
\end{aligned} \tag{39}$$

Since every Q-norm is also an u.i. norm and therefore it is lower semi-continuous, we have

$$\begin{aligned}
\left\|\left|\Delta_{\mathscr{A}} X \Delta_{\mathscr{B}}\right|\right\|_{(2)} &= \left\|\left|w-\lim_{r\nearrow 1} \Delta_{r\mathscr{A}} X \Delta_{r\mathscr{B}}\right|\right\|_{(2)} \leqslant \liminf_{r\nearrow 1}\left\|\left|\Delta_{r\mathscr{A}} X \Delta_{r\mathscr{B}}\right|\right\|_{(2)} \\
&\leqslant \liminf_{r\nearrow 1}\left\|\left|X - r^2\int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\right|\right\|_{(2)} = \left\|\left|X - \int_\Omega \mathscr{A}^* X \mathscr{B}\, d\mu\right|\right\|_{(2)}.
\end{aligned} \tag{40}$$

Taking (15) into account in (40) concludes the proof of (37). The proof for (38) goes by analogy. □

Thus, in the case of Q-norms, Th. 3.5 extends Th. 4.1. of [9] to the situation when only one of families $\{\mathscr{A}_t\}_{t\in\Omega}$ and $\{\mathscr{B}_t\}_{t\in\Omega}$ needs to consist of commuting normal operators. Specially, we have (37) and (38) to hold for Schatten tracial p norms $\|\cdot\|_p$ for all $p \geqslant 2$. In a special case when Ω is a single point, Th 3.5 says that

$$\left\|\left|\sqrt{I - A^*A}\, X \Delta_B\right|\right\|_{(2)} \leqslant \left\|\left|X - AXB\right|\right\|_{(2)},$$

whenever A is a normal contraction and $r(B) \leqslant 1$. This extends Th. 2.3 in [8] to the case of Q-norms.

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