Left-Right Fredholm and Left-Right Browder Linear Relations

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Abstract. In this paper we introduce the notions of left (resp. right) Fredholm and left (resp. right) Browder linear relations. We construct a Kato-type decomposition of such linear relations. The results are then applied to give another decomposition of a left (resp. right) Browder linear relation T in a Banach space as an operator-like sum $T = A + B$, where A is an injective left (resp. a surjective right) Fredholm linear relation and B is a bounded finite rank operator with certain properties of commutativity. The converse results remain valid with certain conditions of commutativity. As a consequence, we infer the characterization of left (resp. right) Browder spectrum under finite rank operator.

1. Introduction

In a paper of 2011 [17], Zivkovic, Djordjevic and Harte consider four types of Fredholm operators. A bounded operator T from a Banach space X to a Banach space Y is said to be left Fredholm if it is upper semiFredholm and $R(T)$ is topologically complemented in Y, right Fredholm if it is lower semiFredholm and $N(T)$ is topologically complemented in X. When in particular $X = Y$ we say that T is a left Browder operator if it is a left Fredholm operator with finite ascent and T is called a right Browder operator if T is a right Fredhom operator with finite descent.

In [17, Theorems 5 and 6] the authors have shown that these operators can be characterized in terms of an algebraic decomposition, the so-called Kato decomposition.

The purpose of this paper is to extend the results above mentioned to the general case of multivalued linear operators. After that, the results obtained are applied to give another decomposition of a left (resp. right) Browder linear relation T in a Banach space as an operator-like sum $T = A + B$, where A is an injective left (resp. a surjective right) Fredholm linear relation and B is a bounded finite rank operator with certain properties of commutativity. The converse results remain valid with certain conditions of commutativity. As a consequence, we infer the characterization of left (resp. right) Browder spectrum under finite rank operator.

We note that the class of operators is unstable under the operations closure, inverse and adjoint. This is not the case if we consider the more general case of linear relations. We emphasize that the linear relations

(sometimes called multivalued linear operators) made their appearance in Functional Analysis motivated by the need to consider adjoints of non-densely defined Fredholm type operators which arise in physical applications (see, for instance [13] [16] among others), and also by the need to consider the inverses of certain operators, used, for example in the study of some Cauchy problems associated to parabolic type equations in Banach spaces (see, for instance [10]).

The investigation of the theory of Fredholm-type linear relations, including the left-right Fredholm and the left-right Browder linear relations, serves a twofold purpose. The first is for it own sake and intrinsic value; the second is that such an investigation may shed light on many problems of the operator theory. We cite two of them.

1.- The study of linear bundles. Let S and T be bounded operators. The map $P(\lambda) := \lambda S - T, \lambda \in \mathbb{C}$ is called a linear bundle. It is known that many problems of mathematical physics (for example, quantum theory, transport theory,...) are reduced to the study of certain reversibility conditions of $\lambda S - T$, and this study is reduced to the analysis of the essential spectra of the linear relations $S^{-1}T$ and TS^{-1} (see, for example [12]).

2.- The study of the existence and uniqueness questions for bounded solutions of the differential equation

$$(*) \; -dx/dt + A(t)x = f(t), t \in \mathbb{J}$$

where $\mathbb{J}$ is an infinite interval of $\mathbb{R}$ and $A(t) : D(A(t)) \subset X \to X, t \in \mathbb{J}$, is a family of closed operators in a complex Banach space X, called the phase space. In a recent paper of 2013 [4], using properties of certain left and right Fredholm linear relations, the author proves some results which are closely connected with the asymptotic behaviour of solutions, the stability of solutions and the problem of the existence of bounded solutions to differential equation (*).

To make the paper easily accessible the exposition is more or less selfcontained. For this, some purely algebraic concepts and properties of linear relations in vector spaces over the field $\mathbb{K}$ of real or complex numbers are recalled in Section 2. In Section 3 some general facts concerning the linear relations in normed spaces are presented; in particular, properties of closed linear relations and the adjoint of a linear relation are studied. Throughout Section 4 we develop a considerable quantity of interesting results needed in the rest of the paper. First, we establish some algebraic properties for linear relations which play an important role in the sequel. After that, we investigate some results concerning the conjugate of a product of linear relations and the relation between $a(T)$ (resp. $d(T)$) and $d(T')$ (resp. $a(T')$) for bounded linear relations. Left Fredholm and left Browder linear relations in a Banach space are considered in Section 5, where also the Kato decomposition results can be found. As an application we prove that a left Browder linear relation T in a Banach space can be expressed in the form $T = A + B$ where A is an injective left Fredholm linear relation and B is a bounded finite rank operator with $BT \subset TB$. The converse of the last result remains valid for everywhere defined linear relation T satisfying T' commutes with itself. Furthermore, we apply the obtained results to study the characterization of left Browder spectrum. The notions of right Fredholm and right Browder linear relation in a Banach space are introduced and studied in Section 6. The Theorems 6.1 and 6.2 characterize the right Fredholm and right Browder linear relations through the Kato decomposition. These Theorems generalize the corresponding results of Zivkovic, Djordjevic and Harte [17, Theorem 6] for bounded operators. As an application of Theorem 6.2 we deduce the third main result of this Section (Theorem 6.3 below), which gives another decomposition of a right Browder linear relation T as an operator-sum $T = A + B$, where A is a surjective right Fredholm linear relation and B is a bounded finite rank operator such that $BT \subset TB$. The converse of the last result remains valid under the condition T commutes with itself. Finally, in Section 7 we study the stability of right Browder linear relations under the class of compact operator perturbations. As a consequence, we obtain the stability of right Browder spectrum under compact operator perturbations.

2. Linear Relations in Vector Spaces

In this Section we present some purely algebraic notions and properties of linear relations in vector spaces which are needed in the sequel, in the attempt of making our paper as selfcontained as possible.

We adhered to the notations and terminology of the monographs [7] and [14]. Let E, F and Z denote vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation or multivalued linear operator T from E to F is any mapping T having domain $D(T)$ a nonempty subspace of E, and taking values in the collection of nonempty subsets of F such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. For $x \in E \setminus D(T)$ we define $Tx = \emptyset$. With this convention, we have that $D(T) := \{x \in E : Tx \neq \emptyset\}$.

The class of all linear relations from E to F is denoted by $LR(E, F)$ and we write $LR(E) := LR(E, E)$. If $T \in LR(E)$ we say that T is a linear relation in E. If $T \in LR(E, F)$ maps the points of its domain to singletons, then T is said to be an operator. We note that a linear relation T is an operator if and only if $T(0) = \{0\}$. Every element $T \in LR(E, F)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) := \{(x, y) \in E \times F : x \in D(T), y \in Tx\},$$

and in the sequel we will identify a linear relation with its graph.

Let $T \in LR(E, F)$. The inverse of T is the linear relation T^{-1} given by $T^{-1} := \{(y, x) : (x, y) \in T\}$. The subspace $T^{-1}(0)$, denoted by $N(T)$, is called the null space of T and we say that T is injective if $N(T) = \{0\}$. The range of T is the subspace $R(T) := T(D(T))$ and T is called surjective if $R(T) = F$. We write $\alpha(T) := \dim N(T)$ and $\beta(T) := \dim F/R(T)$.

For linear relations $S, T \in LR(E, F)$ the linear relations $S + T$, $S\widehat{+}T$ and $S \oplus T$ are defined by

$$S + T := \{(x, y + z) : (x, y) \in S, (x, z) \in T\},$$

$$S\widehat{+}T := \{(x + u, y + v) : (x, y) \in S, (u, v) \in T\}.$$

This last sum is direct when $S \cap T = \{(0, 0)\}$. In such case we write $S \oplus T$.

Let $S \in LR(E, F)$ and let M be a subspace of E. The notation $S\mid_M$ will be used for the linear relation

$$S\mid_M := S \cap (M \times F)$$

and if $E = F$ then S_M is the linear relation given by

$$S_M := S \cap (M \times M).$$

For $\lambda \in \mathbb{K}$ and $S \in LR(E)$ the linear relation λS is given by

$$\lambda S := \{(x, \lambda y) : (x, y) \in S\},$$

while $S - \lambda$ stands for $S - \lambda I$, where I is the identity operator in E.

For $T \in LR(E, F)$ and $S \in LR(F, Z)$ the composition or product ST is the linear relation given by

$$ST := \{(x, z) : (x, y) \in T, (y, z) \in S \text{ for some } y \in F\}.$$

It is useful to observe that the domain of ST coincides with the subspace $\{x \in D(T) : Tx \cap D(S) \neq \emptyset\}$. Furthermore the product of linear relations is clearly associative. Hence if $T \in LR(E)$, then $T^n, n \in \mathbb{Z}$, is defined as usual with $T^0 = I$ and $T^1 = T$. It is easily seen that $(T^{-1})^n = (T^n)^{-1}, n \in \mathbb{Z}$.

The following Lemma is a preliminary result from which information concerning the domain, the range and the null space of powers of a linear relation will follow.

Lemma 2.1. *[14, Lemmas 3.2, 3.4, 3.5, 4.2 and 5.4] Let $T \in LR(E)$. Then*

(i) *For all $n, m \in \mathbb{N} \cup \{0\}$ we have*
   $D(T^{n+m}) \subset D(T^n), R(T^{n+m}) \subset R(T^n),$
   $N(T^n) \subset N(T^{n+m}), T^n(0) \subset T^{n+m}(0),$
   $N(T^n) \subset D(T^m)$ *and* $T^n(0) \subset R(T^m).$

(ii) *If $D(T^k) = D(T^{k+1})$ for some nonnegative integer k, then $D(T^k) = D(T^n)$ for all nonnegative integers $n \geq k$.*

(iii) *If $N(T^r) = N(T^{r+1})$ (resp. $R(T^r) = R(T^{r+1})$) for some $r \in \mathbb{N} \cup \{0\}$, then $N(T^r) = N(T^n)$ (resp. $R(T^r) = R(T^n)$) for all nonnegative integers $n \geq r$.*

(iv) $N(T^{n+m})/(N(T^{n+m}) \cap \{N(T^n) + R(T^m)\}) \cong (N(T^m) \cap R(T^n))/(N(T^m) \cap R(T^{n+m}))$.

(v) *Let $m \in \mathbb{N}$ and assume that $dimN(T) < \infty$. Then $dimN(T^m) \leq m\, dimN(T)$.*

Lemma 2.2. *For $T \in LR(E)$ the following properties are equivalent:*

(i) $N(T) \subset R(T^m)$ *for all $m \in \mathbb{N}$.*

(ii) $N(T^n) \subset R(T)$ *for all $n \in \mathbb{N}$.*

(iii) $N(T^n) \subset R(T^m)$ *for all $n, m \in \mathbb{N}$.*

Proof. (i) $\Rightarrow$ (ii) This implication is proved by induction. The case $n = 1$ is obvious by (i). Assume that $N(T^n) \subset R(T)$ and we shall prove that $N(T^{n+1}) \subset R(T)$. Let $x \in N(T^{n+1})$, that is, $(x, y) \in T^n$ and $(y, 0) \in T$ for some $y \in D(T)$. Hence $y \in N(T) \subset R(T^{n+1})$ by (i), so that $(z, y) \in T^{n+1}$ for some $z \in E$ which implies that there exists $u \in E$ such that $(z, u) \in T$ and $(u, y) \in T^n$. Consequently, $(x - u, 0) \in T^n$ and thus by the induction hypothesis we have that $x - u \in R(T)$. Since $u \in R(T)$, it follows that $x \in R(T)$. Therefore $N(T^{n+1}) \subset R(T)$, as required.

(ii) $\Rightarrow$ (iii) This implication is proved by induction. The cases $m = 0$ is true for all $n \in \mathbb{N}$. Assume that $N(T^n) \subset R(T^m)$ for all $n \in \mathbb{N}$ and we shall prove that $N(T^n) \subset R(T^{m+1})$. Let $x \in N(T^n)$. So $(x, 0) \in T^n$. Due to $x \in R(T)$, it follows that $(y, x) \in T$ for some $y \in D(T)$. This implies that $(y, 0) \in T^{n+1}$ and hence $y \in N(T^{n+1}) \subset R(T^m)$. Using the fact that $(y, x) \in T$ for some $y \in D(T)$ and $(z, y) \in T^m$ for some $z \in D(T^m)$, we get $(z, x) \in T^{m+1}$ for some $z \in D(T^{m+1})$ which shows that $x \in R(T^{m+1})$.

(iii) $\Rightarrow$ (i) This implication is clear. □

The statements in Lemma 2.1 lead to the introduction of the ascent and the descent of T by

$$a(T) := min\{r \in \mathbb{N} \cup \{0\} : N(T^r) = N(T^{r+1})\},$$

$$d(T) := min\{r \in \mathbb{N} \cup \{0\} : R(T^r) = R(T^{r+1})\},$$

respectively, whenever these minima exist. If no such numbers exist the ascent and descent of T are defined to be ∞. Clearly $a(T) = 0$ if and only if T is injective and $d(T) = 0$ if and only if T is surjective. For a linear relation T, the root manifold $N^{\infty}(T)$ is defined by $N^{\infty}(T) = \cup_{n=1}^{\infty} N(T^n)$. Similarly, the root manifold $R^{\infty}(T)$, is defined by $R^{\infty}(T) = \cap_{n=1}^{\infty} R(T^n)$.

In [14] the authors introduce and give a systematic treatment of these notions. They show that many of the results of Taylor and Kaashoek for operators remain valid in the context of linear relations only under the additional condition $R_c(T) = \{0\}$, where for $T \in LR(E)$, $R_c(T) := (\cup_{n \in \mathbb{N}} N(T^n)) \cap (\cup_{n \in \mathbb{N}} T^n(0))$.

We note

Lemma 2.3. *[14, Lemmas 3.1 and 5.5]Let $T \in LR(E)$ such that $R_c(T) = \{0\}$. We have*

(i) *If M is a subspace of E, then $R_c(T_M) = \{0\}$.*

(ii) *If $a(T) \leq p$ for some nonnegative integer p, then $N(T^n) \cap R(T^p) = \{0\}$ for all $n \in \mathbb{N}$.*

The algebraic resolvent set of $T \in LR(E)$ is defined by Sandovici [15, page 2168] as follows:
$\rho(T) := \{\lambda \in \mathbb{K} : T - \lambda$ is injective and surjective $\}$.
The set $\sigma(T) := \mathbb{K} \setminus \rho(T)$ is called the spectrum of T.

Lemma 2.4. *[15, Lemma 6.1] Let $T \in LR(E)$ such that $\rho(T) \neq \emptyset$. Then for all $n, m \in \mathbb{N}$*

$$E = D(T^n) + R(T^m) \text{ and } \{0\} = T^m(0) \cap N(T^n).$$

This Lemma ensures that if T is a linear relation in E with $\rho(T) \neq \emptyset$ then $R_c(T) = \{0\}$.

To prove the main Theorems of this paper some properties concerning the notion of linear relation completely reduced are required.

Assume that M and N are two complementary subspaces of E, that is, $E = M + N$ and $\{0\} = M \cap N$ (in short, $E = M \oplus N$). Following [14, page 25] we say that $T \in LR(E)$ is completely reduced by the pair (M, N) if $T = T_M \oplus T_N$.

Lemma 2.5. *[14, Lemma 8.1 and Theorem 8.2] Let $T \in LR(E)$ be completely reduced by the pair (M, N). Then*

(i) $D(T) = D(T_M) \oplus D(T_N), N(T) = N(T_M) \oplus N(T_N), R(T) = R(T_M) \oplus R(T_N)$ and $T(0) = T_M(0) \oplus T_N(0)$.

(ii) *If $a(T) = p < \infty$ then $a(T_M) \leq p$ and $a(T_N) \leq p$. If $a(T_M)$ and $a(T_N)$ are both finite, then $a(T) = max\{a(T_M), a(T_N)\}$.*

(iii) *If $d(T) = q < \infty$ then $d(T_M) \leq q$ and $d(T_N) \leq q$. If $d(T_M)$ and $d(T_N)$ are both finite, then $d(T) = max\{d(T_M), d(T_N)\}$.*

3. Linear Relations in Normed Spaces

This Section contains some auxiliary notions and properties of linear relations in normed spaces. Throughout this Section X, Y and Z will denote normed spaces and T will always denote an element of $LR(X, Y)$, except where stated otherwise.

We first recall some basic properties for future use.

Lemma 3.1. *[7, Lemma IV.4.7 and IV.5.2] Let M and N be two subspaces of X such that N is closed and $N \subset M$. Then*

(i) *M is closed if and only if M/N is closed.*

(ii) *If M is closed, then*

$$(X/N)/(M/N) = X/M$$

where the equality is a canonical isometry.

(iii) *If M is a closed finite codimensional subspace of X and M_1 is a dense subspace of X, then $M \cap M_1$ is dense in M.*

If K and L are subspaces of X and X' (the dual space of X) respectively, then

$$K^{\perp} := \{x' \in X' : x'(K) = 0\} \text{ and } L^{\top} := \{x \in X : L(x) = 0\}.$$

The adjoint or conjugate of $T \in LR(X, Y)$ is the linear relation T' given by

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'.$$

We note that

Lemma 3.2. *[7, Theorem III.3.9] Let M and N be closed subspaces of the Banach space X. Then $M + N$ is closed if and only if $M + N = (M^{\perp} \cap N^{\perp})^{\top}$.*

Observe that if Q_T denotes the quotient map from Y onto $Y/\overline{T(0)}$, then it is easy to see that $Q_T T$ is an operator and so we can define for $x \in D(T), \| Tx \| := \| Q_T Tx \|$ and the norm of T is defined by $\| T \| := \| Q_T T \|$. We remark that $\| . \|$ is not a true norm since $\| T \| = 0$ does not imply $T = 0$.

Following [7] we say that $T \in LR(X, Y)$ is closed if its graph is a closed subspace of $X \times Y$, continuous if $\| T \| < \infty$, bounded if it is continuous and everywhere defined, open if its inverse is continuous, ϕ_+, denoted by $T \in \phi_+(X, Y)$, if T is closed with closed range and $dimN(T) < \infty$ and T is called a ϕ_- linear relation, denoted by $T \in \phi_-(X, Y)$, if T is closed and its range is a closed finite codimensional subspace of Y. A closed linear relation S in X such that $N(S) \subset R(S^n)$ for all $n \in \mathbb{N}$ and $R(S)$ is closed is called a regular linear relation in X.

We now list some of the known facts about linear relations in normed spaces which will be used in the sequel.

Lemma 3.3. *[7, Chapters II and III] We have:*

(i) *T^{-1} is closed if and only if T is closed if and only if $Q_T T$ is closed and $T(0)$ is closed.*

(ii) *$N(T') = R(T)^{\perp}$. Further $R(T') = N(T)^{\perp}$ if and only if T is open if and only if $\gamma(T) > 0$, where $\gamma(T) := sup\{\epsilon \geq 0 : \epsilon d(x, N(T)) \leq \| Tx \|, x \in D(T)\}$.*

(iii) *Let $S \in LR(X, Y)$ be a bounded operator and assume that T is closed. Then $T + S$ is closed.*

(iv) *Suppose that T is closed and let $S \in LR(Y,Z)$ be open with closed range and $dimN(S) < \infty$. Then ST is closed.*
(v) *If X and Y are Banach spaces and T is closed, then T is open if and only if $R(T)$ is closed if and only if $R(T')$ is closed. Further T is continuous if $D(T)$ is closed.*

We close this Section with the following Lemma concerning the ϕ_+ and ϕ_- linear relations.

Lemma 3.4. *Let X and Y be Banach spaces and let $T \in LR(X,Y)$ be closed. Then*

(i) *$T \in \phi_+(X,Y)$ if and only if $Q_T T \in \phi_+(X, Y/T(0))$. In such case $N(T) = N(Q_T T)$.*
(ii) *$T \in \phi_-(X,Y)$ if and only if $Q_T T \in \phi_-(X, Y/T(0))$. In such case $N(T) = N(Q_T T)$.*

Proof. Follows immediately from Lemmas 3.1 and 3.3. □

Lemma 3.5. *Let $T \in LR(X)$ where X is a Banach space and let $n \in \mathbb{N}$. We have*

(i) *If $T \in \phi_+(X)$, then $T^n \in \phi_+(X)$.*
(ii) *Let $T \in \phi_-(X)$ such that $\overline{D(T)} = X$ and $\rho(T) \neq \emptyset$, then T^n is densely defined, $(T^n)' = (T')^n$ and $T^n \in \phi_-(X)$.*

Proof. (i) Let $n \in \mathbb{N}$. We first show by induction that T^n is closed. For $n = 1$, it is clear. Suppose that T^n is closed. It follows from the conditions (iv) and (v) of Lemma 3.3 that T^{n+1} is closed.

On the other hand, by Lemma 2.1 (v) we infer that $dimN(T^n) \leq ndimN(T)$, so that $dimN(T^n) < \infty$. It remains to prove that $R(T^n)$ is closed. To do this, we proceed by induction. For $n = 1$, it is obvious. Assume that $R(T^n)$ is closed.

Define $T_1 := T\mid_{N(T)+R(T^n)}$.

Since T is closed with finite dimensional null space and $R(T^n)$ is closed we obtain that T_1 is a closed linear relation. Furthermore the linear relations T and T_1 have the same null space and hence $\gamma(T) \leq \gamma(T_1)$. In this situation, applying the conditions (ii) and (v) in Lemma 3.3 we deduce that $R(T_1)$ is closed. But $R(T_1) = R(T^{n+1})$. Indeed, we have that

$R(T_1) := T(N(T) + R(T^n)) = TT^{-1}(0) + TR(T^n) = T(0) + TR(T^n)$ ([7, Corollary I.2.10])$= R(T^{n+1})$.

(ii). See [2, Corollary 2.1] and [9, Proposition 3.1]. □

4. Preliminary Results

Throughout the rest of the paper, we denote by $\mathcal{K}(X)$ the set of compact operators on a Banach space X and $CR(X)$ the set of all closed linear relations in X.

4.1. Algebraic Properties for Linear Relations

This Subsection contains some algebraic properties in the context of linear relations in Banach spaces.

Lemma 4.1. *Let X be a Banach space, $A \in LR(X)$ and $T = A + K$ where $K \in \mathcal{K}(X)$. Assume that $KT \subset TK$. Then*
(i) $KA \subset AK$.
(ii) $TA \subset AT$.
(iii) $KT^n \subset T^n K$.
(iv) $K'T' \subset T'K'$.

Proof. (*i*) $KA = K(T - K) = KT - KK$ [7, Proposition I.4.2] $\subset TK - KK = (T - K)K$ [7, Proposition I.4.2] $= AK$.
(*ii*) $TA = (A + K)A \subset AA + KA$ [7, Proposition I.4.2] $\subset AA + AK \subset A(A + K)$ [7, Proposition I.4.2] $= AT$.
(*iii*) We prove by induction. For $n = 1$ is trivial. Assume the property to be valid for n. Then $KT^{n+1} = KT^n T \subset T^n KT$ (by the induction hypothesis and [7, Proposition I.4.2]) $\subset T^{n+1}K$.
(*iv*) Since $KT \subset TK$, then $-(KT)^{-1} \subset -(TK)^{-1}$. After that, using the definition of adjoint of a linear relation , we obtain $(TK)' \subset (KT)'$. Using [7, Theorem III.1.6 (a), (i)] we have $K'T' \subset (TK)' \subset (KT)'$. The use also of [7, Theorem III.1.6 (a), (ii)] leads to $K'T' \subset (TK)' \subset (KT)' = T'K'$ (since $D(K') = X'$ and $R(T) \subset D(K) = X$). □

Definition 4.1. *Let X be a Banach space and $T \in LR(X)$. We say that T commutes with itself if $TD(T) \subset D(T)$.*

Lemma 4.2. *Let X be a Banach space, $A \in LR(X)$ and $T = A + K$ where $K \in \mathcal{K}(X)$. Assume that A is onto and T commutes with itself. Then*
(i) $R(T^n) = R(T^n) \cap D(T)$.
(ii) $R(T^n) = AR(T^n)$.
(iii) $N(A) \subset R(T^n)$.

Proof. (*i*) It is easy to see that $R(T^n) \subset R(T) := T(D(T)) \subset D(T)$ (as T commutes with itself).
(*ii*) We first show that $R(T^n) \subset A(R(T^n))$. Since A is surjective then $I_{R(A)} = I_X \subset I_X + (AA^{-1} - AA^{-1}) = AA^{-1}$ [7, Proposition I.4.2 (a)] So $T = TI_X \subset TAA^{-1}$ which leads to $R(T) \subset TAR(A^{-1}) = TAD(A) = TAD(T) \subset ATD(T)$ (see Lemma 4.1 (ii)) $= AR(T)$. Now, assume that $R(T^n) \subset AR(T^n)$. Then $R(T^{n+1}) = TR(T^n) \subset TA(R(T^n)) \subset AT(R(T^n))$(see Lemma 4.1 (ii))$= AR(T^{n+1})$. Conversely, let $y \in Ax$, for some $x \in R(T^n) \cap D(T)$. Then $y \in Tx - Kx$ where $Tx \subset TR(T^n) = R(T^{n+1}) \subset R(T^n)$ and $Kx \in KR(T^n) = R(KT^n) \subset R(T^n K)$ (Lemma 4.1 (ii))$\subset R(T^n)$. Hence $y \in R(T^n)$.
(*iii*) Let $Z := R(T^n) = R(T^n) \cap D(T) = R(T^n) \cap D(A) \subset D(A)$. Then
$\dim R(A)/AZ \le \dim D(A)/Z$ [7, Proposition I.6.1] $:= \dim D(A)/R(T^n) \cap D(A) = \dim \frac{D(A)+R(T^n)}{R(T^n)}$ [14, Lemma 2.3] $\le \dim X/R(T^n) = \dim R(A)/R(T^n) = \dim R(A)/AZ$ (see (*ii*)). In consequence $\dim R(A)/AZ = \dim D(A)/Z$ with $Z \subset D(A)$ and in this situation [7, Exercise I.6.5] ensures that $N(A) \subset Z$.

Remark 4.1. *Let $T \in CR(X)$ and $K \in \mathcal{K}(X)$.*
(i) If T commutes with itself, then $T(R(T^m) \cap D(T)) = T(R(T^m)) = R(T^{m+1})$ for all $m \in \mathbb{N}$.
(ii) If $KT \subset TK$, then $K(R(T^m)) = R(KT^m) \subset R(T^m K) \subset R(T^m)$ for all $m \in \mathbb{N}$.

4.2. Further Properties of Ascent and Descent

It is well known for bounded semi-Fredholm operators that the ascent (resp. descent) of T coincides with the descent (resp. ascent) of T'. Recently, T. Alvarez, F. Fakhfakh and M. Mnif improved this result for closed lower semi-Fredholm linear relations. For more information we refer to [2]. Y, Chamkha and M. Mnif [5] give an analogue result for upper semi-Fredholm linear relations.

Let us recall some important results from [5].

Lemma 4.3. *[5, Lemma 9] Let X be a Banach space and $T \in LR(X)$. If T is bounded then $(T^n)' = (T')^n$ for all $n \in \mathbb{N}$.*

Proposition 4.1. *[5, Proposition 10] Let X be a Banach space and $T \in CR(X)$. Assume that T is bounded. If $T \in \Phi_+(X)$, then*
(i) $a(T') = d(T)$.
(ii) $d(T') = a(T)$.

We close this Subsection by the following result which will be used to obtain the main results of this paper.

Lemma 4.4. *Let X be a Banach space and $T \in LR(X)$ such that $N(T) \subset R(T^n)$ for all $n \in \mathbb{N}$. Then*

(i) *If $R_c(T) = \{0\}$, then $a(T) = 0$ or ∞.*
(ii) *If $\rho(T) \ne \emptyset$, then $d(T) = 0$ or ∞.*

Proof. (i) Suppose that $a(T) := p < \infty$. Then by virtue of Lemma 2.3 (ii) we have that $N(T) \cap R(T^p) = \{0\}$ and since $N(T) \subset R(T^p)$ we infer that $N(T) = \{0\}$, so that $a(T) = 0$.
(ii) Assume that $d(T) := q < \infty$ and let $x \in D(T^q)$. Then $T^q x \subset R(T^q) = R(T^{q+1})$ which implies that if $y \in T^q x$, then $(x, y) \in T^q$ and $(z, y) \in T^{q+1}$ for some $z \in D(T^{q+1}) \subset D(T)$. Hence, there is $w \in X$ such that $(x - w, 0) \in T^q$ and $(z, w) \in T$, which implies that $x - w \in N(T^q)$, so that by virtue of Lemma 2.2 we have that $x - w \in R(T)$ and since $w \in Tz \subset R(T)$ we conclude that $x \in R(T)$. Therefore $D(T^q) \subset R(T)$, so that $D(T^q) + R(T) = R(T)$ and now this fact together with Lemma 2.4 ensures that $R(T) = X$ equivalently $d(T) = 0$. □

5. Left Fredholm and Left Browder Linear Relations

Throughout this Section X will denote a Banach space and T will always denote a closed linear relation in X.

We begin this Section with the notion of a left Fredholm linear relation in a Banach space. For this, we recall the following standard definition.

Definition 5.1. *A closed subspace M of X is said to be topologically complemented in X if there exists a closed subspace N of X such that $X = M \oplus N$.*

Note that the finite dimensional subspaces and the closed finite codimensional subspaces of X are topologically complemented in X.

Definition 5.2. *We say that T is a left Fredholm linear relation in X, denoted by $T \in \phi_l(X)$, if T is a ϕ_+ linear relation and $R(T)$ is topologically complemented in X.*

Our next objective is to give a characterization of the class $\phi_l(X)$ in terms of an algebraic decomposition, the so-called Kato decomposition. In order to obtain this characterization, we need a bit of preparation.

Lemma 5.1. *[3, Lemma 2.2] Let X and Y be Banach spaces and let S be a bounded operator from X to Y. Assume that M is a closed subspace of X such that $S_{|M}$ is injective and SM is topologically complemented in Y and let N be a topological complement of SM. Then, $S^{-1}N$ is closed and a topological complement of M.*

Definition 5.3. *[8] A linear relation P in a vector space is said to be a multivalued linear projection if $P^2 = P$ and $R(P) \subset D(P)$.*

A multivalued linear projection can be characterized in terms of a pair of subspaces as follows:

Lemma 5.2. *[8, Proposition 1.1] Let M and N be subspaces of a vector space E. Define $P \in LR(E)$ by $G(P) = \{(m + n, m) : m \in M, n \in N\}$. Then P is a multivalued linear projection satisfying $D(P) = M + N$, $R(P) = M$, $N(P) = N$ and $P(0) = M \cap N$. Conversely, if P is a multivalued linear projection in E, then P determines a pair of subspaces M and N such that $G(P) = \{(m + n, m) : m \in M, n \in N\}$ with $D(P) = M + N$, $R(P) = M$, $N(P) = N$ and $P(0) = M \cap N$.*

The following result shows that the notion of topological complementation may be expressed in terms of multivalued projections under suitable restrictions.

Lemma 5.3. *[8, Proposition 3.13] Suppose M and N are subspaces of a normed space E and let P denote the projection with $D(P) = M + N$, $R(P) = M$, $N(P) = N$ and $P(0) = M \cap N$. If P is continuous and $M + N$ and $M \cap N$ are topologically complemented in E and $M + N$ respectively, then M and N are topologically complemented in E.*

Proposition 5.1. *Let $T \in \phi_l(X)$. Then there is $d \in \mathbb{N} \cup \{0\}$ such that*

  (i) *$N(T) \cap R(T^d) = N(T) \cap R(T^n)$, for all nonnegative integers $n \geq d$.*

 (ii) *$N(T) \cap R(T^d)$ is a finite dimensional subspace of X.*

(iii) *$N(T^d) + R(T)$ is topologically complemented in X.*

    Proof. We first note that

(5.1) For each $n \in \mathbb{N}$, $dim N(T^n) < \infty$ and $R(T^n)$ is closed.

Indeed, since $T \in \phi_+(X)$ the assertion (5.1) follows from Lemma 3.5 (i).

(i) By (5.1) the sequence $dim(N(T) \cap R(T^n))$ is a decreasing sequence and has therefore a limit. Hence, there exists some smallest $d \in \mathbb{N} \cup \{0\}$ for which $N(T) \cap R(T^d) = N(T) \cap R(T^n)$ for all $n \geq d$. Hence (i) holds.

(ii) It is obvious, since $T \in \phi_+$ and hence its null space is finite dimensional.

(iii) Since $R(T)$ is closed and $dim N(T^d) < \infty$, the subspace $N(T^d) + R(T)$ is closed. Furthermore we observe that by Lemma 3.5 (i) $dim N(T^d) < \infty$ which implies that $dim(N(T^d) + R(T))/R(T) = dim N(T^d)/(N(T^d) \cap R(T)) < \infty$ and hence there exists a finite dimensional subspace Z of X such that $N(T^d) + R(T) = R(T) \oplus Z$.

On the other hand, $R(T)$ is topologically complemented in X by hypothesis, so that, there is a closed subspace N such that $X = R(T) \oplus N$. Let I and q denote the isomorphism to $X/R(T)$ onto N and the quotient map from X onto $X/R(T)$. Define $S := Iq$. Then S is a bounded operator from X to N such that $N(S) = R(T)$, $R(S) = N$, $S_{|Z}$ injective, $SZ = S(N(T^d) + R(T))$ and $SZ \oplus R = N$ for some closed subspace R. Indeed, $N(S) = N(q) = R(T)$, $R(S) = N$, $N(S_{|Z}) = N(S) \cap Z = R(T) \cap Z = \{0\}$, $S(N(T^d) + R(T)) = S(R(T) \oplus Z) = SR(T) \oplus SZ = SN(S) \oplus SZ = SZ$ and since SZ is finite dimensional, then there exists a closed subspace R such that $SZ \oplus R = N$. From the reasoning above together with Lemma 5.1, we get $S^{-1}R$ is a topological complement of Z. Using the reasoning above with [7, Proposition I.3.1] , we deduce that

$$S^{-1}S(N(T^d) + R(T)) = ((N(T^d) + R(T)) \cap D(S)) + N(S) = N(T^d) + R(T)$$

and

$$N(T^d) + R(T) + S^{-1}R = R(T) \oplus Z + S^{-1}R = R(T) + X = X.$$

It remains to show that $(N(T^d) + R(T)) \cap S^{-1}R = R(T)$. Indeed, let $x \in (N(T^d) + R(T)) \cap S^{-1}R$. Then $x \in S^{-1}R \cap S^{-1}S(N(T^d) + R(T)) = S^{-1}R \cap S^{-1}SZ$. So that $Sx \in R \cap SZ = \{0\}$. Hence $Sx = 0$ that is, $x \in N(S) = R(T)$. Conversely, let $x \in R(T) = N(S)$, then it is clear that $x \in N(T^d) + R(T)$ and $Sx = 0 \in R$. So that $x \in (N(T^d) + R(T)) \cap S^{-1}R$. Now, by virtue of Lemma 5.2 we can consider the multivalued linear projection P with $D(P) = (N(T^d) + R(T)) + S^{-1}R, R(P) = N(T^d) + R(T), N(P) = S^{-1}R$ and $P(0) = (N(T^d) + R(T)) \cap S^{-1}R$. Further, since $N(T^d) + R(T)$ and $S^{-1}R$ are closed, it follows from [8, Theorem 3.4] that P is continuous and since $P(0) = R(T)$ is topologically complemented in X we infer from Lemma 5.3 that $N(T^d) + R(T)$ is topologically complemented in X. □

We are now in the position to state the first main result of this paper.

Theorem 5.1. *(The Kato decomposition of a left Fredholm linear relation) The following properties are equivalent:*

(i) $T \in \phi_l(X)$.

(ii) *There are two closed subspaces M and N of X such that*
    (a) $X = M \oplus N$ with $N \subset D(T)$ and $dim N < \infty$.
    (b) $T = T_M \oplus T_N$.
    (c) T_M is a regular left Fredholm linear relation in M.
    (d) T_N is a bounded nilpotent operator in N.

Proof. (i) $\Rightarrow$ (ii) Let $d \in \mathbb{N} \cup \{0\}$ as in Proposition 5.1. Since $T \in \phi_l(X)$, then it is easy to see by Proposition 5.1 that T is a quasi-Fredholm linear relation (see Definition 4.1.3 in [6]). Reasoning in the same way as in the proof of Theorem 4.1.1 in [6], we can construct two closed subspaces M and N of X satisfying the following conditions:

(1) $X = M \oplus N$ with $N \subset N(T^d)$.
(2) $T = T_M \oplus T_N$.
(3) T_M is a regular linear relation in M such that $N(T_M) = N(T) \cap R(T^d)$
and $R(T_M) \oplus N = R(T) + N(T^d)$.
(4) T_N is a bounded operator in N and it is nilpotent of degree d (that is, $T_N^d = 0$).
Hence, it only remains to show that $dim N < \infty$ and that $T_M \in \phi_l(M)$.

Since $T \in \phi_+(X)$ so is T^d by virtue of Lemma 3.5 (i), in particular $N(T^d)$ is finite dimensional. This last fact together with the inclusion $N \subset N(T^d)$ ensures that $dim N < \infty$. That $N(T_M)$ is finite dimensional is a direct consequence of the equality $N(T_M) = N(T) \cap R(T^d)$ combined with the condition (ii) in Proposition 5.1. Since T_M is regular, then $R(T_M)$ is closed.

On the other hand, it follows from the identity $R(T) + N(T^d) = R(T_M) \oplus N$ established in (3) and the property (iii) in Proposition 5.1 that there exists a closed subspace Z of X such that $R(T_M) \oplus Z = X$ and hence $R(T_M) \oplus (Z \cap M) = M$, that is, $R(T_M)$ is topologically complemented in M. Therefore $T_M \in \phi_l(M)$, as required.

(ii) $\Rightarrow$ (i) Assume that the properties (a)-(d) in (ii) are satisfied. It is clear that $R(T_M)$ is closed (as T_M is regular) and $N(T_N)$ and $R(T_N)$ are both finite dimensional subspaces (as $dim N < \infty$). Then we deduce from Lemma 2.5 (i) together with [14, Theorem 8.2] that $T \in \phi_+(X)$. On the other hand, there exist closed subspaces M_1 and N_1 of X such that $M_1 \subset M, N_1 \subset N, R(T_M) \oplus M_1 = M$ and $R(T_N) \oplus N_1 = N$. Furthermore

$M_1 + N_1$ is closed (since M_1 is closed and N_1 is finite dimensional), $M_1 \cap N_1 = \{0\}$ (since $M \cap N = \{0\}$) and $R(T) = R(T_M) \oplus R(T_N)$ (Lemma 2.5 (i)). Hence $R(T) \oplus (M_1 \oplus N_1) = X$. The proof is completed. $\square$

Definition 5.4. *We say that T is a left Browder linear relation in X, denoted by $T \in \mathcal{B}_l(X)$, if T is a left Fredholm linear relation with finite ascent.*

For bounded operators this notion was introduced in [17, page 2]. In [17, Theorem 5] the authors prove that the bounded left Browder operators are completely characterized in terms of an algebraic Kato decomposition. Next, this characterization is generalized to the case of linear relations.

Theorem 5.2. *(The Kato decomposition of a left Browder linear relation) Assume that $R_c(T) = \{0\}$. The following properties are equivalent:*

(i) *$T \in \mathcal{B}_l(X)$.*
(ii) *There are two closed subspaces M and N of X such that*
    (a) $X = M \oplus N$ with $N \subset D(T)$ and $\dim N < \infty$.
    (b) $T_M \oplus T_N$.
    (c) T_M is a regular injective linear relation in M with $R(T_M)$ topologically complemented in M.
    (d) T_N is a bounded nilpotent operator in N.

Proof. (i) $\Rightarrow$ (ii) Let M, N, T_M and T_N as in the part (ii) in Theorem 5.1. Then, it only remains to prove that T_M is injective. Observe that by Lemma 2.3 (i) we have that $R_c(T_M) = \{0\}$ and since T_M is regular it follows from Lemma 4.4 (i) that $a(T_M) = 0$ or ∞. The use of this last property combined with Lemma 2.5 (ii) allowed us to conclude that $a(T_M) = 0$, so that T_M is injective, as desired.

(ii) $\Rightarrow$ (i) Using the implication (ii) $\Rightarrow$ (i) in Theorem 5.1 we get $T \in \phi_l(X)$. Furthermore, since $a(T_M) = 0$ and $a(T_N) < \infty$ (as T_N is nilpotent), we infer from Lemma 2.5 that $a(T) = a(T_N) < \infty$. Therefore $T \in \mathcal{B}_l(X)$, as desired. $\square$

We close this Section by another decomposition of a left Browder linear relation T as an operator-like sum $T = A + B$ where A is an injective left Fredholm linear relation and B is a bounded finite rank operator with certain properties of commutativity. The converse result needs that T commutes with itself.

In order to obtain this decomposition we need a bit of preparation.

Proposition 5.2. *Suppose that T commutes with itself. Let $K \in \mathcal{K}(X)$ such that $KT \subset TK$. Assume that there exists $A \in LR(X)$ surjective such that $T = A + K$. Then $d(T) < \infty$.*

Proof. Since A is an onto closed linear relation, then by [7, Theorem III.4.2 (b) and Proposition II.3.2 (b)] there is a positive number γ for which $\|Ax\| \geq \gamma d(x, N(A)) \ \forall \ x \in D(A)$. Suppose that $x \in D(A) = D(T)$ and $z \in R(T^n)$. Therefore by Lemma 4.2 $(i), (ii)$ there is $y \in R(T^n) \cap D(A)$ such that $z \in Ay$ ($Ay = z + A(0) = z + \overline{A(0)}$). Thus we have

(5.2) $\|A(x - y)\| := \|Q_A A(x - y)\| = \|Q_T Ax - Q_T z\| \geq \gamma d(x - y, N(A)) \geq \gamma d(x, R(T^n))$,

since $N(A) \subset R(T^n)$ for all $n \in \mathbb{N}$ (see Lemma 4.2 (iii)). Since this holds for all $z \in R(T^n)$, we obtain

(5.3) $d(Q_T Ax, Q_T R(T^n)) \geq \gamma d(x, R(T^n)) \ \forall \ x \in D(A)$.

Suppose that T has infinite descent. Then there would be a bounded sequence $\{x_n\}$ with $x_n \in R(T^n)$ and $d(x_n, R(T^{n+1})) \geq 1$. Assume $m > n > 0$, then

$$\|Q_T K x_n - Q_T K x_m\| = \|Q_T K x_n - Q_T K x_m + Q_T A x_n - Q_T A x_n\|$$
$$= \|Q_T T x_n - Q_T A x_n - Q_T K x_m\|.$$

The use of Remark 4.1 and Equation (5.3) leads to

(5.4) $\|Q_T K x_n - Q_T K x_m\| \geq d(Q_T A x_n, Q_T R(T^{n+1})) \geq \gamma d(x_n, R(T^{n+1})) \geq \gamma$

which contradicts the compactness of the operator $Q_T K$.□

We are now ready to give the third main result of this paper.

Theorem 5.3. . (*i*) *Let $T \in \mathcal{B}_l(X)$ such that $R_c(T) = \{0\}$. Then there exist two linear relations A and B such that $T = A + B$, A is an injective left Fredholm linear relation and B is a bounded finite rank operator with $BT \subset TB$ and $R(B) \subset D(T)$.*
(*ii*) *If we suppose that T is everywhere defined, T' commutes with itself and there exist two linear relations A and B such that $T = A + B$, A is an injective left Fredholm linear relation and B is a bounded compact operator with $BT \subset TB$, then $T \in B_l(X)$.*

Proof. (*i*) Let M, N, T_M and T_N as in Therorem 5.2 (ii). Define the linear relations A and B in X by

$$A := \{(x, P_M y - P_N x) : (x, y) \in T\}$$

and

$$B := T_N P_N + P_N$$

where P_M and P_N designate the bounded projections of X onto M along N and onto N along M, respectively. According to the definitions of A and B we have that $T = A + B$ and that B is a bounded operator with $R(B) \subset N \subset D(T)$, so that B is finite rank and $R(B) \subset D(T)$. These properties together with Lemma 3.3 (iii) ensure that A is a closed linear relation and that $D(A) = D(T)$.

On the other hand, A is injective and its range is topologically complemented in X. Indeed, let $x \in N(A)$, so that there exists an element $y \in X$ such that $(x, y) \in T$ and $P_M y - P_N x = 0$. Since $P_M y \in M$ and $P_N x \in N$ with $M \cap N = \emptyset$, it follows that $P_M y = P_N x = 0$. Furthermore, since $(x, y) \in T$ we have that $(P_M x, P_M y) \in T_M$ and $(P_N x, P_N y) \in T_N$. Hence $x = P_M x \in N(T_M) = \{0\}$, so that A is injective. The use of the identity $A = T - B$ combined with [1, Theorem 11] allowed us to conclude that $R(A)$ is topologically complemented in X.

In order to complete the proof we will check that $BT \subset TB$. First , we show that $BA \subset AB$. Indeed, let $(x, y) \in BA$. Then there is $z \in X$ for which $(x, z) \in A$ and $(z, y) \in B$. So that $(x, z) = (x, P_M u - P_N x)$ for some $(x, u) \in T$ and $y = Bz$. Consequently
$y = Bz := (T_N P_N + P_N)(P_M u - P_N x) = -T_N P_N x - P_N x := -Bx$.
Define $w := Bx$.
It is clear that $w \in N$, so that $w = P_N w$ and $P_M w = P_M T_N w = 0$. Hence $(w, -w) = (w, P_M T_N w - P_N w)$ and $(P_N w, T_N P_N w) \in T_N \subset T$ and thus it follows from the definition of A that $(w, -w) \in A$. This last property combined with the equalities $w = -y = Bx$ leads to $(x, y) \in AB$. On the other hand, $BT = B(A + B) = BA + B.B$ [7, Proposition I.4.2 (e)] $\subset A.B + B.B = (A + B)B$ [7, Proposition I.4.2 (d)]= TB.
(*ii*) Since $A \in \Phi_l(X)$ and $B \in \mathcal{K}(X)$, then by [1, Theorem 11] we obtain that $T \in \Phi_l(X)$. Hence, it only remains to see that $a(T) < \infty$. By Proposition 4.1, we have $a(T) = d(T')$. Hence, we shall prove that $d(T') < \infty$. For this we shall apply Proposition 5.2. We note that $T' = (A + B)' = A' + B'$(see [7, Proposition III.1.5 (b)] . Since A is a closed linear relation with closed range, then A is open (see Lemma 3.3 (v)). Therefore, by [7, Proposition III.4.6 (b)] we infer $R(A') = N(A)^\perp = \{0\}^\perp$. So A' is surjective. This together with the fact that T' commutes with itself and $B'T' \subset T'B'$ (see Lemma 4.1 (iv)) show $d(T') < \infty$. □

As a consequence, we infer the characterization of left Browder spectrum under finite rank operators with certain properties of commutativity.

Definition 5.5. (*i*) *The left Browder spectrum of $T \in CR(X)$ is the set $\sigma_b^{left}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_l(X)\}$.*
(*ii*) *The left spectrum of $T \in CR(X)$ is the set $\sigma_l(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}_l(X)\}$ where $T \in \mathcal{G}_l(X)$ if and only if T is injective and $R(T)$ is a topologically complemented subspace of X.*

Theorem 5.4. *Let* $G = \{K \in \mathcal{F}_0(X)$ *such that* $KT \subset TK\}$ *where* $\mathcal{F}_0(X)$ *designates the set of bounded finite rank operators.*
(i) If $R_c(T) = \{0\}$, *then*

$$\bigcap_{K \in G} \sigma_l(T + K) \subset \sigma_b^{left}(T).$$

(ii) If T *is everywhere defined and* T' *commutes with itself, then*

$$\sigma_b^{left}(T) \subset \bigcap_{K \in G} \sigma_l(T + K).$$

Proof. *(i)* If $\lambda \notin \sigma_b^{left}(T)$ then $\lambda - T \in \mathcal{B}_l(X)$. From Theorem 5.3 *(i)* (as $R_c(\lambda - T) = \{0\}$ (see [14, Lemma 7.1]), we infer that there exist two linear relations A and B such that $\lambda - T = A + B$, A is an injective left Fredholm linear relation and B is a bounded finite rank operator satisfying $B(T - \lambda) \subset (T - \lambda)B$ and $R(B) \subset D(T)$. Take $K = B \in \mathcal{F}_0(X)$. Since $B(T - \lambda) \subset (T - \lambda)B$ then it is easy to see that $BT \subset TB$. Moreover, $\lambda - T - B = A$ is injective with range space topologically complemented in X. So that $\lambda \notin \bigcap_{K \in G} \sigma_l(T + K)$.

(ii) If $\lambda \notin \bigcap \{\sigma_l(T + K)$ with $K \in G\}$ thus $\exists\ K \in G$ such that $\lambda - T - K$ is injective and $R(\lambda - T - K)$ is topologically complemented in X. So $\lambda - T - K \in \Phi_l(X)$. On the other hand, it easy to see that $K(T - \lambda) \subset (T - \lambda)K$. Hence, applying Theorem 5.3 *(ii)*, we get $\lambda - T = (\lambda - T - K) + K \in \mathcal{B}_l(X)$. So $\lambda \notin \sigma_b^{left}(T)$.☐

6. Right Fredholm and Right Browder Linear Relations

In the sequel we assume that T is a densely defined closed linear relation in a complex Banach space X with $\rho(T) \neq \emptyset$.

Definition 6.1. *We say that* T *is a right Fredholm linear relation in* X, *denoted by* $T \in \phi_r(X)$, *if* $T \in \phi_-(X)$ *and* $N(T)$ *is topologically complemented in* X.

Note that an everywhere defined continuous linear relation in a Banach space may have a spectrum that coincides with the whole complex plane (see [7, Example VI.2.6] ; in contrast, in the case of operators, the spectrum of a bounded operator S in a Banach space is a compact subset contained in a compact disc of radius $\| S \|$. This remark combined with Lemma 3.3 (v) ensures that our definition of right Fredholm linear relation coincides with the definition for bounded operators.

In [17, Theorem 6], the authors give characterizations of bounded right Fredholm and right Browder operators in terms of a Kato decomposition of T. Our objective in this Section is to extend the above charaterizations to the case of multivalued linear operators. To do this, we first prove some auxiliary results.

Proposition 6.1. *Let* $T \in \phi_r(X)$. *Then there is a nonnegative integer* d *satisfying the following properties:*

  (i) $N(T) \cap R(T^d) = N(T) \cap R(T^n)$ *for all nonnegative integers* $n \geq d$.
 (ii) $N(T) \cap R(T^d)$ *is topologically complemented in* X.
(iii) $N(T^d) + R(T)$ *is a closed finite codimensional subspace of* X.

Proof. We first note that
(6.1) For each $n, m \in \mathbb{N}$, $N(T^n) \cap R(T^m)$ and $N(T^n) + R(T^m)$ are closed.
Indeed, from Lemma 3.5 (ii) we have that T^n and T^m belong to $\phi_-(X)$, in particular T^n is closed, so that $N(T^n)$ is closed and $R(T^m)$ is a closed finite codimensional subspace contained in $N(T^n) + R(T^m)$ which implies that $N(T^n) + R(T^m)$ is closed. Hence (6.1) holds.
(6.2) For each $n \in \mathbb{N}$, $N(T^n) + R(T) = (N(T') \cap R((T')^n))^\top$.
In fact, we note that $N(T^n), R(T)$ and $N(T^n) + R(T)$ are closed by Lemma 3.5 (ii) and (6.1) so that we have that

$N(T^n) + R(T) = (N(T^n)^\perp \cap R(T)^\perp)^\top$ (Lemma 3.2) $= (R((T^n)') \cap N(T'))^\top$ (Lemma 3.3 (ii))$= (R((T')^n) \cap N(T'))^\top$ (Lemma 3.5 (ii)). Hence (6.2) holds.

(6.3) There exists a nonnegative integer d for which $N(T^d) + R(T) = N(T^n) + R(T)$ for all nonnegative integers $n \geq d$.

Indeed, since $dimN(T') = dimR(T)^\perp < \infty$ we have that $dim(N(T') \cap R((T')^n))$ is a decreasing sequence and has therefore a limit. Hence there exists some smallest $d \in \mathbb{N} \cup \{0\}$ for which $N(T') \cap R((T')^d) = N(T') \cap R((T')^n)$ for all $n \geq d$. This last equality together with (6.2) allow us to conclude that (6.3) is true.

Let $d \in \mathbb{N} \cup \{0\}$ as in (6.3).

(i) Applying the statement (6.3) and the condition (iv) in Lemma 2.1 with $n = d$ and $m = 1$ we get $N(T) \cap R(T^d) = N(T) \cap R(T^{d+1})$. A repeated application of (6.3) and Lemma 2.1 (iv) leads to the assertion (i).

(ii) Since $N(T) + R(T^d)$ is closed by (6.1) and $R(T^d)$ is closed we deduce from Lemma 3.1 (i) that $(N(T) + R(T^d))/R(T^d)$ is a closed subspace of $X/R(T^d)$. This fact combined with $dimX/R(T^d) < \infty$ (as $T^d \in \phi_-(X)$ by Lemma 3.5 (ii)) yields to $dim(N(T) + R(T^d))/R(T^d) < \infty$ equivalently $dimN(T)/(N(T) \cap R(T^d)) < \infty$, so that there exists a finite dimensional subspace M of X such that $N(T) = (N(T) \cap R(T^d)) \oplus M$ and since $N(T)$ is topologically complemented in X, it follows that $N(T) \cap R(T^d)$ is topologically complemented in X. Hence (ii) holds.

(iii) By (6.1) $N(T^d) + R(T)$ is a closed subspace and since T is a ϕ_- linear relation we infer from Lemma 3.1 (ii) that $dimX/(N(T^d) + R(T)) < \infty$. Therefore (iii) holds. □

We now are in the position to express the fourth main result of this paper.

Theorem 6.1. *(The Kato decomposition of a right Fredholm linear relation) The following properties are equivalent:*

(i) *$T \in \phi_r(X)$.*
(ii) *There are two closed subspaces M and N of X such that*
   (a) $X = M \oplus N$ with $N \subset D(T)$ and $dimN < \infty$.
   (b) $T = T_M \oplus T_N$.
   (c) T_M is a regular right Fredholm linear relation in M.
   (d) T_N is a bounded nilpotent operator in N.

Proof. (i) $\Rightarrow$ (ii) Let $d \in \mathbb{N} \cup \{0\}$ as in Proposition 6.1. Since $T \in \phi_r(X)$, then it is easy to see by Proposition 6.1 that T is a quasi-Fredholm linear relation (see Definition 4.1.3 in [6]). Reasoning in the same way as in the proof of Theorem 4.1.1 in [6], we can construct two closed subspaces M and N of X satisfying the following conditions:

(1) $X = M \oplus N$ with $N \subset N(T^d)$ and $N(T^d) + R(T^d) = N \oplus N(T^d)$.
(2) $T = T_M \oplus T_N$.
(3) T_M is a regular linear relation in M such that $N(T_M) = N(T) \cap R(T^d)$ and $R(T_M) \oplus N = R(T) + N(T^d)$:
(4) T_N is a bounded operator in N and it is nilpotent of degree d.
Hence, it only remains to show that $dimN < \infty$ and that $T_M \in \phi_r(M)$.

Arguing as in the part (ii) in Proposition 6.1, we deduce that $dim(N(T^d) + R(T^d))/R(T^d) < \infty$. This fact together with the equality $N(T^d) + R(T^d) = N \oplus N(T^d)$ established in (1) allow us to conclude that $dimN < \infty$.

On the other hand, that $N(T_M)$ is topologically complemented in M follows immediately from the identity $N(T_M) = N(T) \cap R(T^d)$ combined with the assertion (ii) in Proposition 6.1.

Finally, we deduce that $T_M \in \phi_-(M)$ by noting the following facts:
$dimX/(R(T) + N(T^d)) < \infty$ (Proposition 6.1 (iii)); $dimN < \infty$ and $(X/R(T_M))/(R(T_M) \oplus N/R(T_M)) = (X/R(T_M)/(R(T) + N(T^d)/R(T_M))$ $((3)) = X/(R(T) + N(T^d))$ (Lemma 3.1 (ii)).

(ii) $\Rightarrow$ (i) Assume that the properties (a) - (d) in (ii) are satisfied. From Lemma 2.5 (i) we have $R(T) = R(T_M) \oplus R(T_N)$. So $R(T)$ is closed (as $R(T_M)$ is closed and $dim R(T_N) < \infty$). Using the fact that $T_M \subset T$, we obtain $\beta(T) < \beta(T_M) < \infty$. So $T \in \phi_-(X)$. On the other hand, there exists closed subspaces M_1 and N_1 of X such that $M_1 \subset M, N_1 \subset N, N(T_M) \oplus M_1 = M$ and $N(T_N) \oplus N_1 = N$ and hence $N(T) \oplus (M_1 \oplus N_1) = X$, where $M_1 \oplus N_1$ is a closed subspace of X (as M_1 is closed, $dimN_1 < \infty$ and $M \cap N = \{0\}$), so that $N(T)$ is topologically complemented in X. Therefore $T \in \phi_r(X)$, as required. □

Definition 6.2. *We say that T is a right Browder linear relation in X, denoted by $T \in \mathcal{B}_r(X)$, if T is a right Fredholm linear relation with finite descent.*

Next we characterize the class $\mathcal{B}_r(X)$ in terms of a Kato-type decomposition.

Theorem 6.2. *(The Kato decomposition of a right Browder linear relation) The following properties are equivalent:*

(i) $T \in \mathcal{B}_r(X)$.

(ii) *There are two closed subspaces M and N of X such that*
   (a) $X = M \oplus N$ *with* $N \subset D(T)$ *and* $dimN < \infty$.
   (b) $T = T_M \oplus T_N$.
   (c) T_M *is a regular surjective and right Fredholm linear relation in M.*
   (d) T_N *is a bounded nilpotent operator in N.*

Proof. (i) $\Rightarrow$ (ii) Assume that $T \in \mathcal{B}_r(X)$, in particular $T \in \phi_r(X)$, so that by virtue of Theorem 6.1, there exist two closed subspaces M and N of X verifying the conditions of the part (ii) in Theorem 6.1. Hence, it only remains to prove that $R(T_M) = M$. Note that by virtue of Lemma 2.5 (i) and (iii) we have $\rho(T_M)$ is a nonempty subset and $d(T_M) < \infty$. These properties together with the regularity of T_M allow us to apply Lemma 4.4 (ii) and thus we can deduce that $d(T_M) = 0$, that is $R(T_M) = M$, as desired.

(ii) $\Rightarrow$ (i) Using the implication (ii) $\Rightarrow$ (i) in Theorem 6.1 we get $T \in \phi_r(X)$. Furthermore, since $d(T_M) = 0$ and $d(T_N) < \infty$ (as T_N is nilpotent) we infer from Lemma 2.5 (iii) that T has finite descent. Therefore $T \in \mathcal{B}_r(X)$.

The proof is completed. □

For bounded operators the above Theorem 6.2 was obtained in [17, Theorem 6].

As an application of Theorem 6.2 we deduce another decomposition of a right Browder linear relation T via a special operator-sum decomposition of $T : T = A + B$ where A is a linear relation whose range is the whole space and B is a bounded finite rank operator with certain properties of commutativity. The converse result remains also valid under the condition of commutativity.

Theorem 6.3. (*i*) *Let* $T \in \mathcal{B}_r(X)$. *Then there exist two linear relations A and B in X such that* $T = A + B$, *A is a closed surjective linear relation with null space topologically complemented in X and B is a bounded finite rank operator satisfying* $BT \subset TB$ *and* $R(B) \subset D(T)$.
(*ii*) *If we suppose that T commutes with itself and there exist two linear relations A and B such that A is surjective right Fredholm linear relation and B is a bounded compact operator verifying* $T = A + B$ *and* $BT \subset TB$. *Then* $T \in \mathcal{B}_r(X)$.

Proof. (*i*) Let M, N, T_M and T_N as in the part (ii) in Theorem 6.2. Define the linear relations A and B in X by

$$A := \{(x, P_M y - P_N x) : (x, y) \in T\}$$

and

$$B := T_N P_N + P_N$$

where P_M and P_N designate the bounded projections of X onto M along N and onto N along M, respectively. Then

(6.4) $T = A + B$.

Follows immediately from the definitions of A and B.

(6.5) B is a bounded finite rank operator with $R(B) \subset D(T)$.

Clearly B is a bounded operator with $R(B) \subset N \subset D(T)$, so that $dimR(B) \leq dimN < \infty$.

(6.6) A is closed and $D(A) = D(T)$.

Note that by (6.4) we get $A = T - B$ where T is closed and B is a bounded operator, so that it follows from Lemma 3.3 (iii) that A is closed. Further $D(A) = D(T) \cap D(B) = D(T)$ (by (6.5)).

(6.7) A is surjective.

Let $x \in X$. Then $x = m + n$ for some $m \in M$ and $n \in N$ and $(n, T_N n) \in T_N \subset T$ which implies that $(n, -n) = (n, P_M T_N n - P_N n) \in A$. Further, since $m \in M = R(T_M)$, there is $y \in M$ such that $(y, m) \in T_M \subset T$. Hence $(y, P_M m - P_N y) = (y, m) \in A$. Consequently $(y - n, m + n) = (y - n, x) \in A$, so that $x \in R(A)$. Hence (6.7) holds.

(6.8) $N(A)$ is topologically complemented in X.

Indeed, since A is closed by (6.6) we deduce from Lemma 3.3 (i) that $N(A)$ is closed. In order to show that $N(A)$ is topologically complemented in X it is enough to prove that $N(A) = N(T_M)$. To see this, assume that $N(A) = N(T_M)$, then it follows from Theorem 6.2 (ii) that $N(T_M) \oplus M_1 = M$ for some closed subspace M_1 contained in M and since $X = M \oplus N$ with N finite dimensional we conclude that $N(T_M) \oplus (M_1 \oplus N) = X$. Accordingly, we shall verify that $N(A) = N(T_M)$. Let $x \in N(A)$, that is, $(x, 0) \in A$ so that by the definition of A we have that $(x, 0) = (x, P_M y - P_N x)$ for some $(x, y) \in T$. Hence $P_M y = P_N x$ and since $(P_M x, P_M y) \in T_M$ with $P_M y \in M, P_N x \in N$ and $M \cap N = \{0\}$ we deduce that $P_M y = P_N x = 0$. Consequently $x = P_M x$ and also $(x, 0) = (P_M x, P_M y) \in T_M$ which implies that $x \in N(T_M)$. Therefore $N(A) \subset N(T_M)$.

Conversely, let $x \in N(T_M)$, then $x = P_M x \in M$ and $(x, 0) = (x, P_M 0 - P_N x) \in A$ and thus $x \in N(A)$. Hence $N(T_M) \subset N(A)$.

(6.9) $BT \subset TB$.

This property is obtained proceeding exactly as in Theorem 5.3.

(*ii*) Since $A \in \Phi_r(X)$ and $B \in \mathcal{K}(X)$, then by [1, Theorem 11] we obtain that $T \in \Phi_r(X)$. The use of Proposition 5.2 leads to $\in \mathcal{B}_r(X)$.

Now the proof of Theorem 6.3 is completed. □

As a consequence, we infer the characterization of right Browder spectrum under finite rank operators with certain properties of commutativity.

Definition 6.3. (*i*) *The right Browder spectrum of $T \in CR(X)$ is the set* $\sigma_b^{right}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_r(X)\}$.
(*ii*) *The right spectrum of $T \in CR(X)$ is the set* $\sigma_r(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}_r(X)\}$ *where $T \in \mathcal{G}_r(X)$ if and only if T is onto and $N(T)$ is a topologically complemented subspace of X.*

Theorem 6.4. *We have*
(*i*)
$$\bigcap_{K \in G} \sigma_r(T + K) \subset \sigma_b^{right}(T).$$
(*ii*) *If T commutes with itself, then* $\sigma_b^{right}(T) \subset \bigcap_{K \in G} \sigma_r(T + K).$

Proof. (*i*) If $\lambda \notin \sigma_b^{right}(T)$ then $\lambda - T \in \mathcal{B}_r(X)$. From Theorem 6.3 (*i*) (as $\rho(\lambda - T) \neq \emptyset$), we infer that there exist two linear relations A and B such that $\lambda - T = A + B$, A is a closed surjective linear relation with null space topologically complemented in X and B is a bounded finite rank operator satisfying $B(T - \lambda) \subset (T - \lambda)B$ and $R(B) \subset D(T)$. Take $K = B \in \mathcal{F}_0(X)$. Since $B(T - \lambda) \subset (T - \lambda)B$ then it is easy to see that $KT \subset TK$. Moreover, $\lambda - T - B = A$ is surjective with null space topologically complemented in X. So that $\lambda \notin \bigcap_{K \in G} \sigma_r(T + K)$.

(*ii*) If $\lambda \notin \bigcap \{\sigma_r(T + K) \text{ with } K \in G\}$ thus $\exists K \in G$ such that $\lambda - T - K$ is surjective and $N(\lambda - T - K)$ is topologically complemented in X. So $\lambda - T - K \in \Phi_r(X)$. On the other hand, it easy to see that $K(T - \lambda) \subset (T - \lambda)K$. Hence, applying Theorem 6.3 (*ii*), we get $\lambda - T = (\lambda - T - K) + K \in \mathcal{B}_r(X)$. So $\lambda \notin \sigma_b^{right}(T)$. □

7. Stability of Right Browder Linear Relations

Throughout this section, we investigate the stability of right Browder linear relations under compact operator perturbations. The analysis is essentially based on the results developed in [2].
Let us recall some definitions and results from [2].

Definition 7.1. *[2, Definition 2.1] Let X be a Banach space and let $K, T \in LR(X)$. We say that K commutes with T if*
(*i*) $D(T) \subset D(K)$,
(*ii*) $Kx \subset D(T)$ *whenever $x \in D(T)$,*
(*iii*) $KTx = TKx$ *for $x \in D(T^2)$.*

We denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X.

Lemma 7.1. *[2, Lemma 2.1] Let X be a Banach space, $T \in LR(X)$ and $K \in \mathcal{L}(X)$. If K commutes with T, then K commutes with $T + K$.*

Remark 7.1. *[2, Remark 2.1] Let X be a Banach space, $T \in LR(X)$ and $K \in \mathcal{L}(X)$. If K commutes with T, then K commutes with $T - \lambda$ for all $\lambda \in \mathbb{C}$. Indeed, from [7, Proposition I.4.2, (e), (d)] we infer*

$$K(T - \lambda)x = KTx - \lambda Kx = TKx - \lambda Kx = (T - \lambda)Kx$$

for all $x \in D(T^2) = D((T - \lambda)^2)$.

Proposition 7.1. *[2, Proposition 2.1] Let X be a Banach space, $T \in CR(X)$ and $K \in \mathcal{L}(X)$. Assume that $\overline{D(T)} = X$, $\rho(T) \neq \emptyset$ and $T(0) \subset D(T)$. If K commutes with T, then $KTx = TKx \ \forall x \in D(T)$.*

Proposition 7.2. *[2, Proposition 2.2] Let X be a Banach space and $T \in CR(X)$ such that $T(0) \subset D(T)$, $\rho(T) \neq \emptyset$ and $\overline{D(T)} = X$. Let $K \in \mathcal{L}(X)$ such that K commutes with T. Then*
(i) $K' : D(T') \to D(T')$.
(ii) $K'T'f = T'K'f \ \forall \ f \in D(T')$.
(iii) K' commutes with T'.

Theorem 7.1. *[2, Theorem 2.1] Let X be a Banach space and $T \in CR(X)$ such that $\rho(T) \neq \emptyset$ and $\overline{D(T)} = X$. If $T \in \Phi_-(X)$, then*
(i) $a(T') = d(T)$.
(ii) $d(T') = a(T)$.

Remark 7.2. *Let X be a Banach space and $T \in LR(X)$.*
If $\overline{D(T)} = X$, then it follows from [7, Proposition III.1.4 (b)] , that $T'(0) = D(T)^{\perp} = \overline{D(T)}^{\perp} = \{0\}$ which proves that T' is an operator.

Theorem 7.2. *Let X be a Banach space, $T \in CR(X)$ and $K \in \mathcal{K}(X)$. Assume that*
(i) $\rho(T) \neq \emptyset$ and $\rho(T + K) \neq \emptyset$.
(ii) T is bounded and K commutes with T.
Then
$$T \in \mathcal{B}_r(X) \text{ if and only if } T + K \in \mathcal{B}_r(X).$$

Proof. We first claim that if $T \in \mathcal{B}_r(X)$, then $T + K \in \mathcal{B}_r(X)$. Indeed, let $T \in \Phi_r(X)$ such that $d(T) := p < \infty$. By [1, Theorem 11] we deduce $T + K \in \Phi_r(X)$. It remains to show $d(T + K) < \infty$. Using Theorem 7.1 we infer $a(T') = d(T) = p < \infty$. Since T' is a closed linear relation satisfying $\alpha(T') = \beta(T) < \infty$ and $R(T')$ is a closed subspace (see [7, Propositions II.3.2 (b) and III.1.4 (a) and Theorems III.4.2 (b) and III.5.3 (a)]), then it follows from [7, Proposition II.5.17] that $T'^p \in CR(X') \ \forall \ p \in \mathbb{N}$. So that $((T')^p)^{-1}$ is also closed (see [7, Definitions II.5.1 (2)]). This together with [7, Proposition II.5.3] shows that $((T')^p)^{-1}(0) := N((T')^p)$ is a closed subspace and since $a(T') = d(T) < \infty$ (by Theorem 7.1) we have $N^{\infty}(T') = \overline{N^{\infty}(T')}$. Now, applying [9, Proposition 2.2 (i)] , we get

$$N^{\infty}(T') \cap R^{\infty}(T') = \overline{N^{\infty}(T')} \cap R^{\infty}(T') = \{0\}.$$

Since $\lambda K \in \mathcal{K}(X)$ then it follows by [1, Theorem 11] that $T + \lambda K \in \Phi_-(X)$. Set $A = T_\lambda' = (T + \lambda K)' = T' + \lambda K'$ (see [7, Proposition III.1.5 (b)]) and $B = (\mu - \lambda)K'$ where $\lambda, \mu \in [0, 1]$. It is clear from Remark 7.2 that A is a closed operator satisfying $R(A)$ is a closed subspace and $\alpha(A) = \beta(T_\lambda) < \infty$. The use of Proposition 7.2 makes us to conclude that $D(A) = D(T_\lambda') = D(T') \subset D(AB) = \{x \in D(K') : K'x \in D(T')\}$ and $BAx = (\mu - \lambda)K'T_\lambda'x = (\mu - \lambda)K'(T' + \lambda K')x = (\mu - \lambda)K'T'x + \lambda(\mu - \lambda)K'^2x = (\mu - \lambda)T'K'x + \lambda(\mu - \lambda)K'^2x = (\mu - \lambda)T_\lambda'K'x = ABx$ for $x \in D(T')$. Thus by [11, Theorem 3] , we deduce that there exists $\varepsilon = \varepsilon(\lambda)$ such that

$$(7.1) \quad \overline{N^{\infty}(T_\lambda')} \cap R^{\infty}(T_\lambda') = \overline{N^{\infty}(T_\mu')} \cap R^{\infty}(T_\mu')$$

for all μ in the open disc $S(\lambda)$ with center λ and radius ε.

Equation (7.1) proves that $N^\infty(T_\lambda{}') \cap R^\infty(T_\lambda{}')$ is a locally constant function of λ on the interval $[0,1]$. Since every locally constant function on a connected set like $[0,1]$ is constant, it follows that $\overline{N^\infty((T+K)')} \cap R^\infty((T+K)') = \{0\}$ and hence $N^\infty(T'+K') \cap R^\infty(T'+K') = \{0\}$. Since $\alpha(T'+K') = \beta(T+K) < \infty$, the use, again, of [9, Proposition 2.2 (ii)] proves $a((T+K)') = a(T'+K') < \infty$ which implies by Theorem 7.1 that $d(T+K) < \infty$. Conversely, let $T+K \in \mathcal{B}_r(X)$. [1, Theorem 11] gives $T \in \Phi_r(X)$ and now, by Theorem 7.1 it is enough to prove that $a(T') < \infty$. To do this, we consider

$$(T+K)'_\lambda = T' + K' + \lambda K' = T' + (\lambda+1)K',$$

where $\lambda \in [-1,0]$ and we reasone in the same way as above. $\square$

As a consequence, we infer the invariance of right Browder spectrum under compact operator perturbations

Theorem 7.3. *Let X be a Banach space, $T \in CR(X)$ and $K \in \mathcal{K}(X)$. Assume that*
(i) $\rho(T) \neq \emptyset$ and $\rho(T+K) \neq \emptyset$.
(ii) T is bounded and K commutes with T.
Then
$$\sigma_b^{right}(T) = \sigma_b^{right}(T+K).$$

Proof. We first claim that $\sigma_b^{right}(T+K) \subset \sigma_b^{right}(T)$. Indeed, if $\lambda \notin \sigma_b^{right}(T)$ then $T - \lambda \in \mathcal{B}_r(X)$. From [7, Theorem VI.5.4], we infer $\rho(T-\lambda) \neq \emptyset$ and $\rho(T+K-\lambda) \neq \emptyset$. On the other hand, by Remark 7.1 we have K commutes with $T - \lambda$. This implies by Theorem 7.2 that $T + K - \lambda \in \mathcal{B}_r(X)$. So $\lambda \notin \sigma_b^{right}(T+K)$. Similarly, we show $\sigma_b^{right}(T) \subset \sigma_b^{right}(T+K)$. $\square$

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