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Probability that an Autocommutator Element of a Finite Group Equals to a Fixed Element

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Abstract. In this article we introduce a formula for the probability which an autocommutator element of a finite group G, equals to a fixed element g of G and derive some properties of this formula. Moreover, we obtain a lower bound and an upper bound for this probability in the special cases. This generalizes some results of Das et al. in 2010 and Moghaddam et al. in 2011.

1. Introduction

The concept of commutativity degree started by Gustafson in 1975, which is probability that two elements of a finite group G commute. In this article G denotes a finite group, H a subgroup of G, and g an element of G. In [6], Moghaddam et al. have considered the probability $Paut(H, G)$ for an element randomly chosen of H, which is fixed by an automorphism randomly chosen of $Aut(G)$. On the other hand, in [2], Das and Nath have studied the probability

$$Pr_g(H, G) = \frac{|\{(x, y) \in H \times G : [x, y] = g\}|}{|H||G|},$$

where $[x, y] = x^{-1}y^{-1}xy$. We will study the ratio

$$P_g aut(G) = \frac{|\{(x, \alpha) \in G \times Aut(G) : [x, \alpha] = g\}|}{|G||Aut(G)|},$$

where $[x, \alpha] = x^{-1}x^{\alpha}$, is called *autocommutator element* of G (see also [7]). Moreover, we extend some of the results obtained in [2]. We also develop and characterize a formula for probability of the pair (H, G), which generalizes the formula for $Pr_g(H, G)$ given in [2].
Note that if $H = G$ then $Paut(H, G) = P_g aut(G)$, which coincides with autocommutativty degree $Paut(G)$ of G, if we take $g = 1$, the identity element of G. We note that this case is treated in [7]. It may be recalled that $Paut(G) = \dfrac{k}{|G|}$ where k denotes the number of disjoint orbits of G under the group automorphism $Aut(G)$

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(see also [3–5] for commutativity degree and generalization). Among the other results, we particularly determine an upper bound for relative autocommutativity degree of a finite group.
Let G be a group and $Aut(G)$ be the full automorphisms group of G, for all $\alpha \in Aut(G)$ and $g \in G$, we have the following map

$$\begin{aligned} Aut(G) \times G &\longrightarrow G \\ (\alpha, g) &\longmapsto g^{\alpha}. \end{aligned}$$

For an element $g \in G$, the set of automorphisms $\{\alpha \in Aut(G) \mid g^{\alpha} = g\}$ denoted by $C_{Aut(G)}(g)$ is a subgroup of $Aut(G)$ and the equivalence classes $\{g^{\alpha} \mid \alpha \in Aut(G)\}$ denoted by $Orb_{Aut(G)}(g)$.
Let H be a subgroup of G, we introduce two subgroups of $Aut(G)$ and H as follows, respectively,

$$C_{Aut(G)}(H) = \{\alpha \in Aut(G) \mid h^{\alpha} = h,\ \forall\, h \in H\},$$

$$C_H(Aut(G)) = \{h \in H \mid h^{\alpha} = h,\ \forall\, \alpha \in Aut(G)\}.$$

In this notation we introduce a subgroup of G as follows,

$$L(G) = \{ g \in G \mid [g, \alpha] = 1,\ \forall \alpha \in Aut(G)\},$$

which is called *absolute centre* of G, see also [6].

2. Main Results

In this section we first give the following definition which generalizes definitions of Das et al. and Moghaddam et al., see [2, 6].

Definition 2.1. *Let G be a finite group, H a subgroup of G and g an element of G then the probability $P_g aut(H, G)$ is defined as follows:*

$$P_g aut(H, G) = \frac{|\{(x, \alpha) \in H \times Aut(G) : [x, \alpha] = g\}|}{|H||Aut(G)|}.$$

where $[x, \alpha] = x^{-1}x^{\alpha}$.

Let $[H, Aut(G)]$ be the subgroup of G generated by autocommutators $[x, \alpha] = x^{-1}x^{\alpha}$ with $x \in H$ and $\alpha \in Aut(G)$. Also, for the sake of simplicity let us write $P_1 aut(H, G) = Paut(H, G)$. Clearly,

$$Paut(H, G) = 1 \iff [H, Aut(G)] = \{1\}, and$$

$$P_g aut(H, G) = 0 \iff g \notin \{ [x, \alpha] : x \in H, \alpha \in Aut(G) \}.$$

It is also easy to see that if $C_{Aut(G)}(x) = \{1\}$ for all $x \in H - \{1\}$ then

$$Paut(H, G) = \frac{1}{|H|} + \frac{1}{|Aut(G)|} - \frac{1}{|H||Aut(G)|}.$$

We now derive a computing formula, as generalization of Theorem 2.3 of [2], which plays a key role in the study of $P_g aut(H, G)$.

Theorem 2.2. *The following statement holds*

$$P_g aut(H, G) = \frac{1}{|H||Aut(G)|} \sum_{\substack{x \in H \\ xg \in Orb_{A_G}(x)}} |C_{Aut(G)}(x)| = \frac{1}{|H|} \sum_{\substack{x \in H \\ xg \in Orb_{A_G}(x)}} \frac{1}{|Orb_{A_G}(x)|},$$

where $Orb_{A_G}(x) = \{ x^{\alpha} : \alpha \in Aut(G) \}$.

Proof. We have $\{(x,\alpha) \in H \times Aut(G) : x^{-1}x^{\alpha} = g\} = \bigcup_{x\in H}\{x\} \times T_x$, where $T_x = \{\alpha \in Aut(G) : x^{-1}x^{\alpha} = g\}$. Note that, for any $x \in H$, we have

$$T_x \neq \emptyset \Longleftrightarrow xg \in Orb_{A_G}(x).$$

Let $T_x \neq \emptyset$ for some $x \in H$. Fix an element $\alpha_0 \in T_x$, then $\alpha \mapsto \alpha_0^{-1}\alpha$ defines an one to one correspondence between the set T_x and $C_{Aut(G)}(x)$. This means that $|T_x| = |C_{Aut(G)}(x)|$.
Thus, we have

$$|\{(x,\alpha) \in H \times Aut(G) : x^{-1}x^{\alpha} = g\}| = \sum_{x\in H} |T_x| = \sum_{\substack{x\in H \\ xg\in Orb_{A_G}(x)}} |C_{Aut(G)}(x)|.$$

The first equality in the theorem follows from Definition 2.1.
For the second equality, consider the action of $Aut(G)$ on G by the above discussion. Then, for all $x \in G$, we have

$$|Orb_{A_G}(x)| = |Aut(G) : Stab(x)| = \frac{|Aut(G)|}{|C_{Aut(G)}(x)|}.$$

This completes the proof. □

As an immediate consequence, we have the following generalization of the well-known formula $Paut(G) = \frac{k}{|G|}$, where k is the number of the disjoint orbits of G under the group automorphism $Aut(G)$ (see also [6, 8]).

Corollary 2.3. *If H is a characteristic subgroup of G, then*

$$Paut(H,G) = \frac{k(H)}{|H|},$$

where $k(H)$ is the number of the disjoint orbits of H under the group automorphism $Aut(G)$.

Proof. Note that $Aut(G)$ acts on H, for all $\alpha \in Aut(G)$ and $x \in H$, we have the following map

$$\begin{aligned} Aut(G) \times H &\longrightarrow H \\ (\alpha, x) &\longmapsto x^{\alpha}. \end{aligned}$$

The orbit of any element $x \in H$ under this action is given by $Orb_{A_G}(x)$, hence H is the disjoint union of these classes. Therfore, we have

$$Paut(H,G) = \frac{1}{|H|}\sum_{x\in H}\frac{1}{|Orb_{A_G}(x)|} = \frac{k(H)}{|H|}.$$

Note that for $g = 1$, the condition $xg \in Orb_{A_G}(x)$ is superfluous. □

Lemma 2.4. *(J. H Christopher and L. R. Darren [1]) Let G_1 and G_2 be two groups such that $gcd(|G_1|, |G_2|) = 1$, then*

$$Aut(G_1 \times G_2) \simeq Aut(G_1) \times Aut(G_2).$$

Remark 2.5. *Let $G = G_1 \times G_2$, $H = H_1 \times H_2$, and for all $h_i \in H_i$, $i = 1, 2$. One can easily see that any automorphism α of $Aut(G_i)$, $(i = 1, 2)$ may be extended to an automorphism α^e of $Aut(G)$, in such a way that $(g_1g_2)^{\alpha^e} = g_1^{\alpha}g_2$, for all $g_1, g_2 \in G$. We denote all such extended automorphisms in $Aut(G)$ by $Aut(G_i^e)$, which are one-to-one correspondence with the ones in $Aut(G_i)$. So it is clear that $|Aut(G_i^e)| = |Aut(G_i)|$, for $i = 1, 2$ and $Aut(G_1^e) \cap Aut(G_2^e) =< id_G >$, $Aut(G_1^e)Aut(G_2^e)C_{Aut(G)}(h_1h_2) \subseteq Aut(G)$. Hence*

$$\begin{aligned} \frac{|Aut(G_1^e)Aut(G_2^e)||C_{Aut(G)}(h_1h_2)|}{|Aut(G_1^e)Aut(G_2^e) \cap C_{Aut(G)}(h_1h_2)|} &= \frac{|Aut(G_1^e)||Aut(G_2^e)||C_{Aut(G)}(h_1h_2)|}{|Aut(G_1^e)Aut(G_2^e) \cap C_{Aut(G)}(h_1h_2)|} \\ &= \frac{|Aut(G_1^e)||Aut(G_2^e)||C_{Aut(G)}(h_1h_2)|}{|Aut(G_1^e) \cap C_{Aut(G)}(h_1h_2)||Aut(G_2^e) \cap C_{Aut(G)}(h_1h_2)|} \\ &= \frac{|Aut(G_1^e)||Aut(G_2^e)||C_{Aut(G)}(h_1h_2)|}{|C_{Aut(G_1)}(h_1)||C_{Aut(G_2)}(h_2)|} \\ &\leq |Aut(G)|, \end{aligned}$$

which implies that $\dfrac{|C_{Aut(G)}(h_1h_2)|}{|Aut(G)|} \leq \dfrac{|C_{Aut(G_1)}(h_1)||C_{Aut(G_2)}(h_2)|}{|Aut(G_1^e)||Aut(G_2^e)|}$, *see also [6].*

The following theorem is a generalization of the work of Moghaddam et al. which in turn is similar to Theorem 2.1 of [6].

Theorem 2.6. *Let H_1 and H_2 be subgroups of the finite groups G_1 and G_2, respectively. Let $g_1 \in G_1$ and $g_2 \in G_2$. Then*

$$P_{(g_1,g_2)}aut(H_1 \times H_2, G_1 \times G_2) \leq P_{g_1}aut(H_1 \times G_1)P_{g_2}aut(H_2 \times G_2).$$

In particular, the equality holds when $gcd(|G_1|, |G_2|) = 1$.

Proof. Put $G = G_1 \times G_2$, $H = H_1 \times H_2$ and $g = (g_1, g_2)$, for all $g_i \in G_i$ and $h_i \in H_i, i = 1, 2$. Then from Theorem 2.2 and the above remark, it follows that

$$\begin{aligned}
P_gaut(H,G) &= \frac{1}{|H|} \sum_{\substack{h \in H \\ hg \in Orb_{A_G}(x)}} \frac{|C_{Aut(G)}(h)|}{|Aut(G)|} \\
&= \frac{1}{|H_1||H_2|} \sum_{\substack{h_1h_2 \in H \\ h_ig_i \in Orb_{A_{G_i}}(h_i)}} \frac{|C_{Aut(G)}(h_1h_2)|}{|Aut(G)|} \quad \textit{where } i = 1, 2. \\
&\leq \frac{1}{|H_1|}\frac{1}{|H_2|} \sum_{\substack{h_1 \in H_1 \\ h_1g_1 \in Orb_{A_{G_1}}(h_1)}} \sum_{\substack{h_2 \in H_2 \\ h_2g_2 \in Orb_{A_{G_2}}(h_2)}} \frac{|C_{Aut(G_1)}(h_1)|}{|Aut(G_1)|}\frac{|C_{Aut(G_2)}(h_2)|}{|Aut(G_2)|} \\
&= \frac{1}{|H_1|} \sum_{\substack{h_1 \in H_1 \\ h_1g_1 \in Orb_{A_{G_1}}(h_1)}} \frac{|C_{Aut(G_1)}(h_1)|}{|Aut(G_1)|} \frac{1}{|H_2|} \sum_{\substack{h_2 \in H_2 \\ h_2g_2 \in Orb_{A_{G_2}}(h_2)}} \frac{|C_{Aut(G_2)}(h_2)|}{|Aut(G_2)|} \\
&= P_{g_1}aut(H_1, G_1)P_{g_2}aut(H_2, G_2).
\end{aligned}$$

Now, using the assumption $gcd(|G_1|, |G_2|) = 1$ and Lemma 2.4 the equality yeilds. □

Next, we present the following proposition which is similar to Proposition 3.1 in [2].

Proposition 2.7. *If $g \neq 1$ then*
(i) $P_gaut(H,G) \neq 0 \Longrightarrow P_gaut(H,G) \geq \dfrac{|C_H(Aut(G))||C_{Aut(G)}(H)|}{|H||Aut(G)|}$,
(ii) $P_gaut(G) \neq 0 \Longrightarrow P_gaut(G) \geq \dfrac{|L(G)|}{|G||Aut(G)|}$.

Proof. Let $g = [x, \alpha]$ for some $(x, \alpha) \in H \times Aut(G)$. Since $g \neq 1$, we have $x \notin C_H(Aut(G))$ and $\alpha \notin C_{Aut(G)}(H)$. Consider the left coset $T_{(x,\alpha)} = (x, \alpha)C_H(Aut(G)) \times C_{Aut(G)}(H)$ of $C_H(Aut(G)) \times C_{Aut(G)}(H)$ in $H \times Aut(G)$. Clearly, $|T_{(x,\alpha)}| = |C_H(Aut(G))||C_{Aut(G)}(H)|$, and $[a, \beta] = g$ for all $(a, \beta) \in T_{(x,\alpha)}$. This proves part (i).
Similarly, part (ii) follows with $H = G$. □

Now by using Theorem 2.2, we are able to generalize Proposition 3.2 of Das et al. as follows:

Proposition 2.8. *Let H be a subgroup of group G and $g \in G$ then*

$$P_gaut(H,G) \leq Paut(H,G)$$

with equality if and only if $g = 1$.

Proof. By Theorem 2.2, we have

$$\begin{aligned} P_g aut(H,G) &= \frac{1}{|H||Aut(G)|} \sum_{\substack{x\in H \\ xg\in Orb_{A_G}(x)}} |C_{Aut(G)}(x)| \\ &\leq \frac{1}{|H||Aut(G)|} \sum_{x\in H} |C_{Aut(G)}(x)| = Paut(H,G). \end{aligned}$$

Clearly, the equality holds if and only if $xg \in Orb_{A_G}(x)$ for all $x \in H$, that is the equality holds if and only if $g = 1$. □

As a generalization of Proposition 3.3 of [2], we establish the following theorem.

Theorem 2.9. *Let p be the smallest prime dividing $|Aut(G)|$, and $g \neq 1$. Then,*

$$P_g aut(H,G) \leq \frac{|H| - |C_H(Aut(G))|}{p|H|} < \frac{1}{p}.$$

Proof. Without loss of generality, we may assume that $C_H(Aut(G)) \neq H$. Let $x \in H$ be such that $xg \in Orb_{A_G}(x)$. Then, Since $g \neq 1$, we have $x \neq C_H(Aut(G))$ and $|Orb_{A_G}(x)| > 1$. Since $|Orb_{A_G}(x)| = |Aut(G) : Stab(x)| = \frac{|Aut(G)|}{|C_{Aut(G)}(x)|}$, so $|Orb_{A_G}(x)|$ is a divisor of $|Aut(G)|$. Therefore, $|Orb_{A_G}(x)| \geq p$. Hence, by Theorem 2.2, we have

$$P_g aut(H,G) \leq \frac{1}{|H|} \sum_{\substack{x\in H \\ xg\in Orb_{A_G}(x)}} \frac{1}{p} \leq \frac{|H| - |C_H(Aut(G))|}{p|H|} < \frac{1}{p},$$

which completes the proof. □

Proposition 2.10. *Let H_1 and H_2 be subgroups of a group G and H_1 is contained in H_2, then*

$$P_g aut(H_1,G) \leq |H_2 : H_1| P_g aut(H_2,G).$$

The equality holds if and only if $xg \notin Orb_{A_G}(x)$ for all $x \in H_2$.

Proof. By Theorem 2.2, we have

$$\begin{aligned} |H_1||Aut(G)|P_g aut(H_1,G) &= \sum_{\substack{x\in H_1 \\ xg\in Orb_{A_G}(x)}} |C_{Aut(G)}(x)| \\ &\leq \sum_{\substack{x\in H_2 \\ xg\in Orb_{A_G}(x)}} |C_{Aut(G)}(x)| \\ &= |H_2||Aut(G)|P_g aut(H_2,G). \end{aligned}$$

The condition for equality follows immediately. □

Immediate consequence of the above proposition is as follows:

Corollary 2.11.

$$P_g aut(H,G) \leq |G : H| Paut(G),$$

with equality if and only if $g = 1$ and $H = G$.

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