Filomat 31:20 (2017), 6357–6365
https://doi.org/10.2298/FIL1720357P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Majorization for Subclasses of Multivalent Meromorphic Functions Defined through Iterations and Combinations of the Liu-Srivastava Operator and a Meromorphic Analogue of the Cho-Kwon-Srivastava Operator

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Abstract. In this paper, the authors investigate a majorization problem for certain subclasses of multivalent meromorphic functions defined in the punctured unit disk $\mathbb{U}^*$ having a pole of order p at origin. The subclasses under investigation are defined through iterations and combinations of the Liu-Srivastava operator and a meromorphic analogue of the Cho-Kwon-Srivastava operator for normalized analytic function. Several consequences of the main results in form of corollaries are also pointed out.

1. Introduction and Definition

Let $f(z)$ and $g(z)$ be analytic in the open unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.
We say that f is majorized by g in $\mathbb{U}$ (see [11]) and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \tag{1}$$

if there exists a function $w(z)$, analytic in $\mathbb{U}$ satisfying $|w(z)| \leq 1$ and

$$f(z) = w(z)g(z) \quad (z \in \mathbb{U}). \tag{2}$$

For two analytic functions f and g, we say $f(z)$ is subordinate to $g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < |z|$ $(z \in \mathbb{U})$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{3}$$

We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}). \tag{4}$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C80
Keywords. Meromorphic function, Subordination, Convolution, Liu-Srivastava operator, Cho-Kwon-Srivastava operator, Majorization problem
Received: 25 March 2016; Revised: 25 August 2016; Accepted: 03 September 2016
Communicated by Hari M. Srivastava
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It follows from this definition that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [12]).

$$f(z) \prec g(z) \ \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Further, $f(z)$ is said to be quasi-subordinate to $g(z)$ if there exists an analytic function $w(z)$ $(|w(z)| \leq 1)$ such that $\frac{f(z)}{w(z)}$ is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{w(z)} \prec g(z) \quad (z \in \mathbb{U}). \tag{5}$$

Hence by definition of subordination, (5) is equivalent to (see [1])

$$f(z) = w(z)g(\phi(z)) \quad (|\phi(z)| \leq |z|,\ z \in \mathbb{U}). \tag{6}$$

We denote this quasi-subordination by

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}). \tag{7}$$

If we set $w(z) \equiv 1$ in (6), then (7) becomes the subordination (4).

If we take $\phi(z) = z$ in (6) , then the quasi-subordination (7) becomes the majorization (1).

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, ...\}) \tag{8}$$

that are analytic and p-valent in the punctured unit disk $\mathbb{U}^* := \mathbb{U} \setminus \{0\}$ having a pole of order p at the origin. We note that $\sum_1 = \sum$.

For the functions $f_j \in \sum_p$ given by

$$f_j(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p,j} z^{k-p} \quad (j = 1, 2; z \in \mathbb{U}^*),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p,1} a_{k-p,2} z^{k-p} = (f_2 * f_1)(z). \tag{9}$$

For a function $f \in \sum_p$, let $f^{(q)}$ denote qth order ordinary differential operator given by

$$f^q(z) = (-1)^q \frac{(p+q-1)!}{(p-1)!} z^{-p-q} + \sum_{k=1}^{\infty} \frac{(k-p)!}{(k-p-q)!} a_{k-p} z^{k-p-q}$$
$$(p \in \mathbb{N},\ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}^*). \tag{10}$$

Liu and Srivastava [10] studied meromorphic analogue of the Saitoh operator [16] by introducing the function $\phi_p(z)$ given by

$$\phi_p(a, c, z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \quad (a \in \mathbb{C},\ c \in \mathbb{C} \setminus \mathbb{Z}_0^- := \{0, -1, -2, ...\},\ \ z \in \mathbb{U}^*),$$

where $(\lambda)_n$ is the Pochhammer symbol (or shifted factorial) given by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0, \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1) & (n \in \mathbb{N}, \lambda \in \mathbb{C}). \end{cases}$$

They defined the linear operator $\mathcal{L}(a,c) : \sum_p \longrightarrow \sum_p$ by

$$\mathcal{L}(a,c)f(z) = \phi_p(a,c;z) * f(z).$$

Define the function $\phi_p^+(a,c;z)$, the generalized multiplicative inverse of $\phi_p(a,c;z)$ by the relation

$$\phi_p(a,c;z) * \phi_p^+(a,c;z) = \frac{1}{z^p(1-z)^{\lambda+p}} \quad (a,\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \lambda > -p;\ z \in \mathbb{U}^*). \tag{11}$$

Using this function we define the following family of transforms $\mathcal{L}_p^\lambda(a,c) : \sum_p \longrightarrow \sum_p$ defined by

$$\begin{aligned} \mathcal{L}_p^\lambda(a,c)f(z) &= \phi_p^+(a,c;z) * f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{(a)_k (1)_k} a_{k-p} z^{k-p} \\ &= \frac{{}_2F_1(\lambda+p,c;a;z)}{z^p} * f(z) \quad (z \in \mathbb{U}^*). \end{aligned}$$

The holomorphic analogue of the function $\phi_p^+(a,c;z)$ and the corresponding transform, which is popularly known as the Cho-Kwon-Srivastava operator in literature (see [4]). We remark in passing that a much more general convolution operator, involving the generalized hypergeometric function in defining Hadamard product (or convolution), was introduced recently by various authors [5, 6, 17].

Very recently Mishra et al.[13] (also see [15]) defined the generalized multiplier transformation $\mathcal{L}_{\lambda,p}^{n,m}(a,c,t) : \sum_p \longrightarrow \sum_p$ by

$$\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z) = \mathcal{L}_p^{\lambda,n}(a,c)C^{t,m}f(z).$$

Thus for a function $f(z)$ of the form (8), we have

$$\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left[\frac{(\lambda+p)_k (c)_k}{(a)_k (1)_k}\right]^n \left[\frac{p-kt}{p}\right]^m a_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*). \tag{12}$$

It should be remembered that the operator $\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)$ is the generalized of many other familiar operators considered by earlier authors (for detail, see [13]).

It is easy to verify that

$$z\left(\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z)\right)' = \frac{p}{t}(1-t)\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z) - \frac{p}{t}\mathcal{L}_{\lambda,p}^{n,m+1}(a,c,t)f(z) \quad (t > 0). \tag{13}$$

Now, by making use of the operator $\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)$, we define a new subclass of function $f \in \sum_p$ as follows.

Definition 1.1. *Let $-1 \le B < A \le 1$, $p \in \mathbb{N}$, $j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*$ and $\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)} + |B|\right) < 1$. A function $f \in \sum_p$ is said to be in the class $\mathcal{T}_{p,j}^{n,m}(a,c,t,\alpha,\gamma;A,B)$ of multivalent meromorphic functions of complex order $\gamma \neq 0$ in $\mathbb{U}^*$ if and only if*

$$1 - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z)\right)^{j+1}}{\left(\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z)\right)^{j}} + p + j\right) - \alpha\left|-\frac{1}{\gamma}\left(\frac{z\left(\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z)\right)^{j+1}}{\left(\mathcal{L}_{\lambda,p}^{n,m}(a,c,t)f(z)\right)^{j}} + p + j\right)\right| \prec \frac{1+Az}{1+Bz}. \tag{14}$$

In particular, for $A = 1,\ B = -1$ and $\alpha = 0$, we denote the class

$$\mathcal{T}^{n,m}_{p,j}(a,c,t,0,\gamma;1,-1) = \mathcal{T}^{n,m}_{p,j}(a,c,t;\gamma)$$
$$= \left\{ f \in \sum_p : \Re\left[1 - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z)\right)^{j+1}}{\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z)\right)^{j}} + p + j\right)\right] > 0\right\}. \tag{15}$$

We note that

- for $\gamma = (p-\delta)\ cos\theta\ e^{-i\theta}$ $(|\theta| \leq \frac{\pi}{2},\ \ 0 \leq \delta < p)$, the class $\mathcal{T}^{n,m}_{p,j}(a,c,t;\gamma) = \mathcal{T}^{n,m}_{p,j}(a,c,t;(p-\delta)\ cos\theta\ e^{-i\theta}) = \mathcal{T}^{n,m}_{p,j}(a,c,t,\delta,\theta)$, called the generalized class of meromorphic θ-spiral-like functions of order δ $(0 \leq \delta < p)$ if

$$\Re\left[e^{i\theta}\left\{\frac{z(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z))^{j+1}}{(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z))^{j}} + j\right\}\right] < -\delta\ cos\theta.$$

- for $j = 0,\ n = 0,\ m = 0,\ \mathcal{T}^{0,0}_{p,0}(a,c,t;\gamma)$ reduces to the class $\sum_p(\gamma)$ $(\gamma \in \mathbb{C}^*)$ of p-valently meromorphic starlike function of complex order γ in $\mathbb{U}^*$, where

$$\textstyle\sum_p(\gamma) = \left\{f \in \sum_p : \Re\left(1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} + p\right)\right) > 0,\ \ p \in \mathbb{N},\ \ \gamma \in \mathbb{C}^*\right\};$$

- for $j = 0,\ m = 1,\ t = 1,\ n = 0,\ \mathcal{T}^{0,1}_{p,0}(a,c,1;\gamma)$ reduces to the class $\mathcal{K}_p(\gamma)$ $(\gamma \in \mathbb{C}^*)$ of p-valently meromorphic convex function of complex order γ in $\mathbb{U}^*$, where

$$\mathcal{K}_p(\gamma) = \left\{f \in \textstyle\sum_p : \Re\left(1 - \frac{1}{\gamma}\left(1 + \frac{zf''(z)}{f'(z)} + p\right)\right) > 0,\ \ p \in \mathbb{N}, \gamma \in \mathbb{C}^*\right\};$$

- for $j = 0,\ n = 0,\ m = 0,\ p = 1,\ \mathcal{T}^{0,0}_{1,0}(a,c,t;\gamma) = \mathcal{S}(\gamma)$, the class of meromorphic starlike univalent functions of complex order $\gamma \neq 0$;

- for $j = 0,\ n = 0,\ m = 0,\ p = 1,\ \gamma = 1-\eta,\ \mathcal{T}^{0,0}_{1,0}(a,c,t;1-\eta) = \sum^*(\eta)$ $(0 \leq \eta < 1)$, the class of meromorphic starlike univalent function of order η in $\mathbb{U}^*$ (see [8]);

- for $j = 0,\ n = 0,\ m = 1,\ t = 1,\ p = 1,\ \mathcal{T}^{0,1}_{1,0}(a,c,1;\gamma) = \mathcal{K}(\gamma)$, the class of meromorphic convex univalent function of complex order γ;

- for $j = 0,\ n = 0,\ m = 1,\ t = 1,\ p = 1,\ \gamma = 1-\eta$ $(0 \leq \eta < 1)$, $\mathcal{T}^{0,1}_{1,0}(a,c,1;1-\eta) = \sum_k(\eta)$, the class of meromorphic convex univalent function of order η (see [8]).

  Another subclass of the class $\sum_p$ associated with a linear operators, was studied recently by Srivastava et al. [18] (also see [19, 20]). Also, there is good amount of literature about majorization problems for univalent and multivalent functions discussed by various researchers. A majorization problem for the normalized classes of starlike functions has been investigated by Altintas et al. [2](also see [3]) and MacGreogor [11]. For recent expository work on majorization problems for meromorphic univalent and p-valent functions, see [7, 9, 21].

Motivated by aforementioned works, in this paper the authors investigate majorization problem for the class of multivalent meromorphic functions using iterations and combinations of the Liu-Srivastava operator and a meromorphic analogue of the Cho-Kwon-Srivastava operator for normalized analytic functions.

2. Main Results

Unless otherwise mentioned we shall assume throughout the sequel that

$$-1 \le B < A \le 1, \;\; p \in \mathbb{N}, \;\; j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*; z \in \mathbb{U}^*.$$

Theorem 2.1. *Let the function $f \in \sum_p$ and suppose that $g \in \mathcal{T}^{n,m}_{p,j}(a, c, t, \alpha, \gamma; A, B)$. If $\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)f(z)\right)^j$ is majorized by $\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^j$ in $\mathbb{U}^*$, then*

$$\left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a, c, t)f(z)\right)^j\right| \le \left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a, c, t)g(z)\right)^j\right| \quad (|z| < r_0), \tag{16}$$

where $r_0 = r_0(p, \alpha, t, \gamma; A, B)$ is the smallest positive root of the equation

$$p\left[\frac{(A-B)t|\gamma|}{p(1-\alpha)} + |B|\right] r^3 - (2t|B| + p)r^2 - \left[2t + p\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)} + |B|\right)\right] r + p = 0 \tag{17}$$

Proof. Since $g \in \mathcal{T}^{n,m}_{p,j}(a, c, t, \alpha, \gamma; A, B)$, we find from (14) that

$$1 - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^{j+1}}{\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^j} + p + j\right) - \alpha\left|-\frac{1}{\gamma}\left(\frac{z\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^{j+1}}{\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^j} + p + j\right)\right| = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{18}$$

where $w(z) = c_1z + c_2z^2 + ...$, $w \in \mathcal{P}$, $\mathcal{P}$ denote the well- known class of the bounded analytic functions in $\mathbb{U}$ and satisfies the conditions $w(0) = 0$ and $w(z) < |z|$ $(z \in \mathbb{U})$.

Taking

$$\varpi = 1 - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^{j+1}}{\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^j} + p + j\right) \tag{19}$$

in (18), we have

$$\varpi - \alpha|\varpi - 1| = \frac{1 + Aw(z)}{1 + Bw(z)},$$

which implies

$$\varpi = \frac{1 + \left(\frac{A - B\alpha e^{-i\theta}}{1-\alpha e^{-i\theta}}\right)w(z)}{1 + Bw(z)}. \tag{20}$$

Using (20) in (19), we get

$$\frac{z\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^{j+1}}{\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^j} = -\frac{p + j + \left[\frac{(A-B)\gamma}{1-\alpha e^{-i\theta}} + (p + j)B\right]w(z)}{1 + Bw(z)}. \tag{21}$$

Application of Leibnitz's theorem on (13) gives

$$z\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^{j+1} = \left(\frac{p}{t} - p - j\right)\left(\mathcal{L}^{n,m}_{\lambda,p}(a, c, t)g(z)\right)^j - \frac{p}{t}\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a, c, t)g(z)\right)^j \quad (j > 0). \tag{22}$$

Now, using (22) in (21), we find that

$$\frac{\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j}{\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j} = \frac{1+\left[\frac{(A-B)t\gamma}{p(1-\alpha e^{-i\theta})}+B\right]w(z)}{1+Bw(z)}.$$

Or, equivalently,

$$\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j = \frac{1+Bw(z)}{1+\left[\frac{(A-B)t\gamma}{p(1-\alpha e^{-i\theta})}+B\right]w(z)}\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j. \tag{23}$$

Since $|w(z)| \le |z|$ $(z \in \mathbb{U})$, the formula (23) gives

$$\begin{aligned}\left|\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j\right| &\le \frac{1+|B||z|}{1-\left|\frac{(A-B)t\gamma}{p(1-\alpha e^{-i\theta})}+B\right||z|}\left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j\right| \\ &\le \frac{1+|B||z|}{1-\left[\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right]|z|}\left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j\right| \end{aligned}\tag{24}$$

Further, since $\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z)\right)^j$ is majorized by $\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j$ in the unit disk $\mathbb{U}^*$, from (2), we have

$$\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z)\right)^j = w(z)\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j \tag{25}$$

Differentiating (25) on both sides with respect to z and multiplying by z, we get

$$z\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z)\right)^{j+1} = zw'(z)\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j + zw(z)\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^{j+1}. \tag{26}$$

Using (22) and (25) in (26) yields

$$\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)f(z)\right)^j = -\frac{t}{p}zw'(z)\left(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z)\right)^j + w(z)\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j. \tag{27}$$

Thus, noting that $w \in \mathcal{P}$ satisfies the inequality (see [14])

$$|w'(z)| \le \frac{1-|w(z)|^2}{1-|z|^2} \tag{28}$$

and making use of (24) and (28) in (27), we obtain

$$\left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)f(z)\right)^j\right| \le \left(|w(z)| + \frac{t|z|(1-|w(z)|^2)(1+|B||z|)}{p(1-|z|^2)\left[1-\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right)|z|\right]}\right)\left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j\right|,$$

which, upon setting

$$|z| = r \text{ and } |w(z)| = \rho \quad (0 \le \rho < 1),$$

leads us to the inequality

$$|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)f(z)\right)^j| \le \frac{\psi(\rho)}{p(1-r^2)\left[1-\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right)r\right]}\left|\left(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z)\right)^j\right|$$

where

$$\psi(\rho) = p(1-r^2)\left[1-\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right)r\right]\rho + t(1-\rho^2)(1+|B|r)r$$
$$= -tr(1+|B|r)\rho^2 + p(1-r^2)\left[1-\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right)r\right]\rho + tr(1+|B|r), \tag{29}$$

takes its maximum value at $\rho = 1$ with $r_0 = r_0(p,\alpha,t,\gamma;A,B)$ where r_0 is the smallest positive root of the equation (17). Furthermore, if $0 \leq \delta \leq r_0(p,\alpha,t,\gamma;A,B)$, then the function $\psi(\rho)$ defined by

$$\psi(\rho) = -t\delta(1+|B|\delta)\rho^2 + p(1-\delta^2)\left[1-\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right)\delta\right]\rho + t\delta(1+|B|\delta) \tag{30}$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\psi(\rho) \leq \psi(1) = p(1-\delta^2)\left[1-\left(\frac{(A-B)t|\gamma|}{p(1-\alpha)}+|B|\right)\delta\right] \quad (0 \leq \rho \leq 1, 0 \leq \delta \leq r_0(p,\alpha,t,\gamma;A,B).$$

Hence, upon setting $\rho = 1$ in (30) we conclude that (16) of Theorem 2.1 holds true for $|z| \leq r_0(p,\alpha,t,\gamma;A,B)$, where r_0 is the smallest positive root of the equation (17). This completes the proof of Theorem 2.1. □

3. Corollaries and Concluding Remarks

By letting $A = 1$ and $B = -1$ in Theorem 2.1, we obtain the following corollary.

Corollary 3.1. *Let the functions $f \in \sum_p$ and $g \in \mathcal{T}^{n,m}_{p,j}(a,c,t,\alpha;\gamma)$. If $(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z))^j$ is majorized by $(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z))^j$ in $\mathbb{U}^*$, then*

$$|(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)f(z))^j| \leq |(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z))^j| \quad (|z| \leq r_1),$$

where $r_1 = r_1(p,\alpha,t,\gamma)$ is the smallest positive root of the equation

$$\left(\frac{2t|\gamma|}{1-\alpha}+p\right)r^3 - (2t+p)r^2 - \left[2t+\frac{2t|\gamma|}{1-\alpha}+p\right]r + p = 0,$$

given by $r_1 = \frac{k_1-\sqrt{k_1^2-p\left(p+\frac{2t|\gamma|}{1-\alpha}\right)}}{p+\frac{2t|\gamma|}{1-\alpha}}$ *and* $k_1 = t+p+\frac{t|\gamma|}{1-\alpha}$.

Taking $\alpha = 0$ in Corollary 3.1, we state the following:

Corollary 3.2. *Let the functions $f \in \sum_p$ and $g \in \mathcal{T}^{n,m}_{p,j}(a,c,t;\gamma)$. If $(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)f(z))^j$ is majorized by $(\mathcal{L}^{n,m}_{\lambda,p}(a,c,t)g(z))^j$ in $\mathbb{U}^*$, then*

$$|(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)f(z))^j| \leq |(\mathcal{L}^{n,m+1}_{\lambda,p}(a,c,t)g(z))^j| \quad (|z| \leq r_2),$$

where $r_2 = r_2(p,t,\gamma)$ is the smallest positive root of the equation

$$(2t|\gamma|+p)\,r^3 - (2t+p)r^2 - [2t+2t|\gamma|+p]r + p = 0,$$

given by $r_2 = \frac{k_2-\sqrt{k_2^2-p(p+2t|\gamma|)}}{p+2t|\gamma|}$ *and* $k_2 = t+p+t|\gamma|$.

Taking $n = m = j = 0, t = 1$ in Corollary 3.2, we get

Corollary 3.3. *Let the functions* $f \in \sum_p$ *and* $g \in \sum_p(\gamma)$*. If* $f(z)$ *is majorized by* $g(z)$ *in* $\mathbb{U}^*$*, then*

$$|zf'(z)| \le |zg'(z)| \quad (|z| \le r_3),$$

where $r_3 = r_3(p, \gamma)$ *is the smallest positive root of the equation*

$$(2|\gamma| + p)r^3 - (2 + p)r^2 - [2 + 2|\gamma| + p]r + p = 0$$

given by $r_3 = \frac{k_3 - \sqrt{k_3^2 - (2|\gamma|+p)p}}{2|\gamma|+p}$ *and* $k_3 = |\gamma| + p + 1$.

By setting $\gamma = p - \delta$ $(0 \le \delta < p)$ in Corollary 3.3, we obtain the following results:

Corollary 3.4. *Let the functions* $f \in \sum_p$ *and* $g \in \sum_p(\delta)$*. If* $f(z)$ *is majorized by* $g(z)$ *in* $\mathbb{U}^*$*, then*

$$|zf'(z)| \le |zg'(z)|, \ |z| \le r_4,$$

where $r_4 = r_4(p, \delta)$ *is the smallest positive root of the equation*

$$(p + 2|p - \delta|)r^3 - (2 + p)r^2 - [2 + 2|p - \delta| + p]r + p = 0$$

given by $r_4(p, \delta) = \frac{k_4 - \sqrt{k_4^2 - (p+2|p-\delta|)p}}{p+2|p-\delta|}$ *and* $k_4 = |p - \delta| + p + 1$.

By taking $\gamma = (p - \delta)\, cos\theta\, e^{-i\theta}$ $(|\theta| \le \frac{\pi}{2}, \ \delta \ (0 \le \delta < p))$ in Corollary 3.3 , we get the following:

Corollary 3.5. *Let the functions* $f \in \sum_p$ *and* $g \in \sum_p(\theta, \delta)$*. If* $f(z)$ *is majorized by* $g(z)$ *in* $\mathbb{U}^*$*, then*

$$|zf'(z)| \le |zg'(z)|, \quad |z| \le r_5$$

where $r_5 = r_5(p, \delta, \theta)$ *is given by*
$r_5 = \frac{k_5 - \sqrt{k_5^2 - p(p+2|(p-\delta)cos\theta|)}}{p+2|(p-\delta)cos\theta|}$ *and* $k_5 = p + 1 + |(p - \delta)cos\theta|$.

Letting $p = 1$ and $\gamma = 1$ in Corollary 3.3 leads to the following result:

Corollary 3.6. *Let the functions* $f \in \sum$ *and* $g \in \sum_1(1) = \mathcal{S}(1)$*. If* $f(z)$ *is majorized by* $g(z)$ *in* $\mathbb{U}^*$*, then*

$$|zf'(z)| \le |zg'(z)| \ \textit{for} \ |z| \le \frac{3 - \sqrt{6}}{3}.$$

Concluding Remarks: By specializing different parameters like n, m and t further, one can get various other interesting subclasses of $\sum_p$ containing linear operators and the corresponding corollaries can be easily obtained.

Acknowledgement: The authors would like to thank Professor H.M. Srivastava, university of Victoria, for his valuable suggestions

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