Filomat 31:20 (2017), 6393–6400
https://doi.org/10.2298/FIL1720393G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Signed Graphs with Two Distinct Eigenvalues

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Abstract. Let G^σ be a signed graph with the underlying graph G and with sign function $\sigma : E(G) \longrightarrow \{\pm\}$. In this paper, we characterize the signed graphs with two distinct eigenvalues whose underlying graphs are triangle-free. Also, we classify all 3-regular and 4-regular signed graphs whose underlying graphs are triangle-free and give their adjacency matrices as well.

1. Introduction

Studying applications of graph theory in social psychology was initiated by Cartwright, Norman, and Harary [2] in 1965, when they were doing research on dynamics of population at the University of Michigan. Dynamics of population studies social relations between individuals of a given social group. Graphs which are commonly used in this study are signed graphs. Let $G = (V, E)$ be a simple graph and $\sigma : E(G) \to \{\pm\}$ a mapping on the edges set of G. The graph G together with the sign function σ is called a *signed graph* and is denoted by G^σ [4]. If $\sigma(e) = +$, then the edge e is called *positive*, and if $\sigma(e) = -$, then the edge e is called *negative*. A positive sign between people u and v means that u and v are positively related; namely, they have in common the social characteristics of concern. A negative sign indicates the opposite. A social characteristic can be "friendship", having respect for specific social traditions, etc... A group of people with such relations between them is called a *social system*. A social system is *balanced* if positive relations appear in the system such that positive relations only appear for pairs of people belonging to the same sector and negative relations only appear for pairs of people not belonging to the same sector.

The *adjacency matrix* of a graph G, denoted by $A(G) = [a_{ij}]$, is an $n \times n$ matrix where $a_{ij} = 1$ if v_iv_j is an edge of the graph, and $a_{ij} = 0$ otherwise. Notice that this matrix is real symmetric. The *adjacency matrix of a signed graph*, denoted by $A^\sigma = [a^\sigma_{ij}]$, is an $n \times n$ matrix where $a^\sigma_{ij} = \sigma(ij)a_{ij}$ if v_iv_j is an edge of the graph, and $a^\sigma_{ij} = 0$ otherwise. Thus, if $e = ij$ is an edge and $\sigma(ij) = +$, then $a^\sigma_{ij} = 1$, if $e = ij$ is an edge and $\sigma(ij) = -$, then $a^\sigma_{ij} = -1$, and if $e = ij$ is not an edge, then $a^\sigma_{ij} = 0$. The adjacency matrix of a signed graph is also symmetric (see [1] for some basic results on the adjacency spectrum of signed graphs). A *path of length k* in a graph G is a sequence $v_1e_1, \ldots, v_ke_kv_{k+1}$ with vertices $v_1, \ldots, v_{k+1}$ and edges $e_1, \ldots, e_k$ such that, we have $v_i \neq v_{i+1}$, $1 \leq i \leq k$ and e_i is an edge from v_i to v_{i+1}. In a signed graph G^σ, a path is called *positive* (resp. *negative*) if the number of its negative edges is even (resp. odd) [1]. The number of positive (resp. negative) paths of length

2010 *Mathematics Subject Classification*. 05C50, 15A18.
Keywords. Singed graph, Adjacency matrix, Eigenvalue.
Received: 06 April 2016; Accepted: 08 June 2017
Communicated by Francesco Belardo
Research supported by the University of Kashan (Grant No. 682410/3).
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k from vertex v_i to vertex v_j is denoted by $w^+_{ij}(k)$ (resp. $w^-_{ij}(k)$). Zaslavsky [5] proved that if G^σ is a signed graph with (signed) adjacency matrix A^σ, then the element (i, j) of the matrix $(A^\sigma)^k$ is equal to $w^+_{ij}(k) - w^-_{ij}(k)$. A *cycle* with $n \geq 3$ vertices is a simple graph whose vertices can be sorted as a sequence such that two vertices are adjacent if and only if they are subsequent members of the sequence. In a signed graph G^σ, a cycle is called *balanced* or *positive* (resp. *unbalanced* or *negative*) if the number of its negative edges is even (resp. odd). A signed graph is called *balanced* whenever all its cycles are balanced; otherwise, it is called *unbalanced* [1]. Let G be a graph with adjacency matrix A. We say that λ is an *eigenvalue of* G if there exists a non-zero vector X such that $A(G)X = \lambda X$. A graph of order n is called *regular* if all of the vertices are of the same degree, and in particular, it is called *k-regular* if the degree of each vertex is equal to the integer k. If there exists an edge between each pair of vertices, then the graph is called a *complete graph*. Most of the notions used for simple graphs are used for signed graphs as well: the degree $d(v)$ of a vertex v, the neighbor $N(x)$ of a vertex x , maximum degree of vertices $\Delta(G)$, induced subgraph $G[S]$ for subset S of $V(G)$, complete bipartite graphs $K_{m,n}$, etc... It is well-known that complete graphs are the only graphs with two distinct eigenvalues. Ramezani [3] proved that if G^σ is a signed graph with only two distinct eigenvalues, then the underlying graph G is regular. She made use of properties of bipartite graphs in order to find general properties of matrices of signed graphs with two distinct eigenvalues, but she has not identified specific graphs.

In this paper, we characterize signed graphs with two distinct eigenvalues whose the underlying graphs are triangle-free. Also, we classify all 3-regular and 4-regular signed graphs whose underlying graphs are triangle-free and give their matrices as well.

2. Main Results

In this section, we determine all 3-regular and 4-regular signed graphs whose the underlying graphs are triangle-free.

Theorem 2.1. *[3] Let G^σ be a signed graphs with two distinct eigenvalues. Then the underlying graph G is regular.*

Theorem 2.2. *Let G be triangle-free and k-regular graph. Thus, G^σ has two distinct eigenvalues if and only if the number of positive paths and negative paths of length two between each pair of non-adjacent vertices are equal, in which case $(A^\sigma)^2 = kI_n$.*

Proof. Assume that G^σ has two distinct eigenvalues α and β. Then by diagonalizabality of A^σ:

$$(A^\sigma)^2 - (\alpha + \beta)A^\sigma + \alpha\beta I_n = 0.$$

Since G is a triangle-free, the number of positive paths and negative paths of length two between each pair of adjacent vertices v_i and v_j is zero. Thus, $[(A^\sigma)^2]_{ij} = 0$, $(A^\sigma)_{ij} = \pm 1$ and $\alpha\beta(I_n)_{ij} = 0$. So $(\alpha + \beta)(\pm 1) = 0$. This shows that, $\alpha+\beta = 0$ and $(A^\sigma)^2 = -\alpha\beta I_n$. We know $\alpha\beta = -k$, then $(A^\sigma)^2 = kI_n$. However since $[(A^\sigma)^2]_{ij} = k(I_n)_{ij}$, $(A^\sigma)^2_{ij} = 0$, where $i \neq j$. On the other hand by [5], $[(A^\sigma)^2]_{ij} = w^+_{ij}(2) - w^-_{ij}(2)$. So $w^+_{ij}(2) = w^-_{ij}(2)$. This implies that, the number of positive paths and negative paths of length two between each pair of non-adjacent vertices are equal.

Conversely assume that the number of positive paths and negative paths of length two between each pair of non-adjacent vertices is equal, indeed $w^+_{ij}(2) = w^-_{ij}(2)$. In addition, $(A^\sigma)^2_{ij} = w^+_{ij}(2) - w^-_{ij}(2)$, where $i \neq j$. This implies that, $(A^\sigma)^2_{ij} = 0$. Since G is k-regular, $[(A^\sigma)^2]_{ii} = k$. Hence, $(A^\sigma)^2 = kI_n$. Then minimal polynomial A^σ is $\chi(G^\sigma, x) = x^2 - k$. So the signed graph G^σ has two distinct eigenvalues $\pm\sqrt{k}$. □

Lemma 2.3. *Let A^σ be the adjacency matrix of a signed graph G^σ. If $(A^\sigma)^2 = kI_n$, then the number of common neighbors of pair of non-adjacent vertices x and y in G are even.*

Proof. Since $(A^\sigma)^2 = kI_n$, $w^+_{ij}(2) - w^-_{ij}(2) = 0$. So $w^+_{ij}(2) = w^-_{ij}(2)$. Therefore, half of paths of length two are positive and another half are negative. Then, the number of common neighbors of pair of non-adjacent vertices x and y in G are even. □

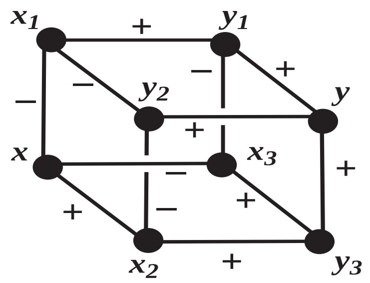

Figure 1: A Signed Hypercube Graph Q_3^σ.

Theorem 2.4. *Let G^σ be 3-regular graphs with two distinct eigenvalues. If the underlying graph G is triangle-free, then G is isomorphic to the hypercube Q_3 in Figure 1.*

Proof. Let $N(x) = \{x_1, x_2, x_3\}$. Since G is triangle-free, $\{x_1, x_2, x_3\}$ is independent set. Suppose that y_1 and y_2 are adjacent to x_1, (x is also adjacent to x_1, so $d(x_1) = 3$), so x_1 is the common neighbor of x and y_1. However by Lemma 2.3, each pair of non-adjacent vertices have an even number of common neighbors. Therefore, there is another common neighbors of x and y_1. Since G is 3-regular and $d(x) = 3$, x_2 or x_3 must be adjacent to y_1. Without loss of generality, let x_3 be the common neighbor of x and y_1. Hence, $y_1x_3 \in E(G)$. Now, consider the common neighbors of x and y_2. Then either $y_2x_2 \in E(G)$ or $y_2x_3 \in E(G)$. Assume that $y_2x_3 \in E(G)$. Then x_1 and x_3 have three common neighbors $\{x, y_1, y_2\}$, a contradiction to Lemma 2.3. So $y_2x_2 \in E(G)$. With the same argument we have $y_1x_2 \notin E(G)$, otherwise x_1 and x_2 have three common neighbors $\{x, y_1, y_2\}$, a contradiction. Now, consider the common neighbors of x_2 and x_3. Moreover, x is the common neighbor of x_2 and x_3 and $y_2x_3 \notin E(G)$ and $y_1x_2 \notin E(G)$. So there has another vertex that the common neighbor of x_2 and x_3. Let y_3 be such common neighbor of x_2 and x_3. So $x_2y_3 \in E(G)$ and $x_3y_3 \in E(G)$. Then we have $d(x) = d(x_1) = d(x_2) = d(x_3) = 3$ and $d(y_1) = d(y_2) = d(y_3) = 2$.

Suppose that y is the third neighbor of y_1. Therefore, $y_1y \in E(G)$. So y_1 is the common neighbor of x_1 and y. However by Lemma 2.3 and considering the common neighbors of y and x_1 and the common neighbors of y and x_3, we have $y_2y \in E(G)$ and $y_3y \in E(G)$,respectively. Moreover y_2 is the unique neighbor of x_1 except y_1, whose degree is less than 3, and y_3 is the unique neighbor x_3 except y_1, whose degree is less than 3. All vertices have already degree 3. So its underlying graph, hypercube Q_3 is shown in Figure 1. □

Corollary 2.5. *Let G be 3-regular and triangle-free graph. Then G^σ has two distinct eigenvalues if and only if G is the three-dimensional hypercube with the unbalanced quadrangles.*

Proof. Suppose that G^σ has two distinct eigenvalues. By Theorem 2.4, G must be hypercube Q_3. By Theorem 2.2, the number of positive paths and negative paths of length two between each pair of non-adjacent vertices are equal. Now, there are two paths between each pair of non-adjacent vertices, then one of them is positive and another is negative. Then all of the quadrangles are unbalanced.

Conversely assume that $G \cong Q_3$ and all of the quadrangles are unbalanced. Then there is a path of length two and positive and a path of length two and negative between each pair of non-adjacent vertices. This proves that the number of positive paths and negative paths of length two between each pair of non-adjacent vertices are equal and by Theorem 2.2, G has two distinct eigenvalues. □

Example 2.6. *Let Q_3^σ be the signed graph as seen in Figure 1. Then it has two distinct eigenvalues. Let the rows of a signed adjacency matrix Q_3^σ correspond successively vertices x, y_1, y_2, y_3, x_1, x_2, x_3 and y. So we have,*

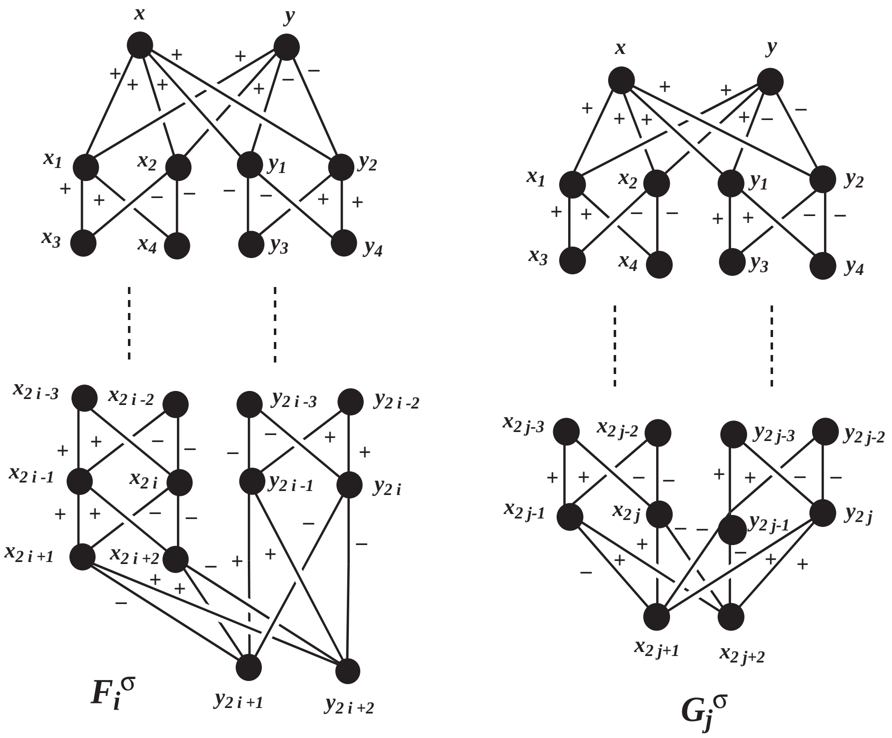

Figure 2: Signed Graphs F_i^σ and G_j^σ.

$$A(Q_3^\sigma) = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & +1 & +1 & +1 \\ -1 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By a tedious calculation we as see that $[A(Q_3^\sigma)]^2 = 3I_8$ *and its eigenvalues are* $\pm\sqrt{3}$.

Theorem 2.7. *Let G be 4-regular graph, triangle-free and G^σ be its signed graphs with two distinct eigenvalues. Then, G is isomorphic to one of F_i, G_j, the hypercube Q_4 and H depicted in Figures 2 and 3.*

Proof. Let $N(x) = \{x_1, x_2, y_1, y_2\}$. Since G is triangle-free, $\{x_1, x_2, y_1, y_2\}$ is independent set. Assume that y, x_3 and x_4 are adjacent to x_1, so x_1 is the common neighbor of x and y. By Lemma 2.3 there must be another one or three neighbors between each pair of vertices x and y in $\{x_2, y_1, y_2\}$, between each pair of vertices x and x_3 in $\{x_2, y_1, y_2\}$ and between each pair of vertices x and x_4 in $\{x_2, y_1, y_2\}$. Now, assume that k_1, k_2 and k_3 are the numbers of common neighbors between each pair of vertices x and y, x and x_3, x and x_4, respectively. Thus $\{k_1, k_2, k_3\} = \{1, 3\}$. Without loss of generality, four cases are happening.

Case 1: $k_1 = 1, k_2 = 1$ and $k_3 = 1$.

Assume that $x_2y \in E(G)$. Since $k_1 = 1$, $y_1y \notin E(G)$ and $y_2y \notin E(G)$. Moreover, x is the common neighbor of x_1 and y_1. By Lemma 2.3, there is the other common neighbor between each pair of vertices x_1 and y_1 in $\{x_3, x_4\}$. Suppose that x_3 is the common neighbor of x_1 and y_1. Since $k_2 = 1$, $x_2x_3 \notin E(G)$ and $y_2x_3 \notin E(G)$. we consider the common neighbors of x_1 and y_2. Since $y_2y \notin E(G)$ and $y_2x_3 \notin E(G)$, $y_2x_4 \in E(G)$. Since $k_3 = 1$, $x_2x_4 \notin E(G)$ and $y_1x_4 \notin E(G)$. As G is 4-regular, x_2 has another two neighbors, say y_3 and y_4. Now, x is the common neighbor of x_2 and y_1 and $y_1y \notin E(G)$, otherwise the number of common neighbors of x_1 and y_1 are 3, a contradiction to Lemma 2.3. Then there is another neighbor between each pair of vertices x_2 and y_1 which belongs to $\{y_3, y_4\}$. Assume that y_3 is the common neighbor of x_2 and y_1, therefore $y_1y_4 \notin E(G)$, otherwise the number of common neighbors of x_2 and y_1 are 3, a contradiction to Lemma 2.3. Since x is the common neighbor of x_2 and y_2 and $y_2y \notin E(G)$, there is another common neighbor between each pair of vertices x_2 and y_2 which belongs to $\{y_3, y_4\}$.

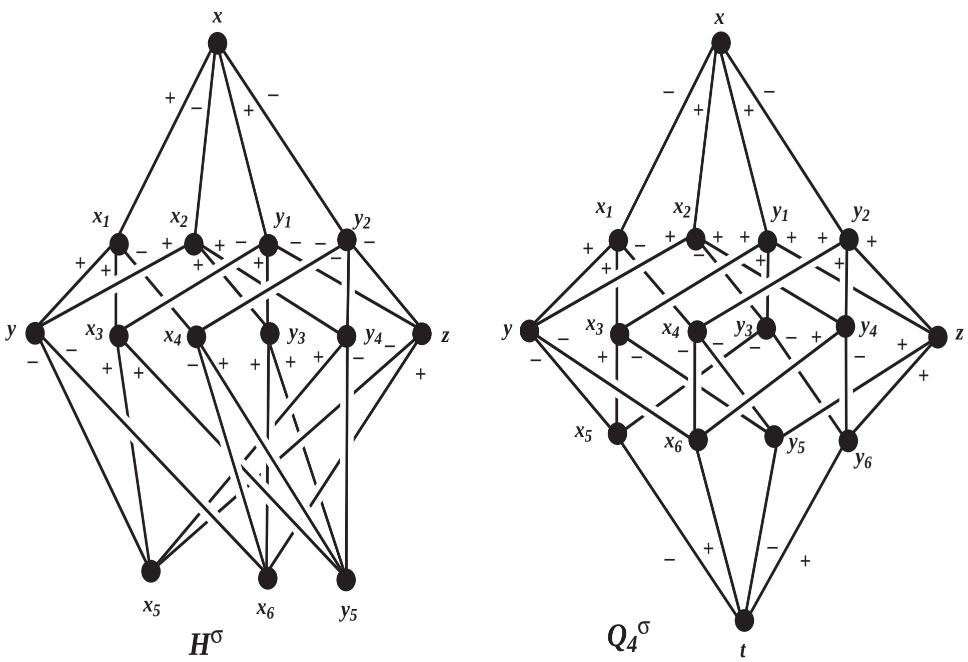

Figure 3: Signed Graphs H^σ and Hypercube Q_4^σ.

We claim that y_3 is not the common neighbor of x_2 and y_2, otherwise the common neighbors of x and y_3 are $\{x_2, y_1, y_2\}$, a contradiction to Lemma 2.3. Then y_4 is the common neighbor of x_2 and y_2. Now, add a new vertex z and consider as the fourth neighbor of y_1. Considering the common neighbors of y_1 and y_2, we have $y_2z \in E(G)$. This follows that $d(x) = d(x_1) = d(x_2) = d(y_1) = d(y_2) = 4$ and $d(y) = d(x_3) = d(x_4) = d(y_3) = d(y_4) = d(z) = 2$.

We claim that $\{y, x_3, x_4, y_3, y_4, z\}$ is independent set, otherwise we have the edges $yz \in E(G)$, $x_3y_4 \in E(G)$ and $x_4y_3 \in E(G)$. If $yz \in E(G)$, then the number of common neighbors of x_1 and z are 1, a contradiction. Then $\{y, x_3, x_4, y_3, y_4, z\}$ is independent set and $yz \notin E(G)$, $x_3y_4 \notin E(G)$ and $x_4y_3 \notin E(G)$.

Suppose that x_5 and x_6 are adjacent to y. Since x_1 is the common neighbor of y and x_3, either $x_3x_5 \in E(G)$ or $x_3x_6 \in E(G)$. Assume that $x_3x_5 \in E(G)$. Therefore $x_3x_6 \notin E(G)$. Now, $x_4x_5 \notin E(G)$, otherwise the common neighbors of x_1 and x_5 are $\{y, x_3, x_4\}$, a contradiction.

Now, consider the common neighbors of x_1 and x_6. Since y is the common neighbor of x_1 and x_6, $x_4x_6 \in E(G)$. Suppose that y_5 is as the fourth neighbor of x_3. So x_3 is the common neighbor of x_1 and y_5. With the same process x_4 is also the common vertex between x_1 and y_5. Moreover, y is the common neighbors of x_2 and x_5. So there is another common neighbor between x_2 and x_5. This shows that $y_3x_5 \in E(G)$ or $y_4x_5 \in E(G)$. However we review both cases.

1) If $y_3x_5 \in E(G)$, then $y_4x_5 \notin E(G)$. By considering the common neighbors of y and y_4 we have $y_4x_6 \in E(G)$. We claim that $y_3x_6 \notin E(G)$ and $y_3y_5 \notin E(G)$, otherwise the common neighbors of y and y_3 are $\{x_2, x_5, x_6\}$ or the common neighbors of x_3 and y_3 are $\{y_1, x_5, y_5\}$, a contradiction to Lemma 2.3.

Then y_3 has a new neighbor, say y_6. Since y_3 is the common neighbor of y_1 and y_6 and z is adjacent to y_1 and with degree less than 4, $zy_6 \in E(G)$. By the same argument and considering the common neighbors of y_2 and y_6, we have $y_4y_6 \in E(G)$. Now, considering the common neighbors of y_2 and y_5 we have $zy_5 \in E(G)$. Then we have $d(x_5) = d(x_6) = d(y_5) = d(y_6) = 3$ and all above vertices are degree 4. Now, since G is a triangle-free, $\{x_5, x_6, y_5, y_6\}$ is independent set. Suppose that t is the fourth neighbor of x_5. Considering the common neighbors of t and y, t and x_3, t and y_3, respectively, we have $x_5t \in E(G)$, $x_6t \in E(G)$, $y_5t \in E(G)$ and $y_6t \in E(G)$. All vertices have already degree 4 and its underlying graph is isomorphic to the hypercube Q_4, in Figure 3.

2) If $y_4x_5 \in E(G)$, then y_4 is the common neighbor of y_2 and x_5. Since z is adjacent to y_2 with degree less than 4, $zx_5 \in E(G)$. Now, x_2 is the common neighbor of y and y_3. By Lemma 2.3, x_6 is another common neighbor between y and y_3. So $x_6y_3 \in E(G)$. By the same argument and considering the common neighbors of x_3 and y_3 we have $y_3y_5 \in E(G)$. Since x_5 is the common neighbor of x_3 and y_4, $y_4y_5 \in E(G)$. However, considering the common neighbors of z and x_4, we have $zx_6 \in E(G)$. All vertices have already degree 4 and its underlying graph is isomorphic to the H, in Figure 3.

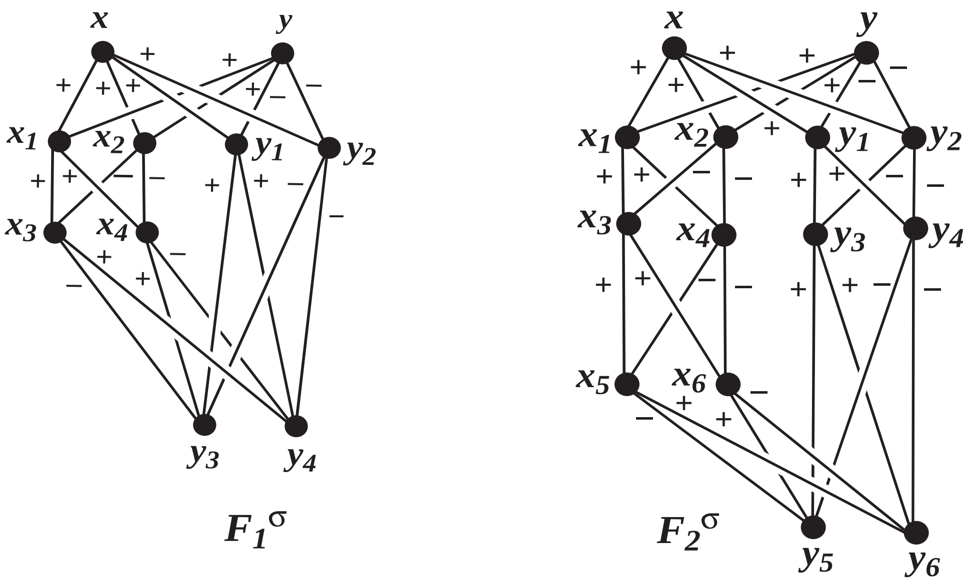

Figure 4: Signed Graphs F_1^σ and F_2^σ.

Case 2: $k_1 = 3, k_2 = 1$ and $k_3 = 1$.

By a similar argument as Case 1, the underlying graph is isomorphic to graph F_1 or F_2 in Figure 4 and the underlying graph G_2 or G_3 in Figure 5.

Case 3: $k_1 = 3, k_2 = 3$ and $k_3 = 1$.

In this case, y and x_3 are adjacent to all vertices of $\{x_2, y_1, y_2\}$ such that x_4 is only adjacent to one of them. Assume that $x_2x_4 \in E(G)$, so $y_1x_4 \notin E(G)$ and $y_2x_4 \notin E(G)$. Hence, the common neighbors of x_1 and y_1 contain $\{x, y, x_3\}$, a contradiction. Then, this case could not occur.

Case 4: $k_1 = 3, k_2 = 3$ and $k_3 = 3$.

In this case, y and x_3 and x_4 are adjacent to all vertices of $\{x_2, y_1, y_2\}$. All vertices have already degree 4 and its underlying graph is isomorphic to the complete bipartite graph $K_{4,4}$, which is also the underlying graph G_1 in Figure 2. □

Corollary 2.8. *Let G be 4-regular and triangle-free graph. Then G^σ has two distinct eigenvalues if and only if*

1) $G \cong Q_4$ or $G \cong H$ and all of the quadrangles G^σ are unbalanced or

2) $G \cong F_1$, F_2, G_1, G_2 or G_3 and there are exactly one positive path and one negative path of length two or two positive paths and two negative paths of length two between each pair of non-adjacent vertices.

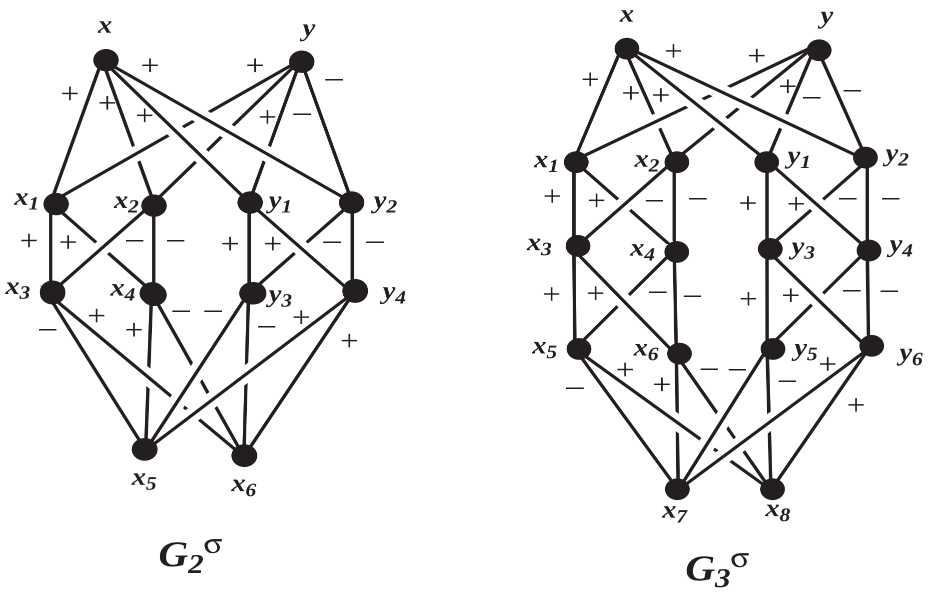

Figure 5: Signed Graphs G_2^σ and G_3^σ.

Proof. Suppose that G^σ has two distinct eigenvalues. By Theorem 2.7, G must be Q_4, H, F_1, F_2, G_1, G_2 or G_3. Assume that $G \cong Q_4$ or $G \cong H$. There are exactly two paths of length two between each pair of non-adjacent vertices. By Theorem 2.2, Q_4 and H have two distinct eigenvalues, if one of the path is positive and other path is negative. Then the number of positive paths and negative paths between each pair of non-adjacent vertices are equal and all of the quadrangles in graphs Q_4 and H are unbalanced. Now, in graphs F_1, F_2, G_1, G_2 and G_3, there are two paths of length two or four paths of length two between each pair of non-adjacent vertices, which the number of positive paths and negative paths of length two between each pair of non-adjacent vertices are equal by Theorem 2.2.

Conversely assume that $G \cong Q_4$ or $G \cong H$ and all of the quadrangles are unbalanced. Then there is only one path of length two and positive and one path of length two and negative between each pair of non-adjacent vertices. Then the number of positive paths and negative paths of length two between each pair of non-adjacent vertices is equal and according to Theorem 2.2, Q_4 and H have two distinct eigenvalues. Now, suppose that $G \cong F_1$, F_2, G_1, G_2 or G_3 and there are exactly one positive path and one negative path of length two or two positive paths and two negative paths of length two between each pair of non-adjacent vertices. Thus, the number of positive paths and negative paths of length two between each pair of non-adjacent vertices is equal. By Theorem 2.2 the graphs F_1, F_2, G_1, G_2 and G_3 have two distinct eigenvalues. □

Example 2.9. *Let F_1^σ be signed graphs as seen in Figures 4. Then it has two distinct eigenvalues. Let the rows of a signed adjacency matrix F_1^σ correspond successively vertices x, y, x_1, x_2, y_1, y_2, x_3, x_4, y_3 and y_4. Then we have,*

$$A(F_1^\sigma) = \begin{bmatrix} 0 & 0 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 \\ +1 & +1 & 0 & 0 & 0 & 0 & +1 & +1 & 0 & 0 \\ +1 & +1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 & 0 & 0 \end{bmatrix}.$$

Finally with a tedious calculation w have $[A(F_1^\sigma)]^2 = 4I_{10}$ *and its eigenvalues are* ± 2.

References

[1] F. Belardo and P. Petecki, Spectral characterizations of signed lollipop graphs, Linear Algebra Appl, 480 (2015) 144-167.
[2] F. Harary, R. Z. Norman and D. Cartwright, Structural models: An Introduction to the Theory of Directed Graphs, John Wiley and Sons, Inc., New York-London-Sydney, 1965.
[3] F. Ramezani, On the signed graphs with two distinct eigenvalues, arXiv: 1511.03511.
[4] T. Zaslavsky, Signed graphs, Discrete Appl. Math, 4 (1982), no 1, 47-74.
[5] T. Zaslavsky, Matrices in the theory of signed simple graphs, in: Advances in Discrete Mathematics and Applications, Mysore, 2008, in: Ramanujan Math. Soc. Lect. Notes Ser, 13 (2010) 207-229.