Extension of the Kantorovich Inequality for Positive Multilinear Mappings

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\textbf{Abstract.} It is known that the power function \(f(t) = t^2\) is not matrix monotone. Recently, it has been shown that \(t^2\) preserves the order in some matrix inequalities. We prove that if \(A = (A_1, \ldots, A_k)\) and \(B = (B_1, \ldots, B_k)\) are \(k\)-tuples of positive matrices with \(0 < m \leq A_i, B_i \leq M\) \((i = 1, \ldots, k)\) for some positive real numbers \(m < M\), then

\[
\Phi^2\left(A_1^{-1}, \ldots, A_k^{-1}\right) \leq \left(\frac{1 + v^k}{4v^k}\right)^2 \Phi^2(A_1, \ldots, A_k)
\]

and

\[
\Phi^2\left(\frac{A_1 + B_1}{2}, \ldots, \frac{A_k + B_k}{2}\right) \leq \left(\frac{1 + v^k}{4v^k}\right)^2 \Phi^2(A_1\#B_1, \ldots, A_k\#B_k),
\]

where \(\Phi\) is a unital positive multilinear mapping and \(v = \frac{M}{m}\) is the condition number of each \(A_i\).

1. Introduction

Throughout the paper, assume that \(\mathcal{M}_n := \mathcal{M}_{n}(\mathbb{C})\) is the algebra of all \(n \times n\) complex matrices and \(I\) denotes the identity matrix. A Hermitian matrix \(A\) is called positive (denoted by \(A \geq 0\)) if all of its eigenvalues are nonnegative. If in addition \(A\) is invertible, then \(A\) is called strictly positive (denoted by \(A > 0\)). For Hermitian matrices \(A, B \in \mathcal{M}_n\), the inequality \(A \leq B\) means that \(B - A \geq 0\). If \(m\) is a real scalar, then by \(m \leq A\) we mean that \(mI \leq A\).

Let \(J \subseteq \mathbb{R}\) be an interval. A continuous real function \(f : J \to \mathbb{R}\) is called matrix monotone if \(A \leq B\) implies that \(f(A) \leq f(B)\) for all Hermitian matrices \(A\) and \(B\) whose eigenvalues are in \(J\). A celebrated result of Löwner–Heinz (see for example [9, 10]) asserts that \(f(t) = t^r\) is matrix monotone for all \(0 \leq r \leq 1\). In fact the converse is also true, if \(f(t) = t^r\) is matrix monotone, then \(0 \leq r \leq 1\). This concludes that the power function \(f(t) = t^r\) does not preserve the matrix order in general except for \(0 \leq r \leq 1\). For example, \(A \leq B\) does not imply \(A^2 \leq B^2\). To see this, it is enough to set \(A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\).

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However, there have recently been some works in which some operator inequalities are squared. Moreover, it has been recently shown that the power function \( f(t) = t^r \) preserves the order in some matrix inequalities even if \( r \geq 1 \). In this section, we take a look at these works.

A linear mapping \( \Phi : M_n \to M_p \) is called positive if \( \Phi \) preserves the positivity, i.e., if \( A \geq 0 \) in \( M_n \), then \( \Phi(A) \geq 0 \) in \( M_p \) and \( \Phi \) is called unital if \( \Phi(I) = I \). Also \( \Phi \) is said to be strictly positive if \( \Phi(A) > 0 \) whenever \( A > 0 \).

A continuous real function \( f : J \to \mathbb{R} \) is said to be matrix convex if

\[
f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)
\]

for all Hermitian matrices \( A, B \) with eigenvalues in \( J \) and all \( \lambda \in [0, 1] \). Positive linear mappings have been used to characterize matrix convex and matrix monotone functions. For example, it is well-known that a continuous real function \( f : J \to \mathbb{R} \) is matrix convex if and only if the Choi–Davis–Jensen inequality \([10]\)

\[
\Phi(f(\Phi(A))) \leq \Phi(f(A))
\]

holds true for every unital positive linear mapping \( \Phi \) and every Hermitian matrix \( A \) whose eigenvalues are in \( J \). Two other special cases of this result are the Kadison inequality and the Choi inequality, see \([2, 10]\):

**Theorem 1.1.** If \( \Phi : M_n \to M_p \) is a unital positive linear mapping, then

1. **The Choi inequality** \( \Phi(A)^{-1} \leq \Phi\left(A^{-1}\right) \) (\( A > 0 \)).
2. **The Kadison inequality** \( \Phi(A)^2 \leq \Phi\left(A^2\right) \).

In what follows, assume that \( m \) and \( M \) are positive real numbers such that \( 0 < m < M \) and \( A, B \in M_n \) are matrices with \( 0 < m \leq A, B \leq M \) except where otherwise clearly indicated. Moreover, assume that

\[
\xi = \sqrt{\frac{(M+m)^2}{4mM}}.
\]

A counterpart to the Choi inequality (1) has been presented by Marshal and Olkin \([15]\) as follows:

\[
\Phi\left(A^{-1}\right) \leq \xi \Phi(A)^{-1}.
\]

A similar result for the Kadison inequality (see \([16]\)) holds true:

\[
\Phi(A^2) \leq \xi \Phi(A)^2.
\]

The constant \( \xi \) is known as the Kantorovich constant. In addition, the inequalities of type (2) and (3), which present reverse of some inequalities, are known as Kantorovich type inequalities. For a recent survey concerning Kantorovich type inequalities the reader is referred to \([17]\).

Regarding the possible squared version of (2), Lin \([13]\) noticed that the inequality

\[
\Phi(A) + Mm\Phi\left(A^{-1}\right) \leq M + m
\]

holds for every unital positive linear mapping \( \Phi \). The inequality (4) turns out to be a tool for squaring matrix inequalities. Using (4) Lin \([13]\) showed that (2) can be squared:

**Theorem 1.2.** \([13, \text{Theorem 2.8}]\) If \( \Phi : M_n \to M_p \) is a unital positive linear mapping, then

\[
\Phi\left(A^{-1}\right)^2 \leq \xi^2 \Phi(A)^{-2}.
\]

As pointed out by Fu and He \([5]\), the inequality (5) and the matrix monotonicity of \( f(t) = t^s \) \((0 \leq s \leq 1)\) imply that

\[
\Phi\left(A^{-1}\right)^r \leq \xi^r \Phi(A)^{-r}
\]

for every \( 0 \leq r \leq 2 \). In the case where \( r \geq 2 \), it was shown in \([5]\) that:
Theorem 1.3. [5, Theorem 3] For every \( r \geq 2 \)
\[
\Phi\left( A^{-1} \right)^r \leq \left( \frac{(M + m)^2}{4^r Mn} \right)^r \Phi(A)^r.
\] (7)

The matrix arithmetic–geometric mean inequality (the A-G mean inequality) (see for example [2, 10])
\[ A \# B \leq \frac{A + B}{2} \]
implies that
\[
\Phi(A \# B) \leq \Phi\left( \frac{A + B}{2} \right)
\]
for every unital positive linear mapping \( \Phi \).

A converse of this inequality reads as follows (see [8])
\[
\Phi\left( \frac{A + B}{2} \right) \leq \sqrt[2]{\xi} \Phi(A \# B) \leq \xi \Phi\left( \frac{A^{-1} + B^{-1}}{2} \right)^{-1}.
\] (8)

Lin [14] has tried to obtain an square version of (8) and proved that
\[
\Phi^2 \left( \frac{A + B}{2} \right) \leq \xi^2 \Phi^2(A \# B)
\] (9)
\[
\Phi^2 \left( \frac{A + B}{2} \right) \leq \xi^2 \left( \Phi(A) \# \Phi(B) \right)^2.
\]

In Section 2, we give an extension of (9) using positive multilinear mappings. As noticed in [5], utilizing the Löwner-Heinz inequality, (9) can be extended as
\[
\Phi^r \left( \frac{A + B}{2} \right) \leq \xi^r \Phi^r(A \# B)
\] (10)
\[
\Phi^r \left( \frac{A + B}{2} \right) \leq \xi^r \left( \Phi(A) \# \Phi(B) \right)^r.
\]
for every \( 0 \leq r \leq 2 \). In the case where \( r \geq 2 \), Fu and He [5] showed that

Theorem 1.4. [5, Theorem 4] If \( r \geq 2 \), then
\[
\Phi^r \left( \frac{A + B}{2} \right) \leq \left( \frac{(M + m)^2}{4^r Mn} \right)^r \Phi^r(A \# B)
\] (11)
\[
\Phi^r \left( \frac{A + B}{2} \right) \leq \left( \frac{(M + m)^2}{4^r Mn} \right)^r \left( \Phi(A) \# \Phi(B) \right)^r.
\]

It is well-known that the arithmetic mean is the biggest and the harmonic mean is the smallest among symmetric means (see [11]). Fu and Hoa in [6] extended the inequalities (10) and (11) to arbitrary means between harmonic and arithmetic means. If \( \sigma, \tau \) be two arbitrary means between harmonic and arithmetic means, then for every positive unital linear mapping \( \Phi \) and \( 0 \leq r \leq 2 \) they proved that
\[
\Phi^r(\sigma A B) \leq \xi^r \Phi^r(\tau A B)
\] (12)
\[
\Phi^r(\sigma A B) \leq \xi^r \left( \Phi(A) \# \Phi(B) \right)^r.
\]

Also for \( r \geq 2 \) they showed that
\[
\Phi^r(\sigma A B) \leq \left( \frac{(M + m)^2}{4^r Mn} \right)^r \Phi^r(\tau A B)
\] (13)
\[
\Phi^r(\sigma A B) \leq \left( \frac{(M + m)^2}{4^r Mn} \right)^r \left( \Phi(A) \# \Phi(B) \right)^r.
Similar results can be found in [18].

Let $G(A_1, \cdots, A_k)$ denote the Ando–Li–Mathias geometric mean of strictly positive $A_i \in \mathcal{M}_n$ ($i = 1, \ldots, k$) [1]. It is known that it satisfies in the arithmetic-geometric-Harmonic mean inequality:

$$
\left( \frac{A_1^{-1} + \cdots + A_k^{-1}}{k} \right)^{-1} \leq G(A_1, \cdots, A_k) \leq \frac{A_1 + \cdots + A_k}{k}.
$$

(14)

The converse of (14) is a Kantorovich type inequality (see [7]) which reads as

$$
\frac{A_1 + \cdots + A_k}{k} \leq \xi G(A_1, \cdots, A_k) \quad \text{and} \quad G(A_1, \cdots, A_k) \leq \xi \left( \frac{A_1^{-1} + \cdots + A_k^{-1}}{k} \right)^{-1}
$$

(15)

where $A_i \in \mathcal{M}_n$ with $0 < m \leq A_i \leq M$ ($i = 1, \ldots, k$).

Lin [14, Theorem 3.2] proved that (15) can be squared:

$$
\left( \frac{A_1 + \cdots + A_k}{k} \right)^2 \leq \xi^2 G(A_1, \cdots, A_k)^2.
$$

(16)

In almost all of the above results, the following two key lemmas have been utilized:

**Lemma 1.5.** [3] Let $A, B \in \mathcal{M}_n$. If $A, B \geq 0$, then

$$
\|AB\| \leq \frac{1}{4} \|A + B\|^2
$$

for every unitarily invariant norm $\| \cdot \|$ on $\mathcal{M}_n$.

**Lemma 1.6.** [2, Theorem 1.6.9] Let $A, B \in \mathcal{M}_n$. If $A, B \geq 0$ and $1 \leq r < \infty$, then

$$
\|A^r + B^r\| \leq \|(A + B)^r\|
$$

(17)

for every unitarily invariant norm $\| \cdot \|$ on $\mathcal{M}_n$.

2. Positive Multilinear Mapping Inequalities

A mapping $\Phi : \mathcal{M}_n^k := \mathcal{M}_n \times \cdots \times \mathcal{M}_n \rightarrow \mathcal{M}_p$ is said to be multilinear if it is linear in each of its variable. A multilinear mapping $\Phi : \mathcal{M}_n^k \rightarrow \mathcal{M}_p$ is called positive if $A_i \geq 0$ for $i = 1, \cdots, k$ implies that $\Phi(A_1, \cdots, A_k) \geq 0$ and $\Phi$ is called unital if $\Phi(I, \ldots, I) = I$. [4].

Recently, an extension of the Choi inequality (1) has been presented in [4] for positive multilinear mappings:

**Lemma 2.1.** If $\Phi : \mathcal{M}_n^k \rightarrow \mathcal{M}_p$ is a unital positive multilinear mapping, then

$$
\Phi(A_1, \cdots, A_k)^{-1} \leq \Phi(A_1^{-1}, \cdots, A_k^{-1})
$$

for all strictly positive matrices $A_i \in \mathcal{M}_n$ ($i = 1, \ldots, k$).

Moreover, a multilinear version of (2), which is a Kantorovich type inequality for positive multilinear mappings, has also been presented in [4] as

**Lemma 2.2.** [4, Corollary 5.3] If $A_i \in \mathcal{M}_n$ ($i = 1, \ldots, k$) are positive matrices with $0 < m \leq A_i \leq M$ for some positive real numbers $m < M$ and $\Phi : \mathcal{M}_n^k \rightarrow \mathcal{M}_p$ is a unital positive multilinear mapping, then

$$
\Phi(A_1^{-1}, \ldots, A_k^{-1}) \leq \frac{(1 + v)^2}{4v} \Phi(A_1, \ldots, A_k)^{-1},
$$

(18)

where $v = \frac{M}{m}$ is the condition number of each $A_i$. 
Unfortunately, there is an error in the above lemma. The Kantorovich constant \( \frac{(1+\epsilon^2)}{4\epsilon^2} \) does not work in (18) in general (see Remark 2.7). We give a correct form of (18) in the next lemma. The proof is quite similar to that of [4, Corollary 5.3] and we omit the proof.

**Lemma 2.3.** Let \( A_i \in M_n \) \((i = 1, \cdots, k)\) with \(0 < m \leq A_i \leq M\) for some positive real numbers \(m \leq M\). If \( \Phi : M_n^k \rightarrow M_p \) is a unital positive multilinear mapping, then

\[
\Phi(A_1^{-1}, \cdots, A_k^{-1}) \leq \frac{(1+\epsilon^2)}{4\epsilon^2} \Phi(A_1, \cdots, A_k)^{-1},
\]

where \( \epsilon = \frac{M}{m} \) is the condition number of each \( A_i \).

The following key lemma which is a direct conclusion of [16, Theorem 2.1] has an important role in obtaining our main results.

**Lemma 2.4.** Let \( f \) be a positive strictly convex twice differentiable function on \([m, M]\) with \(0 < m < M\) and let \( \Phi \in M_\alpha \) such that \( \sum_{i=1}^k C_i C_i = I \). If \( A_i \in M_n \) with \(0 < m \leq A_i \leq M \) \((i = 1, \cdots, k)\), then

\[
\sum_{i=1}^k C_i f(A_i) C_i \leq \alpha f \left( \sum_{i=1}^k C_i A_i C_i \right),
\]

where \( \alpha = \frac{f(M) - f(m)}{M - m} \) and \( b_i = \frac{M f(m) - f(m)}{M - m} \).

**Lemma 2.5.** Let \( A_i \in M_n \) \((i = 1, \cdots, k)\) with \(0 < m \leq A_i \leq M\) for some positive real numbers \(m \leq M \) \((i = 1, \cdots, k)\). If \( \Phi : M_n^k \rightarrow M_p \) is a unital positive multilinear mapping, then

\[
\Phi(A_1', \cdots, A_k') \leq a_r \Phi(A_1, \cdots, A_k) + b_r I
\]

for all \( r \geq 1 \) and \( r \leq 0 \) in which

\[
a_r = \frac{M^{kr} - m^{kr}}{M^k - m^k}, \quad b_r = \frac{M^r m^{kr} - m^r M^{kr}}{M^k - m^k}.
\]

**Proof.** Assume that \( A_i = \sum_{j=1}^n \lambda_{ij} P_{ij} \) \((i = 1, \cdots, k)\) is the spectral decomposition of each \( A_i \in M_k \) for which

\[\sum_{j=1}^n P_{ij} = I.\]

Put \( C(j_1, \cdots, j_k) := (\Phi(P_{j_1}, \cdots, P_{j_k}))^2 \) so that \( \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n C(j_1, \cdots, j_k) C(j_1, \cdots, j_k) = I \).

It is well known that \( f(t) = t^r \) is a positive strictly convex differentiable function on \((0, \infty)\). Then

\[
\Phi(A_1', \cdots, A_k') = \Phi \left( \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n P_{ij_1}, \cdots, \sum_{j_k=1}^n P_{ij_k} \right)
\]

\[
= \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n \Phi(P_{ij_1}, \cdots, P_{ij_k}) \quad \text{(by multilinearity of } \Phi)\\
= \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n C(j_1, \cdots, j_k) (\lambda_{ij_1} \lambda_{ij_2} \cdots \lambda_{ij_k}) C(j_1, \cdots, j_k)
\]

\[
\leq a_r \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n C(j_1, \cdots, j_k) (\lambda_{ij_1} \lambda_{ij_2} \cdots \lambda_{ij_k}) C(j_1, \cdots, j_k) + b_r I \quad \text{(by Lemma 2.4)}\\
= a_r \Phi(A_1, \cdots, A_k) + b_r I.
\]

\[\square\]
Now we give our first main result which is an square version of (19).

**Theorem 2.6.** Let $A_i \in \mathcal{M}_n$ with $0 < m \leq A_i \leq M$ for some positive real numbers $m < M$ ($i = 1, \ldots, k$). If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unitary positive multilinear mapping, then

$$
\Phi^2 \left( A_1^{-1}, \ldots, A_k^{-1} \right) \leq \left( \frac{(1 + \nu^k)^2}{4\nu^k} \right)^2 \Phi^{-2}(A_1, \ldots, A_k)
$$

(22)

in which $\nu = \frac{M}{m}$ is the condition number of each $A_i$.

**Proof.** Assume that the convex function $f$ is defined on $(0, \infty)$ by $f(t) = t^{-1}$. Applying Lemma 2.5 for $r = -1$ we get

$$
a_r = -\frac{1}{M^k m^r}, \quad b_r = \frac{M^k + m^k}{M^k m^k}
$$

and

$$
\Phi \left( A_1^{-1}, \ldots, A_k^{-1} \right) \leq -\frac{1}{M^k m^k} \Phi(A_1, \ldots, A_k) + \frac{M^k + m^k}{M^k m^k} I.
$$

It follows that

$$
\Phi(A_1, \ldots, A_k) + M^k m^k \Phi \left( A_1^{-1}, \ldots, A_k^{-1} \right) \leq M^k + m^k.
$$

(23)

On the other hand Lemma 1.5 yields that

$$
M^k m^k \left\| \Phi(A_1, \ldots, A_k) \Phi \left( A_1^{-1}, \ldots, A_k^{-1} \right) \right\|
\leq \frac{1}{4} \left\| \Phi(A_1, \ldots, A_k) + M^k m^k \Phi \left( A_1^{-1}, \ldots, A_k^{-1} \right) \right\|^2.
$$

(24)

Combining (23) and (24) we obtain

$$
\left\| \Phi(A_1, \ldots, A_k) \Phi \left( A_1^{-1}, \ldots, A_k^{-1} \right) \right\| \leq \frac{(M^k + m^k)^2}{4M^k m^k} = \frac{(1 + \nu)^2}{4\nu^k}.
$$

Therefore

$$
\Phi^2 \left( A_1^{-1}, \ldots, A_k^{-1} \right) \leq \left( \frac{(1 + \nu)^2}{4\nu^k} \right)^2 \Phi^{-2}(A_1, \ldots, A_k).
$$

\[\square\]

**Remark 2.7.** It should be remarked that the number $k$ in the constant $\frac{(1 + \nu)^2}{4\nu}$ is the best possible in (19) and so in (22). To see this, consider the bilinear mapping $\Phi : \mathcal{M}_n^2 \to \mathcal{M}_p$ defined by $\Phi(A, B) = \langle x, \text{diag}(A) \text{diag}(B)x \rangle I_p$, where $x = [1/\sqrt{2}, 1/\sqrt{2}]^t \in \mathbb{C}^2$. If $A = B = \text{diag}(1, 2)$ so that $\nu = 2$, then

$$
\Phi \left( A^{-1}, B^{-1} \right) = 0.625 I_p = \frac{(1 + \nu^2)^2}{4\nu^2} \Phi(A, B)^{-1}.
$$

Let $A_i \in \mathcal{M}_n$ with $0 < m \leq A_i \leq M$ for some real numbers $m < M$ ($i = 1, \ldots, k$). If $\Phi : \mathcal{M}_n^k \to \mathcal{M}_p$ is a unitary positive multilinear mapping, then matrix monotonicity of $f(t) = t^s$ ($0 \leq s \leq 1$) and (22) imply that

$$
\Phi^r \left( A_1^{-1}, \ldots, A_k^{-1} \right) \leq \left( \frac{(1 + \nu)^2}{4\nu^k} \right)^r \Phi^{-r}(A_1, \ldots, A_k)
$$

for every $0 \leq r \leq 2$. By a similar technique used in the proof of Theorem 2.6 and Applying Lemma 1.6 one can obtain the following result as a multilinear version of (7).
Theorem 2.8. Let \( A_i \in \mathcal{M}_n \) with \( 0 < m \leq A_i \leq M \) for some positive real numbers \( m < M \) \((i = 1, \ldots, k)\). If \( \Phi : \mathcal{M}_n \to \mathcal{M}_n \) is a unital positive multilinear mapping and \( r > 2 \), then

\[
\Phi^r \left( A_1^{-1}, \ldots, A_k^{-1} \right) \leq \left( \frac{1 + \rho^r}{4 \rho^r} \right)^r \Phi^{-1}(A_1, \ldots, A_k).
\]

In [12] Lim and Pálfia established the notion of the matrix power means for \( k \) positive definite matrices \((k \geq 3)\). First we recall some basic properties of matrix power means.

Assume that \( A = (A_1, \ldots, A_k) \) is a \( k \)-tuple of strictly positive matrices in \( \mathcal{M}_n \) and \( \omega = (\omega_1, \ldots, \omega_k) \) is a \( k \)-tuple of positive scalars with \( \sum_k \omega_i = 1 \). The matrix power mean of \( A_1, \ldots, A_k \) [12], denoted by \( P_t(\omega; A) \), is the unique positive invertible solution of the non-linear matrix equation

\[
X = \sum_{i=1}^k \omega_i (X^{t} A_i),
\]

where \( t \in (0, 1] \) and \( X \neq A \). \( A^t \) is the \( t \)-weighted geometric mean of strictly positive matrices \( X \) and \( A \). If \( t \in [-1, 0) \), then put \( P_t(\omega; A) := P_{-t}(\omega; A^{-1})^{-1} \), where \( A^{-1} = (A_1^{-1}, \ldots, A_k^{-1}) \). The matrix power mean \( P_t(\omega; A) \) interpolates between the weighted harmonic and arithmetic means. In particular, it satisfies the inequality

\[
\left( \sum_{i=1}^k \omega_i A_i \right)^{-1} \leq P_t(\omega, A) \leq \sum_{i=1}^k \omega_i A_i \quad (t \in [-1, 1 \setminus \{0\}).
\]

The Karcher mean of \( A_1, \ldots, A_k \), denoted by \( G(\omega; A) \), is the unique positive invertible solution of the Karcher equation

\[
\sum_{i=1}^k \omega_i \log(X^{-1/2} A_i X^{-1/2}) = 0.
\]

It is known that the Karcher mean coincides with the limit of matrix power means as \( t \to 0 \). For more information on the matrix power mean the reader is referred to [12]. We are going to present an extension of (9) for positive multilinear mappings.

Theorem 2.9. Let \( \Phi : \mathcal{M}_n \to \mathcal{M}_n \) be a unital positive multilinear mapping and \( A^{(i)} = (A_1^{(i)}, \ldots, A_q^{(i)}) \) \((i = 1, \ldots, k)\), where \( 0 < m \leq A_i^{(i)} \leq M \) for every \( i = 1, \ldots, k \) and every \( j = 1, \ldots, q \) and some positive real numbers \( m < M \). Let \( \omega^{(i)} = (\omega_1^{(i)}, \ldots, \omega_q^{(i)}) \) be a weight vector of positive scalars with \( \sum_q \omega^{(i)}_j = 1 \) for every \( i = 1, \ldots, k \). If \( t \in (0, 1] \), then

\[
\Phi^r \left( \sum_{j=1}^q \omega^{(i)}_j A_j^{(i)}, \ldots, \sum_{j=1}^q \omega^{(i)}_j A_j^{(i)} \right)
\leq \left( \frac{1 + \rho^r}{4 \rho^r} \right)^r \Phi^r \left( P_t(\omega^{(1)}; A^{(1)}), \ldots, P_t(\omega^{(q)}; A^{(q)}) \right),
\]

where \( v = \frac{\rho^r}{m} \) is the condition number of each \( A^{(i)} \).

Proof. Utilizing Lemma 1.5 we have

\[
M^t \left\| \Phi \left( \sum_{j=1}^q \omega^{(i)}_j A_j^{(i)}, \ldots, \sum_{j=1}^q \omega^{(i)}_j A_j^{(i)} \right) \Phi^{-1} \left( P_t(\omega^{(1)}; A^{(1)}), \ldots, P_t(\omega^{(q)}; A^{(q)}) \right) \right\|
\leq \frac{1}{4} \left( \Phi \left( \sum_{j=1}^q \omega^{(i)}_j A_j^{(i)}, \ldots, \sum_{j=1}^q \omega^{(i)}_j A_j^{(i)} \right) + M^t \Phi^{-1} \left( P_t(\omega^{(1)}; A^{(1)}), \ldots, P_t(\omega^{(q)}; A^{(q)}) \right) \right)^2.
\]
Moreover,
\[
\Phi \left( \sum_{j=1}^{q} \omega_j A_j^{(1)}, \ldots, \sum_{j=1}^{q} \omega_j A_j^{(q)} \right) + M^4 m^4 \Phi^{-1} \left( P_i \left( \omega A^{(1)}; \ldots, P_i \left( \omega A^{(q)} \right) \right) \right)
\]
\[
\leq \Phi \left( \sum_{j=1}^{q} \omega_j A_j^{(1)}, \ldots, \sum_{j=1}^{q} \omega_j A_j^{(q)} \right) + M^4 m^4 \Phi \left( \sum_{j=1}^{q} \omega_j \left( A_j^{(1)} \right)^{-1}, \ldots, \sum_{j=1}^{q} \omega_j \left( A_j^{(q)} \right)^{-1} \right)
\]
(by Lemma 2.1)
\[
= \Phi \left( \sum_{j=1}^{q} \omega_j A_j^{(1)}, \ldots, \sum_{j=1}^{q} \omega_j A_j^{(q)} \right) + M^4 m^4 \Phi \left( \left( A_{1}^{(1)} \right)^{-1}, \ldots, \left( A_{q}^{(q)} \right)^{-1} \right)
\]
(by 25)
\[
\leq \sum_{j=1}^{q} \sum_{h=1}^{4} \omega_j A_j^{(1)} \left( A_{1}^{(1)} \right)^{-1} \leq M^4 + m^4
\]
(23)
\[
\leq M^4 + m^4.
\]
Therefore
\[
\left\| \Phi \left( \sum_{j=1}^{q} \omega_j A_j^{(1)}, \ldots, \sum_{j=1}^{q} \omega_j A_j^{(q)} \right) \Phi^{-1} \left( P_i \left( \omega A^{(1)}; \ldots, P_i \left( \omega A^{(q)} \right) \right) \right) \right\|
\]
\[
\leq \frac{\left( M^4 + m^4 \right)^2}{4 M^4 m^4},
\]
which is equivalent to (26).

Remark 2.10. Let \( \Phi : M_n \rightarrow M_p \) be a unital positive linear mapping and let \( A = (A_1, \ldots, A_q) \) is a \( q \)-tuple of matrices in \( M_n \) with \( 0 < m \leq A_i \leq M \) for some positive real numbers \( m < M \). If \( \omega = (\omega_1, \ldots, \omega_q) \) is a weight vector such that \( \omega_i \geq 0 \) \((i = 1, \ldots, q)\) with \( \sum_{j=1}^{q} \omega_i = 1 \), then it follows from Theorem 2.9 that
\[
\Phi^2 \left( \sum_{j=1}^{q} \omega_j A_i \right) \leq \frac{(1 + \varphi)^2}{4 \varphi} \Phi^2 \left( P_i \left( \omega A \right) \right),
\]
(27)
where \( \varphi = \frac{m}{M} \). Tending \( t \) to zero, we get
\[
\Phi^2 \left( \sum_{j=1}^{q} \omega_j A_i \right) \leq \frac{(1 + \varphi)^2}{4 \varphi} \Phi^2 \left( G \left( \omega; A \right) \right),
\]
(28)
where \( G \left( \omega; A \right) \) is the Karcher mean of \( A_1, \ldots, A_q \). Inequality (28) is an extension of (9). Moreover, it follows from (27) that the Kantorovich inequality
\[
\Phi \left( \sum_{j=1}^{q} \omega_j A_i \right) \leq \frac{(M + m)^2}{4 M m} \Phi \left( P_i \left( \omega A \right) \right)
\]
holds true.

Remark 2.11. Define a linear mapping \( \Theta : M_n \oplus \cdots \oplus M_n \rightarrow M_n \oplus \cdots \oplus M_n \) by
\[
\Theta \left( \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_q
\end{pmatrix} \right) = \left( \sum_{j=1}^{q} \omega_j A_i \right) \oplus I_n
\]
where $L_i$ is the identity matrix in $M_i$. Then $\Theta$ is a unital positive linear mapping. Applying (5) to $\Theta$ concludes that
\[
\Theta^2 \left( \begin{array}{cccc}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_q
\end{array} \right) \leq \left( \frac{(M + m)^2}{4Mm} \right)^2 \Theta^2 \left( \begin{array}{cccc}
A_1^{-1} & 0 & \cdots & 0 \\
0 & A_2^{-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_q^{-1}
\end{array} \right).
\]

That is
\[
\left( \sum_{i=1}^q \omega_i A_i \right)^2 \leq \left( \frac{(M + m)^2}{4Mm} \right)^2 \left( \sum_{i=1}^q \omega_i A_i^{-1} \right)^{-2}.
\]

A special case of Theorem 2.9 gives an extension of (9) for multilinear mappings:

**Corollary 2.12.** Suppose that $A_i, B_i \in M_i$ with $0 < m \leq A_i, B_i \leq M$ for some positive real numbers $m < M$ ($i = 1, \ldots, k$). If $\Phi : M_k^p \to M_q$ is a unital positive multilinear mapping, then
\[
d^2 \left( \frac{A_1 + B_1}{2}, \ldots, \frac{A_k + B_k}{2} \right)^2 \leq \left( \frac{(1 + \nu)^2}{4\nu^2} \right)^2 \Phi^2 (A_1 \# B_1, \ldots, A_k \# B_k),
\]
where $\nu = \frac{M}{m}$ is the condition number of each $A_i$ and $B_i$.

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**References**