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Remarks on "Some New Fixed Point Theorems for Contractive and Nonexpansive Mappings"

Rajendra Pant[a], Hemant Kumar Nashine[b], Zoran Kadelburg[c]

[a] *Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India,*
& School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, South Africa
[b] *Department of Mathematics, Texas A & M University - Kingsville - 78363-8202, Texas, USA,*
& Department of Mathematics, Amity School of Applied Sciences Amity University, Chhattisgarh, Raipur - 493225, Chhattisgarh, INDIA
[c] *University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Serbia*

Abstract. Pant [Filomat 28 (2014), no. 2, 313–317] obtained some fixed point results in ultrametric spaces. Unfortunately, the proofs of main results had flaws. We present corrected proofs of his theorems for single valued mappings and correct formulations and proofs in the multivalued case.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. The mapping T is said to be:

(A) a contraction if $d(Tx, Ty) \leq kd(x, y)$ for some $k \in [0, 1)$ and all $x, y \in X$;
(B) contractive if $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$; and
(C) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.

It is well known that *every contraction on a complete metric space has a unique fixed point* [Banach contraction theorem (BCT)]. However, the contractive and nonexpansive mappings need not have a fixed point in a complete metric space. For example the translation mapping $Tx = x + c$ (on any normed space) is nonexpasive but fixed point free. Similarly, the following example illustrates the fact about the contractive mappings.

Example 1.1. [13]. Let $X = (-\infty, +\infty)$ be endowed with the usual metric and $T : X \to X$ be defined by

$$Tx = x + \frac{1}{1 + e^x}$$

for all $x \in X$. Notice that X is complete and T is a contractive mapping but T does not have a fixed point.

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Email addresses: pant.rajendra@gmail.com (Rajendra Pant), drhknashine@gmail.com (Hemant Kumar Nashine), kadelbur@matf.bg.ac.rs (Zoran Kadelburg)

In [4], Edelstein proved that *every contractive mapping on a compact metric space has a unique fixed point.*

On the other hand, study of existence of fixed points of nonexpansive mappings was initiated by Browder [3], Göhde [8] and Kirk [9], independently in 1965 (cf. [7]).

Theorem 1.2. *Every nonexpansive mapping on a nonempty, compact and convex subset K of a Banach space E has a fixed point.*

In 2008, Suzuki [18] obtained the following remarkable generalization of the BCT:

Theorem 1.3. *Let X be a complete metric space and T be a mapping on X. Define a nondecreasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \le r \le (\sqrt{5}-1)/2 \\ (1-r)r^{-2}, & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-\frac{1}{2}} \\ (1+r)^{-1}, & \text{if } 2^{-\frac{1}{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that for all $x, y \in X$,

$$\theta(r)d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le r\, d(x,y). \tag{1}$$

Then T has a unique fixed point in X.

Generalizing nonexpansive mappings, Suzuki [19], introduced the following notion of *condition* (C) and obtained some fixed point theorems for mappings satisfying this condition.

Definition 1.4. *Let K be a nonempty subset of a metric space X. A mapping $T : K \to K$ is said to satisfy the condition (C) if for all $x, y \in K$,*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le d(x,y). \tag{2}$$

The mappings satisfying condition (C) are also known as Suzuki-type generalized nonexpansive mappings or simply generalized nonexpansive mappings. We note that every nonexpansive mapping is a generalized nonexpansive mapping but the converse is not true.

In a subsequent paper, Suzuki [20] introduced a new type of contractive mappings and obtained the following generalization of a result of Edelstein [4].

Theorem 1.5. *Let X be a compact metric space and $T : X \to X$ be a mapping. Assume that for all $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) < d(x,y). \tag{3}$$

Then T has a unique fixed point.

Remark 1.6. We remark that contractions, contractive and nonexpansive mappings are continuous on their domains. However, Suzuki type contraction, contractive and nonexpansive mappings need not be.

As noted above, for the existence of fixed points of contractive type mappings generally the domain need to be compact and for nonexpansive type mappings convex and compact. To weaken the assumptions of compactness and convexity, Petalas and Vidalis [13] replaced the domains of mappings by spherically complete ultrametric spaces and obtained certain fixed point theorems for contractive and nonexpansive mappings.

A metric space X is said to be *ultrametric* if the triangle inequality is replaced by the strong triangle inequality, i.e.,

$$d(x,y) \le \max\{d(x,z), d(y,z)\}$$

for all $x, y, z \in X$.

Example 1.7. Every discrete metric space is an ultrametric space.

An ultrametric space X is spherically complete if every descending collections of closed balls in X has nonempty intersection. Further, an ultrametric space X is compact if every open cover of X is reducible to a finite subcover. We note that:

1. Every compact ultrametric space is spherically complete but the converse is not true.
2. Every spherically complete ultrametric space is complete but the converse is not true in general.

Example 1.8. Every discrete space with infinitely many points is spherically complete but not compact.

Example 1.9. Define an ultrametric d on $\mathbb{N}$ (naturals) as follows. For $m, n \in \mathbb{N}$,

$$d(m,n) = \begin{cases} 1 + 2^{-\min\{m,n\}}, & \text{if } m \neq n \\ 0, & \text{if } m = n. \end{cases}$$

The topology induced by d is then the discrete topology on $\mathbb{N}$, and the Cauchy sequences with respect to d are exactly the sequences which are eventually constant. Therefore $(\mathbb{N}, d)$ is complete. Now consider the chain of balls B_n of the form $\{m \in \mathbb{N} : d(m,n) \leq 1 + 2^{-n}\}$. Then $B_n = \{m : m \geq n\}$ for all $n \in \mathbb{N}$. But $\bigcap_n B_n = \emptyset$. This shows that $(\mathbb{N}, d)$ is not spherically complete.

In 1993, Petalas and Vidalis [13] obtained following theorems:

Theorem 1.10. *Let X be a spherically complete ultrametric space and $T : X \to X$ be a contractive mapping. Then T has a unique fixed point.*

Theorem 1.11. *Let X be a spherically complete ultrametric space and $T : X \to X$ be a nonexpansive mapping. Then either T has at least one fixed point or there exists a ball B of radius $r > 0$ such that $T : B \to B$ and for which $d(b, Tb) = r$ for each $b \in B$.*

A number of extensions and generalizations of Theorems 1.10 and 1.11 have appeared in [5, 6, 10–12, 15–17] and elsewhere.

2. Single Valued Mappings

In this section, we present corrected proofs of main results of [12] for single-valued mappings. We begin with the following lemma, which is modelled on the pattern of [19, Lemma 5].

Lemma 2.1. *Let X be an ultrametric space and $T : X \to X$ be a generalized nonexpansive mapping. Then for all $x, y \in X$:*

(a) $d(Tx, T^2x) \leq d(x, Tx)$;

(b) either $\frac{1}{2}d(x, Tx) \leq d(x, y)$ or $\frac{1}{2}d(T^2x, Tx) \leq d(Tx, y)$;

(c) either $d(Tx, Ty) \leq d(x, y)$ or $d(T^2x, Ty) \leq d(Tx, y)$.

Proof. Since $\frac{1}{2}d(x, Tx) \leq d(x, Tx)$, by (2) it follows that $d(Tx, T^2x) \leq d(x, Tx)$. This proves (a). In order to prove (b), arguing by contradiction, we assume that

$$\frac{1}{2}d(x, Tx) > d(x, y) \text{ and } \frac{1}{2}d(Tx, T^2x) > d(Tx, y).$$

Then by the strong triangle inequality and (a), we have

$$\begin{aligned} d(x, Tx) &\leq \max\{d(x, y), d(y, Tx)\} \\ &< \max\left\{\frac{1}{2}d(x, Tx), \frac{1}{2}d(Tx, T^2x)\right\} \\ &\leq \max\left\{\frac{1}{2}d(x, Tx), \frac{1}{2}d(x, Tx)\right\} = \frac{1}{2}d(x, Tx), \end{aligned}$$

which is a contradiction. This proves (b). Now (c) directly follows from (b). □

The following theorem is Corollary 2.3 in [12].

Theorem 2.2. *Let X be a spherically complete ultrametric space and $T : X \to X$ be a mapping satisfying the condition* (3). *Then T has a unique fixed point.*

Proof. We denote by $\mathcal{B}_a := \mathcal{B}(a, r)$ the closed balls centered at a with radius $r = d(a, Ta)$. Let $\mathcal{A}$ be the collection of these balls for all $a \in X$. The relation introduced by

$$\mathcal{B}_a \le \mathcal{B}_b \text{ iff } \mathcal{B}_b \subseteq \mathcal{B}_a$$

is a partial order. Let $\mathcal{A}_1$ be a totally ordered subfamily of $\mathcal{A}$. From the spherical completeness of X, we have

$$\bigcap_{\mathcal{B}_a \in \mathcal{A}_1} \mathcal{B}_a := B \neq \emptyset.$$

Let $b \in B$ and $\mathcal{B}_a \in \mathcal{A}_1$. Then if $x \in \mathcal{B}_b$,

$$d(x, b) \le d(b, Tb).$$

By Lemma 2.1 (c), either $d(Ta, Tb) \le d(a, b)$ or $d(T^2a, Tb) \le d(Ta, b)$ holds for all $a, b \in X$. In the case $d(Ta, Tb) \le d(a, b)$, the above inequality reduces to

$$\begin{aligned} d(x, b) \le d(b, Tb) &\le \max\{d(b, a), d(a, Ta), d(Ta, Tb)\} \\ &\le \max\{d(b, a), d(a, Ta), d(a, b)\} \\ &= \max\{d(a, b), d(a, Ta)\}. \end{aligned}$$

In the other case when $d(T^2a, Tb) \le d(Ta, b)$, we have

$$\begin{aligned} d(x, b) \le d(b, Tb) &\le \max\{d(b, a), d(a, Ta), d(T^2a, Ta), d(T^2a, Tb)\} \\ &\le \max\{d(b, a), d(a, Ta), d(a, Ta), d(Ta, b)\} \\ &\le \max\{d(b, a), d(a, Ta), d(a, Ta), d(Ta, a), d(a, b)\} \\ &= \max\{d(a, b), d(a, Ta)\}. \end{aligned}$$

Therefore in both cases, we get

$$d(x, b) \le d(b, Tb) \le \max\{d(a, b), d(a, Ta)\}. \tag{4}$$

Now for $x \in \mathcal{B}_b$,

$$d(x, a) \le \max\{d(a, b), d(b, x)\}.$$

By the fact that $d(a, b) \le d(a, Ta)$ and (4), we get

$$d(x, a) \le \max\{d(a, b), d(b, x)\} \le d(a, Ta).$$

Hence $x \in \mathcal{B}_a$ and $\mathcal{B}_b \subseteq \mathcal{B}_a$ for every $\mathcal{B}_a \in \mathcal{A}_1$. Thus $\mathcal{B}_b$ is an upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. By Zorn's lemma, $\mathcal{A}$ has a maximal element, say $\mathcal{B}_z$, for some $z \in X$. We shall show that $z = Tz$. Suppose $z \neq Tz$. Since $\frac{1}{2}d(z, Tz) < d(z, Tz)$ for all $z \in X$, by (3), we get

$$d(Tz, T^2z) < d(z, Tz)$$

and by $Tz \in \mathcal{B}_{Tz} \cap \mathcal{B}_z$, we have $\mathcal{B}_{Tz} \subseteq \mathcal{B}_z$. Since $z \neq Tz$, $\mathcal{B}_{Tz} \subsetneq \mathcal{B}_z$, and this contradicts the maximality of $\mathcal{B}_z$. Therefore T has a fixed point. To prove the uniqueness, if possible, let z and u be two distinct fixed points of T. Then $\frac{1}{2}d(z, Tz) < d(z, u)$. Now by (3),

$$d(z, u) = d(Tz, Tu) < d(u, z),$$

a contradiction unless $z = u$. □

The following theorem is Corollary 3.3 in [12].

Theorem 2.3. *Let X be a spherically complete ultrametric space and $T : X \to X$ be a generalized nonexpansive mapping. Then either T has at least one fixed point or there exists a ball B of radius $r > 0$ such that $T : B \to B$ and for which $d(b, Tb) = r$ for each $b \in B$.*

Proof. Let $\mathcal{B}_a$ and $\mathcal{A}$ be as in the proof of Theorem 2.2. We find a maximal element $\mathcal{B}_z$ of $\mathcal{A}$. By Lemma 2.1 (c), we have either $d(Tb, Tz) \le d(b, z)$ or $d(T^2z, Tb) \le d(Tz, b)$ for all $b, z \in X$. In the case when $d(Tb, Tz) \le d(b, z)$ for any $b \in \mathcal{B}_z$, we have

$$\begin{aligned} d(b, Tb) &\le \max\{d(b, z), d(z, Tz), d(Tz, Tb)\} \\ &\le \max\{d(b, z), d(z, Tz), d(z, b)\} \\ &= \max\{d(b, z), d(z, Tz)\}. \end{aligned}$$

In the other case when $d(T^2b, Tz) \le d(Tb, z)$ for any $b \in \mathcal{B}_z$, we have

$$\begin{aligned} d(b, Tb) &\le \max\{d(b, z), d(z, Tz), d(T^2z, Tz), d(T^2z, Tb)\} \\ &\le \max\{d(b, z), d(z, Tz), d(z, Tz), d(Tz, b)\} \\ &\le \max\{d(b, z), d(z, Tz), d(z, Tz), d(Tz, z), d(z, b)\} \\ &= \max\{d(b, z), d(z, Tz)\}. \end{aligned}$$

Therefore, in both cases, we have

$$d(b, Tb) \le \max\{d(b, z), d(z, Tz)\} = d(z, Tz).$$

Thus $\mathcal{B}_b \subseteq \mathcal{B}_z$ (since $b \in \mathcal{B}_z \cap \mathcal{B}_b$) and $Tb \in \mathcal{B}_z$. If $z = Tz$ then z is a fixed point of T.

Finally, we show that if $z \notin Tz$ then $d(b, Tb) = d(z, Tz)$. Suppose that for some $b \in \mathcal{B}_z$

$$d(b, Tb) < d(z, Tz).$$

Again by Lemma 2.1 (c), we have either $d(Tb, Tz) \le d(b, z)$ or $d(T^2b, Tz) \le d(Tb, z)$ for all $b, z \in X$. In the case when $d(T^2b, Tz) \le d(Tb, z)$, we have

$$\begin{aligned} d(z, Tz) &\le \max\{d(z, b), d(b, Tb), d(T^2b, Tb), d(T^2b, Tz)\} \\ &\le \max\{d(z, b), d(b, Tb), d(b, Tb), d(Tb, z)\} \\ &\le \max\{d(z, b), d(b, Tb), d(b, Tb), d(Tb, b), d(b, z)\} = d(b, z). \end{aligned}$$

Similarly, in the other case, we get

$$d(z, Tz) \le d(b, z).$$

Therefore, in both cases

$$d(z, Tz) \le d(b, z).$$

Hence, we get $d(b, Tb) < d(z, Tz) = d(b, z)$. This implies that $z \notin \mathcal{B}_z$, which is impossible from the maximality of $\mathcal{B}_z$. Thus

$$d(b, Tb) = d(z, Tz) := r, \quad \forall\, b \in \mathcal{B}_z.$$

□

Remark 2.4. Corollaries 2.3 and 3.3 in [12] were obtained as consequences of Theorems 2.2 and 2.3. As we are going to show in the next section, these theorems actually need some additional assumption. However, Corollaries 2.3 and 3.3 are true as they are stated in [12], and this is shown by the proofs given here.

3. Multivalued Mappings

Let X be an ultrametric space and $C(X)$ be the collection of all compact subsets of X. The Hausdorff metric induced by d is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\}$$

for all $A, B \subseteq C(X)$, where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $T : X \to C(X)$ be a set-valued mapping. A point $z \in X$ is said to be a fixed point of T if $z \in Tz$.

Definition 3.1. *A multivalued mapping $T : X \to C(X)$ is said to be upper semi-continuous at $x_0 \in X$ if for any open set $V \in P(X)$, the set $\{x \mid Tx \subset V\}$ is an open set in X.*

If $T : X \to C(X)$ is an upper semi-continuous multivalued mapping then for every nonempty compact subset A of X the set

$$T(A) = \bigcup_{x\in A} Tx$$

is compact [1, 2, 14]. Therefore if $T : X \to C(X)$ is upper semi-continuous then $T^2x = T(Tx) = \bigcup_{y\in Tx} Ty$ is compact.

Now we prove a multivalued analog of Lemma 2.1.

Lemma 3.2. *Let X be an ultrametric space and $T : X \to C(X)$ be an upper semi-continuous multivalued mapping such that for all $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le d(x,y). \tag{5}$$

Then

(i) $H(Tx, T^2x) \le d(x,Tx)$;

(ii) either $\frac{1}{2}d(x,Tx) \le d(x,y)$ *or* $\frac{1}{2}H(T^2x,Tx) \le d(Tx,y)$;

(iii) either $H(Tx,Ty) \le d(x,y)$ *or* $H(T^2x,Ty) \le d(Tx,y)$.

Proof. Since $\frac{1}{2}d(x,Tx) \le d(x,Tx)$, by (5) it follows that $H(Tx,T^2x) \le d(x,Tx)$. This proves (i). In order to prove (ii), arguing by contradiction, we assume that

$$\frac{1}{2}d(x,Tx) > d(x,y) \text{ and } \frac{1}{2}H(Tx,T^2x) > d(Tx,y).$$

Then by the strong triangle inequality and (i), we have

$$\begin{aligned} d(x,Tx) &\le \max\{d(x,y), d(y,Tx)\} \\ &< \max\left\{\frac{1}{2}d(x,Tx), \frac{1}{2}H(Tx,T^2x)\right\} \\ &\le \max\left\{\frac{1}{2}d(x,Tx), \frac{1}{2}d(x,Tx)\right\} = \frac{1}{2}d(x,Tx), \end{aligned}$$

which is a contradiction. This proves (ii). Now (iii) directly follows from (ii). □

Theorem 3.3. (Compare with [12, Th. 2.2]). *Let X be a spherically complete ultrametric space and $T : X \to C(X)$ be an upper semi-continuous multivalued mapping. Assume that for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } H(Tx, Ty) < d(x, y). \tag{6}$$

Then T has a fixed point.

Proof. We will use denotations $\mathcal{B}_a$, $\subseteq$, $\mathcal{A}$ and $\mathcal{A}_1$ as in the proof of Theorem 2.2.

By Lemma 3.2 (iii), we have either $H(Ta, Tb) \leq d(a, b)$ or $H(T^2a, Tb) \leq d(Ta, b)$ for all $a, b \in X$. In the case $H(Ta, Tb) \leq d(a, b)$, the above inequality reduces to

$$\begin{aligned} d(x, b) \leq d(b, Tb) &\leq \max\{d(b, a), d(a, Ta), H(Ta, Tb)\} \\ &\leq \max\{d(b, a), d(a, Ta), d(a, b)\} \\ &= \max\{d(a, b), d(a, Ta)\}. \end{aligned}$$

In the case $H(T^2a, Tb) \leq d(Ta, b)$, we have

$$\begin{aligned} d(x, b) \leq d(b, Tb) &\leq \max\{d(b, a), d(a, Ta), H(T^2a, Ta), H(T^2a, Tb)\} \\ &\leq \max\{d(b, a), d(a, Ta), d(a, Ta), d(Ta, b)\} \\ &\leq \max\{d(b, a), d(a, Ta), d(a, Ta), d(Ta, a), d(a, b)\} \\ &= \max\{d(a, b), d(a, Ta)\}. \end{aligned}$$

Therefore, in both cases, we have

$$d(x, b) \leq d(b, Tb) \leq \max\{d(a, b), d(a, Ta)\}. \tag{7}$$

Now for $x \in \mathcal{B}_b$,

$$d(x, a) \leq \max\{d(a, b), d(b, x)\}.$$

By the fact that $d(a, b) \leq d(a, Ta)$ and (7), we get

$$d(x, a) \leq \max\{d(a, b), d(b, x)\} \leq d(a, Ta).$$

Hence $x \in \mathcal{B}_a$ and $\mathcal{B}_b \subseteq \mathcal{B}_a$ for every $\mathcal{B}_a \in \mathcal{A}_1$. Thus $\mathcal{B}_b$ is an upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. By Zorn's lemma, $\mathcal{A}$ has a maximal element, say $\mathcal{B}_z$, for some $z \in X$. We shall show that $z \in Tz$. Suppose that $z \notin Tz$. Then the compactness of Tz implies that there exists $w \in Tz$ with $w \neq z$ such that $d(w, z) = d(z, Tz)$. We show that $\mathcal{B}_w \subseteq \mathcal{B}_z$.

If $u \in \mathcal{B}_w$ then $d(w, u) \leq d(w, Tw)$. Since $w \in Tz$ and $\frac{1}{2}d(z, Tz) < d(w, z)$ for all $w, z \in X$, we have

$$d(w, u) \leq d(w, Tw) \leq H(Tz, Tw) < d(z, w) = d(z, Tz).$$

Also

$$d(u, z) \leq \max\{d(u, w), d(w, z)\} \leq d(z, Tz).$$

Therefore $u \in \mathcal{B}_z$ and $\mathcal{B}_w \subseteq \mathcal{B}_z$. But as

$$d(w, Tw) \leq H(Tw, Tz) < d(w, z),$$

$z \notin \mathcal{B}_w$, so $\mathcal{B}_w \subsetneq \mathcal{B}_z$. This contradicts the maximality of $\mathcal{B}_z$. Therefore T has a fixed point. □

Theorem 3.4. (Compare with [12, Th. 3.2]). *Let X be a spherically complete ultrametric space and $T : X \to C(X)$ be an upper semi-continuous multivalued mapping. Assume that for all $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le d(x,y). \tag{8}$$

Then either T has at least one fixed point or there exists a ball B of radius $r > 0$ such that $T : B \to B$ and for which $d(b,Tb) = r$ for each $b \in B$.

Proof. Let $\mathcal{B}_a$ and $\mathcal{A}$ be as in the proof of Theorem 3.3. We will find a maximal element $\mathcal{B}_z$ of $\mathcal{A}$.
By Lemma 3.2 (iii) for all $b, z \in X$, we have either

$$H(Tb,Tz) \le d(b,z) \quad \text{or} \quad H(T^2z,Tb) \le d(Tz,b).$$

In the case when $H(Tb,Tz) \le d(b,z)$ for any $b \in \mathcal{B}_z$, we have

$$\begin{aligned} d(b,Tb) &\le \max\{d(b,z), d(z,Tz), H(Tz,Tb)\} \\ &\le \max\{d(b,z), d(z,Tz), d(z,b)\} \\ &= \max\{d(b,z), d(z,Tz)\}. \end{aligned}$$

In the other case when $H(T^2z,Tb) \le d(Tz,b)$ for any $b \in \mathcal{B}_z$, we have

$$\begin{aligned} d(b,Tb) &\le \max\{d(b,z), d(z,Tz), H(T^2z,Tz), H(T^2z,Tb)\} \\ &\le \max\{d(b,z), d(z,Tz), d(z,Tz), d(Tz,b)\} \\ &\le \max\{d(b,z), d(z,Tz), d(z,Tz), d(Tz,z), d(z,b)\} \\ &= \max\{d(b,z), d(z,Tz)\}. \end{aligned}$$

Therefore in both cases, we have

$$d(b,Tb) \le \max\{d(b,z), d(z,Tz)\} = d(z,Tz).$$

Thus $\mathcal{B}_b \subseteq \mathcal{B}_z$ (since $b \in \mathcal{B}_z \cap \mathcal{B}_b$) and $Tb \in \mathcal{B}_z$. If $z \in Tz$ then z is a fixed point of T.
Finally, we show that if $z \notin Tz$ then $d(b,Tb) = d(z,Tz)$. Suppose that for some $b \in \mathcal{B}_z$

$$d(b,Tb) < d(z,Tz).$$

Again by Lemma 3.2 (iii), we have either $H(Tb,Tz) \le d(b,z)$ or $H(T^2b,Tz) \le d(Tb,z)$ for all $b, z \in X$. It is easy to show (as above) that in both cases

$$d(z,Tz) \le d(b,z).$$

Hence we get $d(b,Tb) < d(z,Tz) = d(b,z)$. This implies that $z \notin \mathcal{B}_z$, which is impossible from the maximality of $\mathcal{B}_z$. Thus,

$$d(b,Tb) = d(z,Tz) := r \ \ \forall b \in \mathcal{B}_z.$$

□

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