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Variational Inclusion Governed by $\alpha\beta$-$H((.,.),(.,.))$-Mixed Accretive Mapping

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Abstract. In this paper, we look into a new concept of accretive mappings called $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings in Banach spaces. We extend the concept of proximal-point mappings connected with generalized m-accretive mappings to the $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings and discuss its characteristics like single-valuable and Lipschitz continuity. Some illustration are given in support of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings. Since proximal point mapping is a powerful tool for solving variational inclusion. Therefore, As an application of introduced mapping, we construct an iterative algorithm to solve variational inclusions and show its convergence with acceptable assumptions.

1. Introduction

Variational inequality theory is providing mathematical models to some problems make an appearance in optimization and control, economics, and engineering sciences. Many heuristic has been widely used these applications of variational inequalities, e.g., we refer to see [18], [20]-[22],[24]. The proximal-point mapping technique is an important powerful tool to study varitional inequalities and their generalization.

Firstly, Huang and Fang [6] investigated the generalized m-accretive mapping and defined its proximal-point mapping in Banach spaces. Since then a number of mathematician presented various classes of

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generalized m-accretive mappings, see for examples [5, 17], [20, 21]. Sun et al. [22] presented a new class of M-monotone mapping in Hilbert spaces. In the past few days, Zou and Huang [24], Kazmi et al. [14, 15] investigated $H(.,.)$-accretive mappings, Ahmad et. al investigated $H(.,.)$-cocoercive mapping [2] and Husain and Gupta [7] investigated $H((.,.),(.,.))$-mixed cocoercive mappings in Banach (Hilbert) spaces, a natural extension of m-accretive (M-monotone) mapping and focussed on variational inclusions involving these mappings. In recent past, the techniques through different classes of proximal-point mappings have been developed to work on the existence of solutions and to analyze convergence and stability of iterative algorithms for several classes of variational inclusions, see for example [2, 4], [7]-[18], [20, 21], [24].

Very recently, Luo and Huang [18] introduced and studied a class of B-monotone and Kazmi et al. [14] introduced and studied a class of generalized $H(.,.)$-accretive mappings in Banach spaces which is generalization of H-monotone mappings [5]. They showed its proximal-point mapping properties connected with B-monotone and generalized $H(.,.)$-accretive mapping.

This work is motivated and inspired by the research works mentioned above. We look into a new notion of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings and give its proximal-point mapping. Further, we will discuss its characteristics that is single-valued as well as Lipschitz continuity. As an application, we attempt to solve generalized set-valued variational inclusions in real q-uniformly smooth Banach spaces. By using the proximal-point mapping technique, we construct an iterative algorithm and prove its convergence with acceptable assumptions. The results presented in this paper can be viewed as an extension and generalization of some known results [2, 7], [14]-[16], [18, 24]. Some illustrations are given in support of introduced results.

2. Preliminaries

Let us consider a real Banach space E with norm $\|.\|$ and topological dual space E^*. We use inner product $\langle .,. \rangle$ denote the dual pair between E and E^* and 2^E be the power set of E.

Definition 2.1. *[23] A mapping $J_q : E \multimap E^*$, where $q > 1$, is said to be generalized duality mapping, if it is given as*

$$J_q(u) \;=\; \{f^* \in E^* : \langle u, f^* \rangle = \|u\|^q,\ \|f^*\| = \|u\|^{q-1}\}, \quad \forall\, u \in E.$$

If J_2 is the usual normalized duality mapping on E, given as

$$J_q(u) = \|u\|^{q-1} J_2(u)\ \forall\, u(\neq 0)\ \in E.$$

If $E \equiv X$, a real Hilbert space, then J_2 becomes identity mapping on X.

Definition 2.2. *[23]* A Banach space E is called *smooth* if for every $u \in E$ with $\|u\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(u) = 1$.
The *modulus of smoothness* of E is a function $\rho_E : [0,\infty) \to [0,\infty)$, defined by

$$\rho_E(t) \;=\; \sup\left\{\frac{1}{2}(\|u+v\| + \|u-v\|) - 1 :\ \|u\| \le 1,\ \|v\| \le t\right\}.$$

Definition 2.3. *[23]* A Banach space E is called
(i) *uniformly smooth* if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} \;=\; 0;$$

(ii) *q-uniformly smooth*, for $q > 1$, if there exists $c > 0$ such that

$$\rho_E(t) \le c\, t^q,\ t \in [0,\infty).$$

Note that J_q is single-valued if E is uniformly smooth.

Lemma 2.4. *[23] Let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists $c_q > 0$ such that, for all $u, v \in E$,*

$$\|u + v\|^q \leq \|u\|^q + q\langle v, J_q(u)\rangle + c_q\|v\|^q.$$

In order to proceed our next step, we write basic important concepts and definitions, which will be used in this work.

Lemma 2.5. A mapping $f : E \to E$ is said to be
(i) ξ-*strongly accretive* with $\xi > 0$, if

$$\langle f(x) - f(y),\ J_q(x - y)\rangle \geq \xi\, \|x - y\|^q,\ \ \forall x,\ y \in E;$$

(ii) μ-*cocoercive* with $\mu > 0$, if

$$\langle f(x) - f(y),\ J_q(x - y)\rangle \geq \mu\, \|f(x) - f(y)\|^q,\ \ \forall x,\ y \in E;$$

(iii) γ-*relaxed cocoercive* with $\gamma > 0$, if

$$\langle f(x) - f(y),\ J_q(x - y)\rangle \geq -\gamma\, \|f(x) - f(y)\|^q,\ \ \forall x,\ y \in E;$$

(iv) β-*Lipschitz continuous* with $\beta > 0$, if

$$\|f(x) - f(y)\| \leq \beta\, \|x - y\|,\ \ \forall x,\ y \in E;$$

(v) α-*expansive* with $\alpha > 0$, if

$$\|f(x) - f(y)\| \geq \alpha\, \|x - y\|,\ \ \forall x,\ y \in E;$$

if $\alpha = 1$, then it is *expansive*.

Definition 2.6. *[7]* Let $H : (E\times E)\times(E\times E) \to E$, and $A, B, C, D : E \to E$ be the single-valued mappings. Then

(i) $H((A, .), (C, .))$ is said to be (μ_1, γ_1)-*strongly mixed cocoercive* regarding (A, C) with $\mu_1, \gamma_1 > 0$, if

$$\langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)),\ J_q(x - y)\rangle \geq \mu_1\, \|Ax - Ay\|^q + \gamma_1\, \|x - y\|^q,\ \forall x,\ y,\ u \in E;$$

(ii) $H((., B), (., D))$ is said to be (μ_2, γ_2)-*relaxed mixed cocoercive* regarding (B, D) with $\mu_2, \gamma_2 > 0$, if

$$\langle H((u, Bx), (u, Dx)) - H((u, By), (u, Dy)),\ J_q(x - y)\rangle \geq -\mu_2\, \|Bx - By\|^q + \gamma_2\, \|x - y\|^q,\ \forall x,\ y,\ u \in E;$$

(iii) $H((A, B), (C, D))$ is said to be *symmetric mixed cocoercive* regarding (A, C) and (B, D) if $H((A, .), (C, .))$ is (μ_1, γ_1)-strongly mixed cocoercive regarding (A, C) and $H((., B), (., D))$ is (μ_2, γ_2)-relaxed mixed cocoercive regarding (B, D);

(iv) $H((A, B), (C, D))$ is said to be τ-*mixed Lipschitz continuous* regarding A, B, C and D with $\tau > 0$, if

$$\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\| \leq \tau\, \|x - y\|,\ \ \forall x,\ y \in E.$$

Definition 2.7. *[18]* Let $S : E \multimap E$ and $M : E \times E \multimap E$ be the set-valued mapping. Then

(i) S is said to be *accretive* if

$$\langle u - v,\ J_q(x - y)\rangle \geq 0\ \ \forall x,\ y \in E,\ u \in Sx,\ v \in Sy;$$

(ii) S is said to be *strictly accretive* if

$$\langle u - v,\ J_q(x - y)\rangle > 0\ \ \forall x,\ y \in E,\ u \in Sx,\ v \in Sy;$$

and equality holds if and only if $x = y$.

(iii) S is said to be *μ'-strongly accretive* with $\mu' > 0$, if

$$\langle u - v,\ J_q(x - y)\rangle \ \geq\ \mu'\|x - y\|^q \ \ \forall x,\ y \in E,\ u \in Sx,\ v \in Sy;$$

(iv) S is said to be *γ'-relaxed accretive* with $\gamma' > 0$, if

$$\langle u - v,\ J_q(x - y)\rangle \ \geq\ -\gamma'\|x - y\|^q \ \ \forall x,\ y \in E,\ u \in Sx,\ v \in Sy;$$

(v) $M(f, .)$ is said to be *α-strongly accretive* regarding f with $\alpha > 0$, if

$$\langle u - v,\ J_q(x - y)\rangle \ \geq\ \alpha\|x - y\|^q \ \ \forall x,\ y,\ w \in E,\ u \in M(f(x), w)\ v \in M(f(y), w);$$

(vi) $M(., g)$ is said to be *β-relaxed accretive* regarding g with $\beta > 0$, if

$$\langle u - v,\ J_q(x - y)\rangle \ \geq\ -\beta\|x - y\|^q \ \ \forall x,\ y,\ w \in E,\ u \in M(w, g(x))\ v \in M(w, g(y));$$

(vii) $M(., .)$ is said to be *$\alpha\beta$-symmetric accretive* regarding f and g if $M(f, .)$ is α-strongly accretive regarding f and $M(., g)$ is β-relaxed accretive regarding g with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$.

3. $\alpha\beta$-$H((.,.),(.,.))$-Mixed Accretive Mappings

Firstly we consider the following assumptions, then we will introduce $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings and its proximal-point mapping. Later we will discuss the properties of its proximal point mapping properties.

Let $H : (E\times E)\times(E\times E) \to E$, $f, g : E \to E$ and $A, B, C, D : E \to E$ be single-valued mappings and $M : E\times E \multimap E$ be a set-valued mapping.
Assumption (a_1)**:** Let H is symmetric mixed cocoercive regarding (A, C) and (B, D).
Assumption (a_2)**:** Let A is α_1-expansive and B is β_1-Lipschitz continuous.

Definition 3.1. Let assumption (a_1) holds, then M is said to be *$\alpha\beta$-$H((.,.),(.,.))$-mixed accretive* regarding (A, C), (B, D) and (f, g) if

(i) M is $\alpha\beta$-symmetric accretive regarding f and g;
(ii) $(H((.,.),(.,.)) + \rho M(f, g))(E) \ =\ E$, for all $\rho > 0$.

The following example illustrate the Definitions (2.6) and (3.1).

Example 3.2. Let $q = 2$ and $E = \mathbb{R}^2$ with usual inner product defined by

$$\langle (x_1, x_2), (y_1, y_2)\rangle = x_1 y_1 + x_2, y_2.$$

Let $A, B, C, D : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} 4x_1 \\ 4x_2 \end{pmatrix},\ Bx = \begin{pmatrix} -3x_1 \\ -3x_2 \end{pmatrix},\ Cx = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix},\ Dx = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \forall\ x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $H : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2$ is defined by

$$H((Ax, Bx), (Cx, Dx)) \ =\ Ax + Bx + Cx + Dx.$$

In addition, let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{pmatrix} 5x_1 - \frac{2}{3}x_2 \\ \frac{2}{3}x_1 + 5x_2 \end{pmatrix}, \; g(x) = \begin{pmatrix} \frac{7}{4}x_1 + \frac{3}{4}x_2 \\ -\frac{3}{4}x_1 + \frac{7}{4}x_2 \end{pmatrix}, \; \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

and $M : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$M(fx, gx) = fx - gx, \quad \forall \; x = (x_1, x_2) \in \mathbb{R}^2.$$

Then, constants in Definition 2.6 and 3.1 having values $(\mu_1, \gamma_1) = (\frac{1}{4}, 2)$, $(\mu_2, \gamma_2) = (\frac{1}{3}, 1)$, $\tau = 4$ $\alpha = 5$ and $\beta = \frac{7}{4}$. It shows that H is symmetric mixed cocoercive regarding (A, C) and (B, D), M is symmetric accretive regarding f and g, and H is mixed Lipschitz continuous regarding A, B, C and D. Further, it can be obtained easily that $[H((A, B), (C, D)) + \rho\, M(f, g)](\mathbb{R}^2) = \mathbb{R}^2$. Thus M is $\alpha\beta$-mixed accretive with respect to (A, B), (C, D) and (f, g).

Remark 3.3. (i) If $H((A, B), (C, D)) = H(A, B)$, then $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive mapping reduces to generalized $H(.,.)$-accretive mapping considered in [16].

(ii) If $H((A, B), (C, D)) = B$, then $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive mapping reduces to generalized B-monotone mapping considered in [18].

(iii) If $H((A, B), (C, D)) = H(A, B)$, $M(.,.) = M$ and M is accretive, then $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive mapping reduces to $H(.,.)$-accretive mapping considered in [24].

(iv) If E is Hilbert space, $M(f, g) = M$ and M is m-relaxed monotone, then $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive mapping reduces to $H((.,.), (.,.))$-mixed cocoercive mapping considered in [7].

Since $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive mapping is a generalization of the maximal accretive mapping, it is logical that they have similar properties. The next result guarantee this supposition.

Proposition 3.4. *Let M be a $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive mapping regarding (A, C), (B, D) and (f, g). If assumptions (a_1) and (a_2) hold with $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and γ_1, $\gamma_2 > 0$. If the following inequality*

$$\langle u - v,\; J_q(x - y)\rangle \;\geq\; 0,$$

satisfied for all $(v,\; y) \in$ Graph$(M(f, g))$, implies $(u,\; x) \in\; M(f, g)$, where

$$\text{Graph}(M(f, g)) \;=\; \{(u,\; x) \in\; E \times E :\; (u,\; x) \in M(f(x), g(x))\}.$$

Proof. Assume on the contrary that there exists $(u_0,\; x_0) \notin$ Graph$(M(f, g))$ such that

$$\langle u_0 - v,\; J_q(x_0 - y)\rangle \;\geq\; 0, \forall\, (y, v) \in\; \text{Graph}(M(f, g)). \tag{1}$$

By definition of $\alpha\beta$-$H((.,.), (.,.))$-mixed accretive, we know that $(H((.,.), (.,.)) + \rho\, M(f, g))(E) \;=\; E$, holds for all $\rho > 0$. So there exists $(u_1, x_1) \in$ Graph$(M(f, g))$ such that

$$H((Ax_1, Bx_1), (Cx_1, Dx_1)) + \rho u_1 = H((Ax_0, Bx_0), (Cx_0, Dx_0)) + \rho u_0 \in\; E. \tag{2}$$

$$\begin{aligned}
\text{Now,}\quad \rho\, u_0 - \rho\, u_1 \;&=\; H((Ax_1, Bx_1), (Cx_1, Dx_1)) - H((Ax_0, Bx_0), (Cx_0, Dx_0)) \in\; E. \\
\langle \rho\, u_0 - \rho\, u_1,\; J_q(x_0 - x_1)\rangle \;&=\; -\langle H((Ax_0, Bx_0), (Cx_0, Dx_0)) - H((Ax_1, Bx_1), (Cx_1, Dx_1)), \\
&\qquad J_q(x_0 - x_1)\rangle.
\end{aligned}$$

Since M is $\alpha\beta$-symmetric accretive regarding f and g, we obtain

$$\begin{aligned}
(\alpha - \beta)\, \|x_0 - x_1\|^q &\leq \rho\, \langle u_0 - u_1,\; J_q(x_0 - x_1)\rangle \\
&= -\langle H((Ax_0, Bx_0), (Cx_0, Dx_0)) - H((Ax_1, Bx_1), (Cx_1, Dx_1)),\; J_q(x_0 - x_1)\rangle \\
&= -\langle H((Ax_0, Bx_0), (Cx_0, Dx_0)) - H((Ax_1, Bx_0), (Cx_1, Dx_0)),\; J_q(x_0 - x_1)\rangle \\
&\quad -\langle H((Ax_1, Bx_0), (Cx_1, Dx_0)) - H((Ax_1, Bx_1), (Cx_1, Dx_1)),\; J_q(x_0 - x_1)\rangle.
\end{aligned}$$

(3)

Since assumption (a_1) holds, we have from (3)

$$(\alpha-\beta)\,\|x_0-x_1\|^q \le -\mu_1\|Ax_0-Ax_1\|^q-\gamma_1\|x_0-x_1\|^q+\mu_2\|Bx_0-Bx_1\|^q-\gamma_2\|x_0-x_1\|^q. \tag{4}$$

Since assumption (a_2) holds, we have from (4)

$$\begin{aligned}(\alpha-\beta)\,\|x_0-x_1\|^q &\le -\mu_1\alpha_1^q\|x_0-x_1\|^q-\gamma_1\|x_0-x_1\|^q+\mu_2\beta_1^q\|x_0-x_1\|^q-\gamma_2\|x_0-x_1\|^q\\ &= -[(\mu_1\alpha_1^2-\mu_q\beta_1^q)+(\gamma_1+\gamma_2)]\,\|x_0-x_1\|^2\\ 0\le\ (\alpha-\beta)\,\|x_0-x_1\|^q &\le -[(\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)]\,\|x_0-x_1\|^q\\ 0 &\le -(\ell+\kappa)\,\|x_0-x_1\|^q\ \le\ 0,\end{aligned}$$

where $\ell\ =\ (\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)$ and $\kappa\ =\ (\alpha-\beta)$,

which gives $x_0=x_1$ since $\alpha>\beta,\ \mu_1>\mu_2,\ \alpha_1>\beta_2$, and $\gamma_1,\gamma_2>0$. By (1), we have $u_0=u_1$, a contradiction. This complete the proof. □

Theorem 3.5. *Let M be a $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping regarding (A,C), (B,D) and (f,g). If assumptions (a_1) and (a_2) hold with $\alpha>\beta,\ \mu_1>\mu_2,\ \alpha_1>\beta_1$ and $\gamma_1,\ \gamma_2>0$, then $(H((A,B),(C,D))+\rho M(f,g))^{-1}$ is single-valued.*

Proof. For any given $x\in E$, let $u,v\in(H((A,B),(C,D))+\rho M(f,g))^{-1}(x)$. It follows that

$$\begin{cases}-H((Au,Bu),(Cu,Du))+x\in\ \rho M(f,g)u,\\ -H((Av,Bv),(Cv,Dv))+x\in\ \rho M(f,g)v.\end{cases}$$

Since M is $\alpha\beta$-symmetric accretive with respect to f and g, we have

$$\begin{aligned}(\alpha-\beta)\|u-v\|^q\ &\le\ \frac{1}{\rho}\langle -H((Au,Bu),(Cu,Du))+x-(-H((Av,Bv),(Cv,Dv))+x),\ J_q(u-v)\rangle\\ (\alpha-\beta)\|u-v\|^q\ &\le\ \langle -H((Au,Bu),(Cu,Du))+x-(-H((Av,Bv),(Cv,Dv))+x),\ J_q(u-v)\rangle\\ &= -\langle H((Au,Bu),(Cu,Du))-H((Av,Bv),(Cv,Dv)),\ J_q(u-v)\rangle\\ &= -\langle H((Au,Bu),(Cu,Du))-H((Av,Bu),(Cv,Du)),\ J_q(u-v)\rangle\\ &\quad -\langle H((Av,Bu),(Cv,Du))-H((Av,Bv),(Cv,Dv)),\ J_q(u-v)\rangle.\end{aligned} \tag{5}$$

Since assumption (a_1) holds, we have from (5)

$$\rho(\alpha-\beta)\|u-v\|^q\le-\mu_1\|Au-Av\|^q-\gamma_1\|u-v\|^q+\mu_2\|Bu-Bv\|^q-\gamma_2\|u-v\|^q. \tag{6}$$

Since assumption (a_2) holds, we have from (6)

$$\begin{aligned}\rho(\alpha-\beta)\|u-v\|^q &\le -\mu_1\alpha_1^q\|u-v\|^q-\gamma_1\|u-v\|^q+\mu_2\beta_1^q\|u-v\|^q-\gamma_2\|u-v\|^q\\ &= -[(\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)]\,\|u-v\|^q\\ 0\le\ (\alpha-\beta)\,\|u-v\|^q &\le -(\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)\,\|u-v\|^q\\ 0 &\le -(\ell+\rho\kappa)\,\|u-v\|^q\ \le\ 0,\end{aligned}$$

where $\ell\ =\ (\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)$ and $\kappa\ =\ (\alpha-\beta)$.

Since $\alpha>\beta,\ \mu_1>\mu_2,\ \alpha_1>\beta_2$ and $\gamma_1,\gamma_2>0$, it follows that $\|u-v\|\ \le\ 0$. This implies that $u=v$ and so $(H((A,B),(C,D))+\rho M(f,g))^{-1}$ is single-valued. □

Definition 3.6. Let M be a $\alpha\beta$- $H((.,.),(.,.))$-mixed accretive mapping regarding (A,C), (B,D) and (f,g). If assumptions (a_1) and (a_2) hold with $\alpha>\beta,\ \mu_1>\mu_2,\ \alpha_1>\beta_1$ and $\gamma_1,\ \gamma_2>0$, then the *proximal-point mapping* $R^{H((.,.),(.,.))}_{\rho,\ M(.,.)} : E \to\ E$ is defined by

$$R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(u)\ =\ (H((A,B),(C,D))+\rho M(f,g))^{-1}(u),\ \ \forall\, u\in E. \tag{7}$$

Now we prove that the proximal-point mapping defined by (7) is Lipschitz continuous.

Theorem 3.7. *Let $M : E\times E \multimap E$ be a $\alpha\beta$- $H((.,.),(.,.))$-mixed accretive mapping with respect to (A,C), (B,D) and (f,g). If assumptions (a_1) and (a_2) hold with $\alpha>\beta,\ \mu_1>\mu_2,\ \alpha_1>\beta_1$ and $\gamma_1,\ \gamma_2>0$, then the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,\ M(.,.)} : E\to\ E$ is $\frac{1}{\ell+\rho\kappa}$-Lipschitz continuous, that is,*

$$\|R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(u)-R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(v)\|\le \frac{1}{\ell+\rho\kappa}\|u-v\|,\ \forall\, u,v\in E.$$

Proof. For given points $u,\ v\in E$, It proceed from Definition 3.6 that

$$\begin{aligned} R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(u) &= (H((A,B),(C,D))+\rho M(f,g))^{-1}(u),\\ R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(v) &= (H((A,B),(C,D))+\rho M(f,g))^{-1}(v).\end{aligned}$$

Let $w_1\ =\ R^{H((.,.)(.,.))}_{\rho,\ M(.,.)}(u)$ and $w_2\ =\ R^{H((.,.)(.,.))}_{\rho,\ M(.,.)}(v)$.

$$\begin{cases} \frac{1}{\rho}\Big(u-H\Big(\Big(A\Big(w_1\Big),B\Big(w_1\Big)\Big),\Big(C\Big(w_1\Big),\ D\Big(w_1\Big)\Big)\Big)\Big)\in\ M\left(f\Big(w_1\Big),g\Big(w_1\Big)\right)\\ \frac{1}{\rho}\Big(v-H\Big(\Big(A\Big(w_2\Big),B\Big(w_2\Big)\Big),\Big(C\Big(w_2\Big),\ D\Big(w_2\Big)\Big)\Big)\Big)\in\ M\left(f\Big(w_2\Big),g\Big(w_2\Big)\right).\end{cases}$$

Since M is $\alpha\beta$-symmetric accretive with respect to f and g, we have

$$\begin{cases} \langle\frac{1}{\rho}(u-H((A(w_1),B(w_1)),(C(w_1),\ D(w_1)))-(v-H((A(w_2),B(w_2)),(C(w_2),\ D(w_2))))),J_q(w_1-w_2)\rangle\\ \qquad\qquad\qquad\geq(\alpha-\beta)\ \|w_1-w_2\|^q,\\ \\ \langle\frac{1}{\rho}(u-v-H((A(w_1),B(w_1)),(C(w_1),\ D(w_1))))+H((A(w_2),B(w_2)),(C(w_2),\ D(w_2))),J_q(w_1-w_2)\rangle\\ \qquad\qquad\qquad\geq(\alpha-\beta)\ \|w_1-w_2\|^q,\end{cases}$$

which implies

$$\begin{aligned}\langle u-v,\ J_q(w_1-w_2)\rangle\ \geq\ &\langle H((A(w_1),B(w_1)),(C(w_1),D(w_1)))-H((A(w_2),B(w_2)),(C(w_2),\ D(w_2))),J_q(w_1-w_2)\rangle\\ &+\rho(\alpha-\beta)\ \|w_1-w_2\|^q.\end{aligned}$$

Now, we have

$$\begin{aligned}&\|u-v\|\ \|w_1-w_2\|^{q-1}\\ &\quad\geq\ \langle u-v,\ w_1-w_2\rangle\\ &\quad\geq\ \langle H((A(w_1),B(w_1)),(C(w_1),D(w_1)))-H((A(w_2),B(w_2)),(C(w_2),D(w_2))),J_q(w_1-w_2)\rangle+\rho(\alpha-\beta)\ \|w_1-w_2\|^q\\ &\quad=\ \langle H((A(w_1),B(w_1)),(C(w_1),D(w_1)))-H((A(w_2),B(w_1)),(C(w_2),D(w_1))),\ J_q(w_1-w_2)\rangle\\ &\quad+\langle H((A(w_2),B(w_1)),(C(w_2),D(w_1)))-H((A(w_2),B(w_2)),(C(w_2),D(w_2))),J_q(w_1-w_2)\rangle+\rho(\alpha-\beta)\ \|w_1-w_2\|^q.\end{aligned}$$

Since assumption (a_1) holds, we have

$$\begin{aligned}\|u-v\|\ \|w_1-w_1\|^{q-1}\geq\ &\mu_1\|A(w_1)-A(w_2)\|^q+\gamma_1\|w_1-w_2\|^q-\mu_2\|B(w_1)-B(w_2)\|^q+\gamma_2\|w_1-w_2\|^q\\ &+\rho(\alpha-\beta)\ \|w_1-w_2\|^q.\end{aligned}$$

Since assumption (a_2) holds, we have

$$\begin{aligned}\|u-v\|\,\|w_1-w_2\|^{q-1} &\geq [(\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)]\,\|w_1-w_2\|^q+\rho(\alpha-\beta)\,\|w_1-w_2\|^q\\ &\geq\ (\ell+\rho\kappa)\,\|w_1-w_2\|^q,\end{aligned}$$

where $\ell\ =\ (\mu_1\alpha_1^q-\mu_2\beta_1^q)+(\gamma_1+\gamma_2)$ and $\kappa=(\alpha-\beta)$.
Hence,

$$\|u-v\|\,\|w_1-w_2\|^{q-1}\ \geq\ (\ell+\rho\kappa)\,\|w_1-w_2\|^q,$$

that is

$$\|R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(u)-R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(v)\|\leq\frac{1}{\ell+\rho\kappa}\,\|u-v\|,\ \forall\ u,v\in E.$$

This completes the proof. □

4. An Application of *αβ*-*H*((., .), (., .))-Mixed Accretive Mappings.

Here we shall show that the $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping under acceptable assumptions can be used as a powerful tool to solve variational inclusion problem in Banach space.

Let $S,T:E\multimap CB(E)$ be the set-valued mappings, and let $f,g:E\to E$, $A,B,C,D:E\to E$, $F:E\times E\to E$ and $H:(E\times E)\times(E\times E)\to E$ be single-valued mappings. Suppose that set-valued mapping $M:E\times E\multimap E$ be a $\alpha\beta$- $H((.,.),(.,.))$-mixed accretive mapping regarding (A,C), (B,D) and (f,g). We consider the following generalized set-valued variational inclusion: for given $\lambda\in E$, find $u\in E$, $v\in S(u)$ and $w\in T(u)$ such that

$$\lambda\in\ F(v,w)+M(f(u),g(u)). \tag{8}$$

If $S,T:E\to E$ be single-valued mappings and $M(.,.)=\rho N(.)$, where $\rho>0$ is a constant, then the problem (8) reduces to the following problem: find $u\in E$ such that

$$\lambda\in\ F(S(u),T(u))+\rho N(u). \tag{9}$$

If M is an (A,η)-accretive mapping, then the problem (9) was introduced and studied by Lan et al. [17].

If $\rho=1$, $\lambda=0$ and $F(S(u),T(u))=T(u)$ for all $u\in E$, where $T:E\to E$ is a single-valued mapping, then the problem (9) reduces to the following problem: find $u\in E$ such that

$$0\in\ T(u)+N(u). \tag{10}$$

If N is an $H(.,.)$-accretive mapping, then the problem (10) was studied by Zou and Huang [24]; and N is a generalized m-accretive mapping, then the problem (10) was studied by Bi et al. [4].

If E is a Hilbert space and N is an H-monotone mappings, then the problem (10) was introduced and studied by Fang and Huang [5] and includes many variational inequalities (inclusions) and complementarity problems as special cases. For example, see [20, 21].

Lemma 4.1. *Let $u\in E$, $v\in S(u)$ and $w\in T(u)$ is a solution of problem (8) if and only if $u\in E$, $v\in S(u)$ and $w\in T(u)$ satisfies the following relation:*

$$u\ =\ R^{H((.,.),(.,.))}_{\rho,M(.,.)}\,[H((Au,Bu),(Cu,Du))-\rho F(v,w)+\rho\lambda],\ \rho>0. \tag{11}$$

Proof. Observe that for $\rho > 0$,

$$\begin{aligned}
&\lambda \in\ F(w,v) + M(f(u),g(u))\\
\Leftrightarrow\ & [H((Au,Bu),(Cu,Du)) - \rho F(v,w) + \rho\lambda] \in H((Au,Bu),(Cu,Du)) + \rho M(f(u),g(u))\\
\Leftrightarrow\ & [H((Au,Bu),(Cu,Du)) - \rho F(v,w) + \rho\lambda] \in (H((A,B),(C,D)) + \rho M(f,g))u\\
\Leftrightarrow\ & u = (H((A,B),(C,D)) + \rho M(f,g))^{-1}[H((Au,Bu),(Cu,Du)) - \rho F(v,w) + \rho\lambda]\\
\Leftrightarrow\ & u = R^{(H(.,.),(.,.))}_{\rho\ M(.,.)}[H((Au,Bu),(Cu,Du)) - \rho F(v,w) + \rho\lambda].
\end{aligned}$$

□

Remark 4.2. We can rewrite the equality (11) as:

$$z\ =\ H((Au,Bu),(Cu,Du)) - \rho F(v,w) + \rho\lambda, \quad u = R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(z).$$

By using the result of Nadler [19], this fixed point formulation allow us to construct the iterative algorithm as given below:

Algorithm 4.3. For any given $z_0 \in E$, we can choose $u_0 \in E$ such that sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ satisfy

$$\begin{cases}
u_n = R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}(z_n),\\
v_n \in S(u_n),\ \ \|v_n - v_{n+1}\| \le\ \left(1 + \frac{1}{n+1}\right)\ \mathcal{D}(S(u_n),S(u_{n+1})),\\
w_n \in T(u_n),\ \|w_n - w_{n+1}\|\ \le\ \left(1 + \frac{1}{n+1}\right)\ \mathcal{D}(T(u_n),T(u_{n+1})),\\
z_{n+1}\ =\ H((Au_n,Bu_n),(Cu_n,Du_n)) - \rho F(v_n,w_n) + \rho\lambda + e_n,\\
\sum\limits_{j=1}^{\infty} \|e_j - e_{j-1}\|\,\varpi^{-j}\ <\ \infty,\ \forall\,\varpi \in\ (0,1),\ \lim_{n\to\infty} e_n = 0,
\end{cases}$$

where $\rho > 0$ is a constant, $\lambda \in E$ is any given element and $e_n \subset E$ is an error to take into account a possible inexact computation of the proximal-point mapping point for all $n \ge 0$, and $\mathcal{D}(.,.)$ is the Hausdorff metric on CB(E).

Next, we need the following definitions which will be used to state and prove the main result.

Definition 4.4. A set-valued mapping $G : E \multimap CB(E)$ is said to be *$\mathcal{D}$-Lipschitz continuous* with constant $l > 0$, if

$$\mathcal{D}(Gx,Gy) \le\ l\,\|x - y\|,\ \ \forall x,\ y \in E.$$

Definition 4.5. Let $S,T : E \multimap E$ be the set-valued mappings, $A,B,C,D : E \to E$, $F : E \times E \to E$ and $H : (E \times E) \times (E \times E) \to E$ be single-valued mappings. Then

(i) F is said to be *σ-strongly accretive* regarding S and $H((A,B),(C,D))$ in the first component with constant $\sigma > 0$, if

$$\begin{aligned}
&\langle F(v_1,.) - F(v_2,.),\ J_q(H((Au,Bu),(Cu,Du)) - H((Av,Bv),(Cv,Dv)))\rangle\\
&\qquad \ge\ \sigma\,\|H((Au,Bu),(Cu,Du)) - H((Av,Bv),(Cv,Dv))\|^q,\\
&\qquad\quad \forall\,u,v \in\ E \text{ and } v_1 \in S(u),\ v_2 \in S(v);
\end{aligned}$$

(ii) F is said to be *δ-strongly accretive* regarding T and $H((A,B),(C,D))$ in the second component with $\delta > 0$, if

$$\begin{aligned}
&\langle F(.,w_1) - F(.,w_2),\ J_q(H((Au,Bu),(Cu,Du)) - H((Av,Bv),(Cv,Dv)))\rangle\\
&\qquad \ge\ \delta\,\|H((Au,Bu),(Cu,Du)) - H((Av,Bv),(Cv,Dv))\|^q,\\
&\qquad\quad \forall\,u,v \in\ E \text{ and } w_1 \in T(u),\ w_2 \in T(v);
\end{aligned}$$

(iii) F is said to be ϵ_1-*Lipschitz continuous* in the first component with $\epsilon_1 > 0$, if

$$\|F(u, v') - F(v, v')\| \leq \epsilon_1 \|u - v\|, \ \forall u, \ v, \ v' \in E;$$

(iv) F is said to be ϵ_2-*Lipschitz continuous* in the second component with $\epsilon_2 > 0$, if

$$\|F(v', u) - F(v', v)\| \leq \epsilon_2 \|u - v\|, \ \forall u, \ v, \ v' \in E.$$

Next, we find the convergence of iterative algorithm for generalized set-valued variational inclusion (8).

Theorem 4.6. *Let us consider the problem (8) and assume that*

(i) S and T are l_1 and l_2 $\mathcal{D}$-Lipschitz continuous, respectively;

(ii) $H((A, B), (C, D))$ is τ-mixed Lipschitz continuous regarding A, B, C and D;

(iii) F is is σ-strongly accretive regarding S and $H((A, B), (C, D))$ in the first component and δ-strongly accretive regarding T and $H((A, B), (C, D))$ in the second component;

(iv) F is is ϵ_1, ϵ_2-Lipschitz continuous in the first and second component, respectively;

(v) $$0 < \sqrt[q]{\tau^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q - \rho q(\sigma + \delta)\tau^q} < \ell + \rho\kappa; \tag{12}$$

where $\ell = (\mu_1\alpha_1^q - \mu_2\beta_1^q) + (\gamma_1 + \gamma_2)$ and $\kappa = \alpha - \beta$, and $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \ \gamma_2, \rho > 0$.

Then problem (8) has a solution (u, v, w), where $u \in E$, $v \in S(u)$ and $w \in T(u)$, and the iterative sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$, generated by Algorithms 4.3 converges strongly to u, v and w, respectively.

Proof. Using the Lipschitz continuity of S and T, it follows from Algorithms 4.3 such that

$$\|v_{n+1} - v_n\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(S(u_{n+1}), S(u_n)) \leq \left(1 + \frac{1}{n+1}\right) l_1 \|u_{n+1} - u_n\|, \tag{13}$$

$$\|w_{n+1} - w_n\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(T(u_{n+1}), T(u_n)) \leq \left(1 + \frac{1}{n+1}\right) l_2 \|u_{n+1} - u_n\|, \tag{14}$$

for $n = 0, 1, 2, ...$
From (11) and Theorem 3.7, we have

$$\|u_{n+1} - u_n\| \leq \|R_{\rho, M(.,.)}^{H((.,.),(.,.))}(z_{n+1}) - R_{\rho, M(.,.)}^{H((.,.),(.,.))}(z_n)\| = \frac{1}{\ell + \rho\kappa}\|z_{n+1} - z_n\|. \tag{15}$$

Now, we estimate $\|z_{n+1} - z_n\|$ by using Algorithms 4.3, we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|[H((Au_n, Bu_n), (Cu_n, u_n)) - \rho F(v_n, w_n) + \rho\lambda + e_n] \\ &\quad - [H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, u_{n-1})) - \rho F(v_{n-1}, w_{n-1}) + \rho\lambda + e_{n-1}]\| \\ &\leq \|H((Au_n, Bu_n), (Cu_n, u_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, u_{n-1})) \\ &\quad + (\rho F(v_n, w_n) - \rho F(v_{n-1}, w_{n-1})\| + \|e_n - e_{n-1}\|. \end{aligned} \tag{16}$$

By Lemma 2.4, we have

$$\begin{aligned} &\|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1})) - \rho(F(v_n, w_n) - F(v_{n-1}, w_{n-1})\|^q \\ &\leq \|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))\|^q + c_q\rho^q\|F(v_n, w_n) - F(v_{n-1}, w_{n-1})\|^q \\ &\quad - \rho q\langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), \ J_q(H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1})))\rangle. \end{aligned} \tag{17}$$

From (ii), we get

$$\|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))\| \leq \tau \|u_n - u_{n-1}\|. \tag{18}$$

By Algorithm 4.3, and conditions (i) and (iv), we get

$$\begin{aligned}\|F(v_n,w_n)-F(v_{n-1},w_{n-1})\| &\le \|F(v_n,w_n)-F(v_{n-1},w_n)\|+\|F(v_{n-1},w_n)-F(v_{n-1},w_{n-1})\|\\ &\le \epsilon_1\|v_n-v_{n-1}\| + \epsilon_2\|w_n-w_{n-1}\|\\ &\le \epsilon_1\left(1+\frac{1}{n}\right)\mathcal{D}(S(u_n),S(u_{n-1}))+\epsilon_2\left(1+\frac{1}{n}\right)\mathcal{D}(T(u_n),T(u_{n-1}))\\ &\le \left(\epsilon_1 l_1\left(1+\frac{1}{n}\right) + \epsilon_2 l_2\left(1+\frac{1}{n}\right)\right)\|u_n-u_{n-1}\|. \end{aligned}\tag{19}$$

Using conditions (iii), we get

$$\begin{aligned}&\langle F(v_n,w_n)-F(v_{n-1},w_{n-1}),\ J_q(H((Au_n,Bu_n),(Cu_n,Du_n))-H((Au_{n-1},Bu_{n-1}),(Cu_{n-1},Du_{n-1})))\rangle\\ &\le \langle F(v_n,w_n)-F(v_{n-1},w_n),\ J_q(H((Au_n,Bu_n),(Cu_n,Du_n))-H((Au_{n-1},Bu_{n-1}),(Cu_{n-1},Du_{n-1})))\rangle\\ &\quad+\langle F(v_{n-1},w_n)-F(v_{n-1},w_{n-1}),\ J_q(H((Au_n,Bu_n),(Cu_n,Du_n))-H((Au_{n-1},Bu_{n-1}),(Cu_{n-1},Du_{n-1})))\rangle\\ &\le (\sigma+\delta)\,\|H((Au_n,Bu_n),(Cu_n,Du_n))-H((Au_{n-1},Bu_{n-1}),(Cu_{n-1},Du_{n-1}))\|^q\\ &\le (\sigma+\delta)\tau^q\,\|u_n-u_{n-1}\|^q. \end{aligned}\tag{20}$$

From (17)-(19), we have

$$\begin{aligned}&\|H((Au_n,Bu_n),(Cu_n,Du_n))-H((Au_{n-1},Bu_{n-1}),(Cu_{n-1},Du_{n-1}))-\rho(F(v_n,w_n)-F(v_{n-1},w_{n-1})\|\\ &\qquad\le \sqrt[q]{\tau^q+c_q\rho^q\left(\epsilon l_1\left(1+\frac{1}{n}\right)+\epsilon_2 l_2\left(1+\frac{1}{n}\right)\right)^q-\rho q(\sigma+\delta)\tau^q}\,\|u_n-u_{n-1}\|. \end{aligned}\tag{21}$$

Combining (15), (16) and (21), we have

$$\|u_{n+1}-u_n\| \le \|R^{H((.,.),(.,.))}_{\rho,\,M(.,.)}(z_{n+1})-R^{H((.,.),(.,.))}_{\rho,\,M(.,.)}(z_n)\| \le \varphi_n\,\|u_n-u_{n-1}\| + \frac{1}{\ell+\rho\kappa}\,\|e_n-e_{n-1}\|,\tag{22}$$

where

$$\varphi_n = \frac{1}{\ell+\rho\kappa}\sqrt[q]{\tau^q+c_q\rho^q\left(\epsilon_1 l_1\left(1+\frac{1}{n}\right)+\epsilon_2 l_2\left(1+\frac{1}{n}\right)\right)^q-\rho q(\sigma+\delta)\tau^q}.\tag{23}$$

Let

$$\varphi = \frac{1}{\ell+\rho\kappa}\sqrt[q]{\tau^q+c_q\rho^q(\epsilon_1 l_1+\epsilon_2 l_2)^q-\rho q(\sigma+\delta)\tau^q}.\tag{24}$$

Since $\varphi_n \to \varphi$ as $n\to\infty$. By (12), we know that $0<\varphi<1$ and hence there exist $n_0>0$ and $\varphi_0\in(0,1)$ such that $\varphi_n\le\varphi_0$ for all $n\ge n_0$. Therefore, by (22), we have

$$\|u_{n+1}-u_n\| \le \varphi_0\,\|u_n-u_{n-1}\| + \frac{1}{\ell+\rho\kappa}\|e_n-e_{n-1}\|\ \ \forall\, n\ge n_0.\tag{25}$$

(25) implies that

$$\|u_{n+1}-u_n\| \le \varphi_0^{n-n_0}\,\|u_{n_0+1}-u_{n_0}\| + \frac{1}{\ell+\rho\kappa}\sum_{j=1}^{n-n_0}\varphi_0^{j-1}t_{n-(n-1)},\tag{26}$$

where $t_n=\|e_n-e_{n-1}\|$ for all $n\ge n_0$. Hence, for any $m\ge n>n_0$, we have

$$\|u_m-u_n\| \le \sum_{p=n}^{m-1}\|u_{p+1}-u_p\| \le \sum_{p=n}^{m-1}\varphi_0^{p-n_0}\|u_{n_0+1}-u_{n_0}\| + \frac{1}{\ell+\rho\kappa}\sum_{p=n}^{m-1}\varphi_0^p\sum_{j=1}^{p-n_0}\left[\frac{t_{p-(j-1)}}{\varphi_0^{p-(j-1)}}\right].\tag{27}$$

Since $\sum_{j=1}^{\infty} \|e_j - e_{j-1}\| \, \varpi^{-j} \ < \ \infty, \ \forall \ \varpi \in \ (0,1)$ and $0 < \varphi_0 < 1$, it follows that $\|u_m - u_n\| \to 0$ as $n \to \infty$, and so $\{u_n\}$ is a Cauchy sequence in E. From (13) and (14), it follows that $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in E. Thus, there exist u, v and w such that $u_n \to u$, $v_n \to v$ and $w_n \to w$ as $n \to \infty$. In the sequel, we will prove that $v \in S(u)$. In fact, since $v_n \in S(u_n)$, we have

$$\begin{aligned} d(v,\ S(u)) \ &\leq \ \|v - v_n\| \ + \ d(v_n,\ S(u)) \\ &\leq \ \|v - v_n\| \ + \ \mathcal{D}(S(u_n),\ S(u)) \\ &\leq \ \|v - v_n\| \ + \ \rho \, \|u_n - u\| \to \ 0, \ \text{as } n \to \infty, \end{aligned}$$

which implies that $d(v,\ S(u)) \ = \ 0$. Since $S(u) \in CB(E)$, it follows that $v \in S(u)$. Similarly, it is easy to see that $w \in T(u)$.

By the continuity of $R^{H((.,.),(.,.))}_{\rho,\ M(.,.)}$, $A,\ B,\ C,\ D,\ S,\ T$ and F and Algorithms 4.3, we know that $u,\ v$ and w satisfy

$$u \ = \ R^{H((.,.),(.,.))}_{\rho,M(.,.)} \, [H((Au, Bu), (Cu, Du)) - \rho F(u, z) + \rho\lambda].$$

By Lemma 4.1, (u, v, w) is a solution of the problem (8). This completes the proof □

The following example shows that assumptions (i) to (v) of Theorem 4.6 are satisfied for variational inclusion problem (8).

Example 4.7. Let $q = 2$ and $E \ = \ \mathbb{R}^2$ with usual inner product.

(i) Let $S, T : \mathbb{R}^2 \multimap \mathbb{R}^2$ are identity mappings, then R, S are n-Lipschitz continuous for $n = 1, 2$.

Let $A, B, C, D : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} \frac{1}{10}x_1 \\ \frac{1}{10}x_2 \end{pmatrix},\ Bx = \begin{pmatrix} -\frac{1}{5}x_1 \\ -\frac{1}{5}x_2 \end{pmatrix},\ Cx = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix},\ Dx = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \ \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $H : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2$ is defined by

$$H((Ax, By), (Cx, Dy)) = Ax + Bx + Cx + Dx, \quad \forall\, x \in \mathbb{R}^2.$$

Then, it is easy to cheek that

$H((.,.),(.,.))$ is $(10, 2)$-strongly mixed cocoercive regarding (A, C) and $(5, 1)$-relaxed mixed cocoercive regarding (B, D), and A is $\frac{1}{n}$-expansive for $n = 10, 11$ and B is $\frac{1}{n}$-Lipschitz continuous for $n = 4, 5$.

(ii) $H((A, B), (C, D))$ is $\frac{29}{n}$-mixed Lipschitz continuous regarding A, B, C and D for $n = 9, 10$.
Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{pmatrix} \frac{1}{2}x_1 - \frac{4}{3}x_2 \\ \frac{4}{3}x_1 + \frac{1}{2}x_2 \end{pmatrix},\ g(x) = \begin{pmatrix} \frac{1}{4}x_1 - \frac{3}{4}x_2 \\ \frac{3}{4}x_1 + \frac{1}{4}x_2 \end{pmatrix},\ \forall\, x = (x_1, x_2), \in \mathbb{R}^2.$$

Suppose that $M : (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2$ is defined by

$$M(fx, gx) = fx - gx, \quad \forall\ x = (x_1, x_2), \in \mathbb{R}^2.$$

Then, it is easy to check that $M(f, g)$ is $\frac{1}{n}$-strongly accretive regarding f for $n = 2, 3$ and $\frac{1}{n}$-relaxed accretive regarding g for $n = 3, 4$. Moreover, for $\rho = 1$, M is $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive regarding (A, C), (B, D) and (f, g).

Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$F(x, y) = \frac{x}{4} + \frac{y}{5}, \quad \forall\, x,\ y, \in \mathbb{R}^2.$$

Then, it is easy to check that

(iii) F is is $\frac{29}{n}$-strongly accretive regarding S and $H((A, B), (C, D))$ in the first component for $n = 30, 40$ and $\frac{29}{n}$-strongly accretive with respect to T and $H((A, B), (C, D))$ in the second component for $n = 40, 50$;

(iv) F is is $\frac{1}{n}$-Lipschitz continuous in the first component for $n = 3, 4$ and $\frac{1}{n}$-Lipschitz continuous in the second component for $n = 4, 5$.

Therefore, for the constants

$$\begin{aligned}
&l_1 = l_2 = 1,\ \mu_1 = 10,\ \gamma_1 = 2,\ \mu_2 = 5,\ \gamma_2 = 1,\ \alpha_1 = 0.1,\ \beta_1 = 0.2,\\
&\alpha = 0.5,\ \beta = 0.25,\ \sigma = 0.725,\ \delta = 0.580,\ \epsilon_1 = 0.25,\ \epsilon_2 = 0.2,\ \tau = 2.9,\\
&q = 2,\ \ell = 2.9,\ \kappa = 0.25.
\end{aligned}$$

obtained in (i) to (v) above, all the conditions of the Theorem 4.7 is satisfied for the generalized mixed variational inclusion problem (8) for $\rho = 0.35$ and $c_q = 1$.

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