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Ideal Convergence of Nets of Functions with Values in Uniform Spaces

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Abstract. We consider the pointwise, uniform, quasi-uniform, and the almost uniform $\mathcal{I}$-convergence for a net $(f_d)_{d\in D}$ of functions from a topological space X into a uniform space $(Y, \mathcal{U})$, where $\mathcal{I}$ is an ideal on D. The purpose of the present paper is to provide ideal versions of some classical results and to extend these to nets of functions with values in uniform spaces. In particular, we define the notion of $\mathcal{I}$-equicontinuous family of functions on which pointwise and uniform $\mathcal{I}$-convergence coincide on compact sets. Generalizing the theorem of Arzelà, we give a necessary and sufficient condition for a net of continuous functions from a compact space into a uniform space to $\mathcal{I}$-converge pointwise to a continuous function.

Introduction

In recent years, a lot of papers have been written on statistical convergence and ideal convergence in metric and topological spaces (see, for instance, [14, 15, 17–20, 22, 23]). Recently, several researchers have been working on sequences of real functions and of functions between metric spaces by using the idea of statistical and $\mathcal{I}$-convergence (see, for instance, [2, 3, 6–9]).

On the other hand, classical results about sequences and nets of functions have been extended from metric to uniform spaces (see, for example, [5, 16, 21]).

In this paper, we investigate the pointwise, uniform, quasi-uniform, and the almost uniform $\mathcal{I}$-convergence for a net $(f_d)_{d\in D}$ of functions in a topological space X with values in a uniform space Y, where $\mathcal{I}$ is an ideal on D. Particularly, the continuity of the limit of the net $(f_d)_{d\in D}$ is studied. Since each metric space is a uniform space, the results of the paper remain valid in the case that Y is a metric space.

The rest of this paper is organized as follows. Section 1 contains preliminaries. In Section 2 we introduce the pointwise, uniform and quasi-uniform $\mathcal{I}$-convergence for nets of functions with values in uniform spaces. In Section 3 we present a modification of the classical result which states that equicontinuity on a compact metric space turns pointwise to uniform convergence. In Section 4 we extend the classical result of Arzelà [1] to the quasi-uniform $\mathcal{I}$-convergence of nets of functions with values in uniform spaces. Finally, the concept of almost uniform $\mathcal{I}$-convergence of a net of functions with values in a uniform space is investigated in Sections 5 and 6.

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1. Preliminaries and General Notations

First, we recall some of the basic concepts related to the uniform spaces and nets of functions. We refer to [16] for more details.

A *uniformity* on a set Y is a collection $\mathcal{U}$ of subsets of $Y \times Y$ satisfying the following properties:

($\mathcal{U}_1$) $\Delta \subseteq U$, for every $U \in \mathcal{U}$, where $\Delta = \{(y, y) : y \in Y\}$.

($\mathcal{U}_2$) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(y_1, y_2) : (y_2, y_1) \in U\}$.

($\mathcal{U}_3$) If $U \in \mathcal{U}$ and $U \subseteq V \subseteq Y \times Y$, then $V \in \mathcal{U}$.

($\mathcal{U}_4$) If $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$.

($\mathcal{U}_5$) For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$, where
$V \circ V = \{(y_1, y_2) :$ there exists $y \in Y$ such that $(y_1, y) \in V$ and $(y, y_2) \in V\}$.

A *uniform space* is a pair $(Y, \mathcal{U})$ consisting of a set Y and a uniformity $\mathcal{U}$ on the set Y. The elements of $\mathcal{U}$ are called *entourages*. An entourage V is called *symmetric* if $V^{-1} = V$. For every $U \in \mathcal{U}$ and $y_0 \in Y$ we use the following notation:

$$U[y_0] = \{y \in Y : (y_0, y) \in U\}.$$

For every uniform space $(Y, \mathcal{U})$ the *uniform topology* $\tau_{\mathcal{U}}$ on Y is the family consisting of the empty set and all subsets O of Y such that for each $y \in O$ there is $U \in \mathcal{U}$ with $U[y] \subseteq O$.

If (Y, ρ) is a metric space, then the collection $\mathcal{U}_\rho$ of all $U \subseteq Y \times Y$ for which there is $\varepsilon > 0$ such that $\{(y_1, y_2) : \rho(y_1, y_2) < \varepsilon\} \subseteq U$ is a uniformity on Y which generates a uniform space with the same topology as the metric topology induced by ρ. For the special case in which $Y = [0, 1]$ and $\rho(y_1, y_2) = |y_1 - y_2|$, then we call $\mathcal{U}_\rho$ the *usual uniformity* for $[0, 1]$.

A mapping f from a topological space X into a uniform space $(Y, \mathcal{U})$ is called *continuous at* x_0 if for each $U \in \mathcal{U}$ there exists an open neighbourhood O_{x_0} of x_0 such that $f(O_{x_0}) \subseteq U[f(x_0)]$ or equivalently $(f(x_0), f(x)) \in U$, for every $x \in O_{x_0}$. The mapping f is called *continuous* if it is continuous at every point of X.

Let D be a nonempty set. A family $\mathcal{I}$ of subsets of D is called an *ideal on* D if $\mathcal{I}$ has the following properties:

(1) $\emptyset \in \mathcal{I}$.

(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

(3) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

An ideal $\mathcal{I}$ on D is said to be *non-trivial* if $\mathcal{I} \neq \{\emptyset\}$ and $D \notin \mathcal{I}$. The ideal $\mathcal{I}$ is called *admissible* if it contains all finite subsets of D.

A partially preordered set D is called *directed* if every two elements of D have an upper bound in D. Let $(D, \leqslant)$ be a directed set. We consider the family

$$\{A \subseteq D : A \subseteq \{d \in D : d \not\geqslant d_0\} \text{ for some } d_0 \in D\}.$$

This family is an ideal on D which will be denoted by $\mathcal{I}_D$.

A *net* in the set Y^X of all functions $f : X \to Y$ is an arbitrary function s from a nonempty directed set D to Y^X. If $s(d) = f_d$, for all $d \in D$, then the net s will be denoted by the symbol $(f_d)_{d \in D}$.

If $(f_d)_{d \in D}$ is a net in Y^X, then a net $(g_\lambda)_{\lambda \in \Lambda}$ in Y^X is said to be a *semi-subnet* of $(f_d)_{d \in D}$ if there exists a function $\varphi : \Lambda \to D$ such that $g_\lambda = f_{\varphi(\lambda)}$, for every $\lambda \in \Lambda$. We write $(g_\lambda)^\varphi_{\lambda \in \Lambda}$ to indicate the fact that φ is the function mentioned above.

Suppose that $(g_\lambda)^\varphi_{\lambda \in \Lambda}$ is a semi-subnet of the net $(f_d)_{d \in D}$. For every ideal $\mathcal{I}$ of the directed set D, we consider the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$. This family is an ideal on Λ which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

Now, we recall some basic types of convergence of sequences and nets of functions from a set into a metric space.

Definition 1.1. ([3]) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a nonempty set X into a metric space (Y, ρ), and let $\mathcal{I}$ be an ideal on D.

(1) $(f_n)_{n\in\mathbb{N}}$ is said to *$\mathcal{I}$-pointwise converge to f on X* if for every $x \in X$ and for every $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for every $n \notin A$ we have $\rho(f(x), f_n(x)) < \varepsilon$.

(2) $(f_n)_{n\in\mathbb{N}}$ is said to *$\mathcal{I}$-uniform converge to f on X* if for every $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for every $x \in X$ and for every $n \notin A$ we have $\rho(f(x), f_n(x)) < \varepsilon$.

Definition 1.2. ([1]; see also [4]) A net $(f_d)_{d\in D}$ of functions from a nonempty set X into a metric space (Y, ρ) is said to *converge quasi uniformly to f on X* if it converges pointwise to f, and for every $\varepsilon > 0$ and for every $d_0 \in D$, there exists a finite number of indices $d_1, \dots, d_k \geqslant d_0$ such that for each $x \in X$ at least one of the following inequalities holds:

$$\rho(f(x), f_{d_i}(x)) < \varepsilon, \ i = 1, \dots, k.$$

Definition 1.3. ([12]; see also [10]) A net $(f_d)_{d\in D}$ of functions from a nonempty set X into a metric space (Y, ρ) is said to *converge almost uniformly to f on X* if for every $x \in X$, for every $\varepsilon > 0$, and for every $d \in D$, there exist $d_x \geqslant d$ and an open neighbourhood O_x of x such that for every $t \in O_x$ we have $\rho(f(t), f_{d_x}(t)) < \varepsilon$.

Finally, we give some definitions that will be used in the last section of the paper. For more details we refer the reader to [11, 24].

A topological space X is called *completely regular* if X is a T_1-space and for every closed subset F of X and for every point $x \in X \setminus F$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

A topological space X (not necessarily Hausdorff) is called *locally compact* if for each $x \in X$ there exist an open neighbourhood U of x and a compact subset C of X such that $U \subseteq C$.

A topological space X (not necessarily completely regular) is called *pseudocompact* if every continuous real-valued function on X is bounded. A completely regular space X is pseudocompact if and only if every locally finite collection of nonempty open subsets of X is finite.

2. Basic Concepts

In this section we consider the pointwise, uniform and quasi-uniform $\mathcal{I}$-convergence for nets of functions with values in uniform spaces. In what follows we consider a net $(f_d)_{d\in D}$ of functions from a topological space X into a uniform space $(Y, \mathcal{U})$, and an ideal $\mathcal{I}$ on D.

Definition 2.1. The net $(f_d)_{d\in D}$ is said to *$\mathcal{I}$-converge pointwise to f on X* if for every $x \in X$ and for every $U \in \mathcal{U}$ there exists $A \in \mathcal{I}$ such that for every $d \notin A$ we have $(f(x), f_d(x)) \in U$. In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$. We shall say that the net $(f_d)_{d\in D}$ *$\mathcal{I}$-converges pointwise on* X if there is a function to which the net $\mathcal{I}$-converges pointwise.

Proposition 2.2. *If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, then for every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ we have $(g_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)} f$.*

Proof. Let $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ be a semi-subnet of $(f_d)_{d\in D}$, $x \in X$, and $U \in \mathcal{U}$. There is $A \in \mathcal{I}$ such that for every $d \notin A$ we have $(f(x), f_d(x)) \in U$. We set

$$A_\Lambda = \{\lambda \in \Lambda : \varphi(\lambda) \in A\}.$$

Since $\varphi(A_\Lambda) \subseteq A$ and $A \in \mathcal{I}$, we have $\varphi(A_\Lambda) \in \mathcal{I}$ and, hence, $A_\Lambda \in \mathcal{I}_\Lambda(\varphi)$. If $\lambda \notin A_\Lambda$, then $\varphi(\lambda) \notin A$ and, therefore, $(f(x), g_\lambda(x)) = (f(x), f_{\varphi(\lambda)}(x)) \in U$. Thus, $(g_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)} f$. □

Definition 2.3. The net $(f_d)_{d\in D}$ is said to *$\mathcal{I}$-converge uniformly to f on X* if for every $U \in \mathcal{U}$ there exists $A \in \mathcal{I}$ such that for every $x \in X$ and for every $d \notin A$ we have $(f(x), f_d(x)) \in U$. In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$. We shall say that the net $(f_d)_{d\in D}$ *$\mathcal{I}$-converges uniformly on* X if there is a function to which the net $\mathcal{I}$-converges uniformly.

Proposition 2.4. *If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, then for every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ we have $(g_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)\text{-}u} f$.*

Proof. Similar to the proof of Proposition 2.2. □

Lemma 2.5. *Let $(Y, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$.*

Proof. By property $(\mathcal{U}_5)$ of the uniformity $\mathcal{U}$, there exists $V' \in \mathcal{U}$ such that $V' \circ V' \subseteq U$. In the same manner we find $V'' \in \mathcal{U}$ such that $V'' \circ V'' \subseteq V'$. Hence, $(V'' \circ V'') \circ (V'' \circ V'') \subseteq U$ and, therefore,

$$V'' \circ V'' \circ V'' \subseteq (V'' \circ V'') \circ (V'' \circ V'') \subseteq U$$

in view of property $(\mathcal{U}_1)$ of the uniformity $\mathcal{U}$. We set $V = V'' \cap (V'')^{-1}$. Then, V is a symmetric entourage such that $V \circ V \circ V \subseteq U$. □

Proposition 2.6. *If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, the functions f_d, $d \in D$ are continuous, and the ideal $\mathcal{I}$ is non-trivial, then the function f is continuous.*

Proof. Suppose that $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$ and let $x_0 \in X$. We prove that f is continuous at x_0. Let $U \in \mathcal{U}$. By Lemma 2.5 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, there exists $A \in \mathcal{I}$ (because $\mathcal{I}$ is non-trivial, $A \neq D$) such that for every $x \in X$ and $d \notin A$ we have $(f(x), f_d(x)) \in V$. Let $d_0 \notin A$. Then,

$$(f(x_0), f_{d_0}(x_0)) \in V. \tag{2.1}$$

Since f_{d_0} is continuous at x_0, there exists an open neighbourhood O_{x_0} of x_0 such that $(f_{d_0}(x_0), f_{d_0}(x)) \in V$, for every $x \in O_{x_0}$. Let $x \in O_{x_0}$. Then,

$$(f_{d_0}(x_0), f_{d_0}(x)) \in V \tag{2.2}$$

and

$$(f(x), f_{d_0}(x)) \in V. \tag{2.3}$$

Therefore, using successively the relations (2.1), (2.2), and (2.3), we have $(f(x_0), f(x)) \in V \circ V \circ V$ and the continuity of f is proved. □

Definition 2.7. The net $(f_d)_{d\in D}$ is said to *$\mathcal{I}$-converge quasi-uniformly to f on X* if $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$ and for every $U \in \mathcal{U}$ and for every $A \in \mathcal{I} \setminus \{D\}$, there exists a finite subset $\{d_1, \ldots, d_n\}$ of $D \setminus A$ such that for each $x \in X$ at least one of the following relations holds:

$$(f(x), f_{d_i}(x)) \in U, \ i = 1, \ldots, n.$$

In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}qu} f$. We shall say that the net $(f_d)_{d\in D}$ *$\mathcal{I}$-converges quasi-uniformly on X* if there is a function to which the net $\mathcal{I}$-converges quasi-uniformly.

Proposition 2.8. *If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$ and $(g_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)\text{-}qu} f$ for some semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$, where $\mathcal{I}_\Lambda(\varphi)$ is a non-trivial ideal on Λ, then $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}qu} f$.*

Proof. Similar to the proof of Proposition 2.2. □

3. $\mathcal{I}$-equicontinuity and Uniform $\mathcal{I}$-Convergence

In this section we present a modification of the classical result which states that equicontinuity on a compact metric space turns pointwise to uniform convergence. An extension of this result has been obtained in [3].

Definition 3.1. ([16]) A family $\{f_i : i \in I\}$ of functions from a topological space X into a uniform space $(Y, \mathcal{U})$ is called *equicontinuous at a point x_0 of X* if for every $U \in \mathcal{U}$ there exists an open neighbourhood O_{x_0} of x_0 such that $(f_i(x_0), f_i(x)) \in U$ for all $i \in I$ and for all $x \in O_{x_0}$. The family $\{f_i : i \in I\}$ is called *equicontinuous* if it is equicontinuous at each point of X.

Definition 3.2. Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. The family $\{f_d : d \in D\}$ is called *$\mathcal{I}$-equicontinuous at a point x_0 of X* if for every $U \in \mathcal{U}$ there exist $A \in \mathcal{I}$ and an open neighbourhood O_{x_0} of x_0 such that $(f_d(x_0), f_d(x)) \in U$ for all $d \in D \setminus A$ and for all $x \in O_{x_0}$. The family $\{f_d : d \in D\}$ is called *$\mathcal{I}$-equicontinuous* if it is equicontinuous at each point of X.

Theorem 3.3. *Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D such that the family $\{f_d : d \in D\}$ is $\mathcal{I}$-equicontinuous. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, then the function f is continuous. Moreover, the $\mathcal{I}$-convergence is uniform on every compact subset of X.*

Proof. Suppose that $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to a function f. We prove that f is continuous. Let $x_0 \in X$ and $U \in \mathcal{U}$. In view of Lemma 2.5 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. By the $\mathcal{I}$-equicontinuity of the family $\{f_d : d \in D\}$ at the point x_0, there exist $A_0 \in \mathcal{I}$ and an open neighbourhood O_{x_0} of x_0 such that $(f_d(x_0), f_d(x)) \in V$ for all $d \in D \setminus A_0$ and for all $x \in O_{x_0}$. Let $x \in O_{x_0}$. Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, there exist $A_1, A_2 \in \mathcal{I}$ such that:

(1) $(f(x_0), f_d(x_0)) \in V$, for every $d \notin A_1$.

(2) $(f(x), f_d(x)) \in V$, for every $d \notin A_2$.

We set $A = A_0 \cup A_1 \cup A_2$. Then, $A \in \mathcal{I}$. Since the ideal $\mathcal{I}$ is non-trivial, we have $A \neq D$. Let $d_0 \notin A$. Then, we have $(f_{d_0}(x_0), f_{d_0}(x)) \in V$, $(f(x_0), f_{d_0}(x_0)) \in V$, and $(f(x), f_{d_0}(x)) \in V$. Hence, $(f(x), f(x_0)) \in V \circ V \circ V$ and the function f is continuous.

Now, let C be a compact subset of X and $U \in \mathcal{U}$. By Lemma 2.5 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Let $c \in C$. Since the family $\{f_d : d \in D\}$ is $\mathcal{I}$-equicontinuous at the point c, there exist $A_c \in \mathcal{I}$ and an open neighbourhood O_c of c such that $(f_d(c), f_d(x)) \in V$ for all $d \in D \setminus A_c$ and for all $x \in O_c$. Also, since the function f is continuous at c, there exists an open neighbourhood O'_c of c such that $(f(c), f(x)) \in V$ for all $x \in O'_c$. We set $H_c = O_c \cap O'_c \cap C$. The family

$$\{H_c : c \in C\}$$

is an open cover of C. By compactness of C, there are $c_1, \dots, c_n \in C$ such that

$$C = \bigcup_{i=1}^{n} H_{c_i}.$$

Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, for every $i = 1, \dots, n$ there exists $A_i \in \mathcal{I}$ such that for every $d \notin A_i$,

$$(f(c_i), f_d(c_i)) \in V.$$

We set

$$A = \bigcup_{i=1}^{n} A_i \cup \bigcup_{i=1}^{n} A_{c_i}.$$

Then, $A \in \mathcal{I}$. Since the ideal $\mathcal{I}$ is non-trivial, $A \neq D$. Let $x \in C$ and $d \notin A$. Then, for proper choice of i, we have $(f_d(c_i), f_d(x)) \in V$, $(f(c_i), f(x)) \in V$, and $(f(c_i), f_d(c_i)) \in V$. Hence, $(f(x), f_d(x)) \in V \circ V \circ V$ and the proof is complete. □

Corollary 3.4. *Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$, where the family $\{f_d : d \in D\}$ is equicontinuous and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, then the function f is continuous. Moreover, the $\mathcal{I}$-convergence is uniform on every compact subset of X.*

4. Ideal Version of Arzelà's Theorem for Uniform Spaces

The theorem of Arzelà [1] which gives a necessary and sufficient condition for a net of continuous functions to converge to a continuous function plays an important role in functional analysis. In this section we extend the classical result of Arzelà (see for instance [4]) to the quasi-uniform $\mathcal{I}$-convergence of nets of functions with values in uniform spaces.

Lemma 4.1. (see [16, Theorem 6, Chapter 6]) Let $(Y, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then, there exists a symmetric entourage $W \in \mathcal{U}$ such that:

(1) $W \subseteq U$.

(2) W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Lemma 4.2. *Let f and g be two continuous functions of a topological space X into a uniform space $(Y, \mathcal{U})$. The following statements are true:*

(1) *The function $m : X \to (Y \times Y, \tau_{\mathcal{U}} \times \tau_{\mathcal{U}})$ defined by $m(x) = (f(x), g(x))$, for every $x \in X$ is continuous.*

(2) *If W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$, then the set $\{x \in X : (f(x), g(x)) \in W\}$ is open.*

Proof. (1) Let $x \in X$ and let $V_1[f(x)] \times V_2[g(x)]$ be an open neighbourhood of $m(x)$. Since f is continuous at x, there exists an open neighbourhood O_x of x such that $f(O_x) \subseteq V_1[f(x)]$. Since g is continuous at x, there exists an open neighbourhood O'_x of x such that $g(O'_x) \subseteq V_2[g(x)]$. We set $O''_x = O_x \cap O'_x$. Hence, $m(O''_x) \subseteq f(O_x) \times g(O'_x) \subseteq V_1[f(x)] \times V_2[g(x)]$.

(2) Let W be open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$. Since $\{x \in X : (f(x), g(x)) \in W\} = m^{-1}(W)$, by statement (1) we obtain the desired result. □

Lemma 4.3. *Let f be a continuous function of a topological space X into a uniform space $(Y, \mathcal{U})$ and let $x_0 \in X$.*

(1) *The function $m : X \to (Y \times Y, \tau_{\mathcal{U}} \times \tau_{\mathcal{U}})$ defined by $m(x) = (f(x), f(x_0))$, for every $x \in X$ is continuous.*

(2) *If W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$, then the set $\{x \in X : (f(x_0), f(x)) \in W\}$ is open.*

Proof. It is similar to the proof of Lemma 4.2. □

Theorem 4.4. *Let $(f_d)_{d\in D}$ be a net of continuous functions from a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. If the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to a continuous limit, then the $\mathcal{I}$-convergence is quasi-uniform on every compact subset of X. Conversely, if the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges quasi-uniformly on a subset of X, then the limit is continuous on this subset.*

Proof. Suppose that $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to a continuous function f on X. Let C be a compact subset of X. We prove that the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges quasi-uniformly to f on C. In view of Definition 2.7, it suffices to prove that for every $U \in \mathcal{U}$ and for every $A \in \mathcal{I} \setminus \{D\}$, there exists a finite subset $\{d_1, \ldots, d_n\}$ of $D \setminus A$ such that for each $x \in C$ at least one of the following relations holds:

$$(f(x), f_{d_i}(x)) \in U,\ i = 1, \ldots, n.$$

For that purpose we take arbitrary elements $U \in \mathcal{U}$ and $A \in \mathcal{I}$ (since $\mathcal{I}$ is non-trivial, $A \neq D$). By Lemma 4.1 there exists a symmetric entourage $W \in \mathcal{U}$ such that:

(1) $W \subseteq U$.

(2) W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Let $c \in C$. Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, there exists $A_c \in \mathcal{I}$ such that for every $d \notin A_c$,

$$(f(c), f_d(c)) \in W. \tag{4.1}$$

Then, $A_c \cup A \in \mathcal{I}$. Since the ideal $\mathcal{I}$ is non-trivial, $A_c \cup A \neq D$. Choose $d_c \notin A_c \cup A$ and set

$$O_c = \{x \in X : (f(x), f_{d_c}(x)) \in W\}.$$

Since f_{d_c} and f are continuous, by Lemma 4.2, O_c is an open set. Moreover, by relation (4.1), the set O_c contains c. Thus, the family

$$\{O_c \cap C : c \in C\}$$

is an open cover of C. By compactness of C, there are $c_1, \ldots, c_n \in C$ such that

$$C = \bigcup_{i=1}^{n} O_{c_i} \cap C.$$

By the choice of the elements $d_{c_1}, \ldots, d_{c_n}$ of D, the set $\{d_{c_1}, \ldots, d_{c_n}\}$ is a finite subset of $D \setminus A$. Moreover, for each $x \in C$ at least one of the following relations holds:

$$(f(x), f_{d_{c_i}}(x)) \in W, \ i = 1, \ldots, n.$$

Indeed, let $x \in C$. Then, there exists $i \in \{1, \ldots, n\}$ such that $x \in O_{c_i} \cap C$. Therefore, by the definition of the set O_{c_i}, we have $(f(x), f_{d_{c_i}}(x)) \in W$. Since $W \subseteq U$, for each $x \in C$ it holds $(f(x), f_{d_{c_i}}(x)) \in U$ for at least one $i = 1, \ldots, n$. Thus, $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}qu} f$ on C.

Conversely, suppose that $(f_d)_{d\in D}$ $\mathcal{I}$-converges quasi-uniformly to f on a subset X' of X. Let $x_0 \in X'$. We prove that f is continuous at x_0. Let $U \in \mathcal{U}$. By Lemma 2.5 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. By Lemma 4.1 there exists a symmetric entourage $W \in \mathcal{U}$ such that:

(1) $W \subseteq V$.

(2) W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Let

$$A_0 = \{d \in D : (f(x_0), f_d(x_0)) \notin W\}.$$

Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, there exists $A \in \mathcal{I}$ such that for every $d \notin A$ we have $(f(x_0), f_d(x_0)) \in W$. Hence, $A_0 \subseteq A$ and, therefore, $A_0 \in \mathcal{I}$. Moreover, since the ideal $\mathcal{I}$ is non-trivial, $A_0 \neq D$. By assumption, there exists a finite subset $\{d_1, \ldots, d_n\}$ of $D \setminus A_0$ such that for each $x \in X'$ at least one of the following relations holds:

$$(f(x), f_{d_i}(x)) \in W, \ i = 1, \ldots, n. \tag{4.2}$$

Since $\{d_1, \ldots, d_n\} \subseteq D \setminus A_0$, by the definition of A_0, we have

$$(f(x_0), f_{d_i}(x_0)) \in W, \ i = 1, \ldots, n. \tag{4.3}$$

Let

$$O_i = \{x \in X : (f_{d_i}(x_0), f_{d_i}(x)) \in W\}, \ i = 1, \ldots, n. \tag{4.4}$$

Since the functions f_{d_i}, $i = 1, \ldots, n$ are continuous, by Lemma 4.3, the sets O_i, $i = 1, \ldots, n$ are open in X and contain x_0. We set

$$O_{x_0} = X' \cap \bigcap_{i=1}^{n} O_i.$$

The set O_{x_0} is open in X' and contains x_0. Let $x \in O_{x_0}$. Then, $x \in O_i$, for every $i = 1, \ldots, n$. Since $x \in X'$, by relation (4.2), there exists $i \in \{1, \ldots, n\}$ such that $(f(x), f_{d_i}(x)) \in W$. For this i, by relations (4.3) and (4.4), we have $(f(x_0), f_{d_i}(x_0)) \in W$ and $(f_{d_i}(x_0), f_{d_i}(x)) \in W$. Therefore, for every $x \in O_{x_0}$, using relations (4.2), (4.3), and (4.4), for proper choice of i, we obtain

$$(f(x_0), f(x)) \in W \circ W \circ W \subseteq V \circ V \circ V \subseteq U.$$

Thus, $f(O_{x_0}) \subseteq U[f(x_0)]$. We conclude that f is continuous at x_0 completing the proof of the theorem. □

Corollary 4.5. *On a compact topological space, the limit of a pointwise $\mathcal{I}$-convergent net $(f_d)_{d\in D}$ of continuous functions from a topological space into a uniform space is continuous if and only if the $\mathcal{I}$-convergence is quasi-uniform, when $\mathcal{I}$ is a non-trivial ideal on D.*

Corollary 4.6. *Let X be a compact topological space, and suppose that the net $(f_d)_{d\in D}$ of continuous functions from the topological space X into a uniform space $(Y, \mathcal{U})$ $\mathcal{I}$-converges pointwise to a continuous function f, where $\mathcal{I}$ is a non-trivial ideal on D. Then, f is continuous in any topology on X in which all the functions f_d, $d \in D$ are continuous.*

Proof. By Corollary 4.5, $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}qu} f$. Let τ be a topology on X which makes all the functions f_d, $d \in D$ continuous. By Theorem 4.4, the function $f : (X, \tau) \to (Y, \mathcal{U})$ is continuous. □

Lemma 4.7. (see [16, Theorem 8, Chapter 6]) Let $(Y, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then, there exists a symmetric entourage $K \in \mathcal{U}$ such that:

(1) $K \subseteq U$.

(2) K is closed in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Theorem 4.8. *Let M be a dense subset of a compact topological space X, and suppose that the net $(f_d)_{d\in D}$ of continuous functions from X into the uniform space $(Y, \mathcal{U})$ $\mathcal{I}$-converges pointwise to a continuous limit f on M, where $\mathcal{I}$ is a non-trivial ideal on D. The following statements are true:*

(1) *If $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to f on X, then every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ $\mathcal{I}_\Lambda(\varphi)$-converges quasi-uniformly to f on X, in the case where $\mathcal{I}_\Lambda(\varphi)$ is a non-trivial ideal on Λ.*

(2) *If every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ $\mathcal{I}_\Lambda(\varphi)$-converges quasi-uniformly to f on M, then $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to f on X.*

Proof. (1) Suppose that $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to f on X. Then, by Proposition 2.2, every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ $\mathcal{I}_\Lambda(\varphi)$-converges pointwise to f on X. Therefore, in view of Theorem 4.4, every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ $\mathcal{I}_\Lambda(\varphi)$-converges quasi-uniformly to f on X, in the case where $\mathcal{I}_\Lambda(\varphi)$ is a non-trivial ideal on Λ.

(2) Suppose that $(f_d)_{d\in D}$ does not $\mathcal{I}$-converge pointwise to f on X. Then, there exist $x_0 \in X \setminus M$ and $U \in \mathcal{U}$ such that for every $A \in \mathcal{I}$ there exists $d_A \notin A$ with

$$(f(x_0), f_{d_A}(x_0)) \notin U. \tag{4.5}$$

By Lemma 4.7 there exists a symmetric entourage $K \in \mathcal{U}$ such that:

(i) $K \subseteq U$.

(ii) K is closed in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Let Λ be the set $\mathcal{I}$. For $A_1, A_2 \in \Lambda$, define $A_1 \leqslant A_2$ if and only if $A_1 \subseteq A_2$. Then, $(\Lambda, \leqslant)$ is a directed set. We consider the function $\varphi : \Lambda \to D$, where $\varphi(A) = d_A$, for every $A \in \Lambda$ and the net $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$, where $g_\lambda = f_{\varphi(\lambda)}$, for every $\lambda \in \Lambda$. Let $A_\Lambda \in \mathcal{I}_\Lambda(\varphi) \setminus \{\Lambda\}$. Since $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ $\mathcal{I}_\Lambda(\varphi)$-converges quasi-uniformly to f on M, there exists a finite subset $\{A_1, \ldots, A_n\}$ of $\Lambda \setminus A_\Lambda$ such that for each $x \in M$ at least one of the following relations holds:

$$(f(x), g_{A_i}(x)) = (f(x), f_{\varphi(A_i)}(x)) = (f(x), f_{d_{A_i}}(x)) \in K,\ i = 1, \ldots, n. \tag{4.6}$$

We set

$$O_i = \{x \in X : (f(x), f_{d_{A_i}}(x)) \in (Y \times Y) \setminus K\},\ i = 1, 2, \dots, n.$$

Since the functions $f_{d_{A_i}}$ and f are continuous, by Lemma 4.2, O_i is an open set for $i = 1, \dots, n$. Moreover, since $K \subseteq U$, by relation (4.5) we have

$$(f(x_0), f_{d_{A_i}}(x_0)) \in (Y \times Y) \setminus U \subseteq (Y \times Y) \setminus K,\ i = 1, 2, \dots, n.$$

Hence, O_i is an open set containing x_0 for $i = 1, \dots, n$. Sine M is dense in X, there exists $m \in M \cap O_1 \cap \dots \cap O_n$. For this point m we have

$$(f(m), f_{d_{A_i}}(m)) \notin K,\ i = 1, \dots, n,$$

but this contradicts the relations (4.6). Thus, $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to f on X. □

5. Almost Uniform $\mathcal{I}$-Convergence

In this section we give the notion of almost uniform $\mathcal{I}$-convergence of a net of functions with values in a uniform space and we prove that the almost uniform $\mathcal{I}$-convergence preserves continuity of its limit. Modifications of this result have been obtained in [10, 13].

Definition 5.1. A net $(f_d)_{d\in D}$ of functions in a topological space X with values in a uniform space $(Y, \mathcal{U})$ is said to *$\mathcal{I}$-converge almost uniformly to f on X* if for every $x \in X$ and for every $U \in \mathcal{U}$ there exist $A \in \mathcal{I}$ and an open neighbourhood O_x of x such that for every $d \notin A$ and for every $t \in O_x$ we have $(f(t), f_d(t)) \in U$. In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$. We shall say that the net $(f_d)_{d\in D}$ *$\mathcal{I}$-converges almost uniformly on X* if there is a function to which the net $\mathcal{I}$-converges almost uniformly.

Theorem 5.2. *Let $(f_d)_{d\in D}$ be a net of continuous functions from a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$, then the function f is continuous.*

Proof. Suppose that $(f_d)_{d\in D}$ $\mathcal{I}$-converges almost uniformly to a function f. We prove that f is continuous. Let $x \in X$ and $U \in \mathcal{U}$. By Lemma 2.5 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$, there exist $A \in \mathcal{I}$ and an open neighbourhood O_x of x such that for every $d \notin A$ and for every $t \in O_x$ we have $(f(t), f_d(t)) \in V$. Let $d_0 \notin A$. Then, $(f(x), f_{d_0}(x)) \in V$. Since the function f_{d_0} is continuous at x, there exists an open neighbourhood O'_x of x such that $(f_{d_0}(x), f_{d_0}(t)) \in V$, for all $t \in O'_x$. We set $H_x = O_x \cap O'_x$. Then, H_x is an open neighbourhood of x. For every $t \in H_x$ we have $(f(t), f_{d_0}(t)) \in V$. Therefore, $(f(x), f(t)) \in V \circ V \circ V$ and the continuity of f is proved. □

Theorem 5.3. *Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D such that the family $\{f_d : d \in D\}$ is $\mathcal{I}$-equicontinuous. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, where the function f is continuous, then the $\mathcal{I}$-convergence is almost uniform.*

Proof. Suppose that $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to a continuous function f. Let $x \in X$ and $U \in \mathcal{U}$. By Lemma 2.5 there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Since $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, there exists $A_x \in \mathcal{I}$ such that

$$(f(x), f_d(x)) \in V, \text{ for every } d \notin A_x.$$

By the $\mathcal{I}$-equicontinuity of the family $\{f_d : d \in D\}$ at the point x, there exist $A'_x \in \mathcal{I}$ and an open neighbourhood O_x of x such that

$$(f_d(x), f_d(t)) \in V, \text{ for all } d \in D \setminus A'_x \text{ and for all } t \in O_x.$$

Since the function f is continuous at x, there exists an open neighbourhood O'_x of x such that

$$(f(x), f(t)) \in V, \text{ for all } t \in O'_x.$$

We set

$$H_x = O_x \cap O'_x.$$

Then, H_x is an open neighbourhood of x. We set

$$A = A_x \cup A'_x \in \mathcal{I}.$$

For every $d \notin A$ and for every $t \in H_x$ we have $(f(t), f_d(t)) \in V \circ V \circ V$. Thus, the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges almost uniformly to f. $\square$

Corollary 5.4. *Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$, where the family $\{f_d : d \in D\}$ is equicontinuous and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, where the function f is continuous, then the $\mathcal{I}$-convergence is almost uniform.*

6. Comparison of the Uniform and Almost Uniform $\mathcal{I}$-Convergence

In this section we prove that the uniform $\mathcal{I}$-convergence and the almost uniform $\mathcal{I}$-convergence coincide on compact spaces. We will give examples to indicate that this is not true for non-compact spaces.

Proposition 6.1. *Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, then $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$.*

Proof. Follows easily from Definitions 2.3 and 5.1. $\square$

Theorem 6.2. *Let $(f_d)_{d\in D}$ be a net of functions from a compact space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$, then $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$.*

Proof. Suppose that $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$ and let $U \in \mathcal{U}$. For every $x \in X$ there exist $A_x \in \mathcal{I}$ and an open neighbourhood O_x of x such that for every $d \notin A_x$ and for every $t \in O_x$ we have $(f(t), f_d(t)) \in U$. Hence, the family $\{O_x : x \in X\}$ is an open cover of X. By compactness of X, there are $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} O_{x_i}.$$

We set

$$A = \bigcup_{i=1}^{n} A_{x_i}.$$

Then, $A \in \mathcal{I}$. For every $x \in X$ and for every $d \notin A$ we have $(f(x), f_d(x)) \in U$, so $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$. $\square$

Based on the ideas of Theorems 2.2 and 2.5 of [12], consider the following two examples.

Example 6.3. Let X be a completely regular non-pseudocompact space. We construct a net $(f_d)_{d\in D}$ of functions from X into the real unit interval $[0, 1]$ endowed with the usual uniformity such that for every admissible non-trivial ideal $\mathcal{I}$ on D, $(f_d)_{d\in D}$ $\mathcal{I}$-converges almost uniformly to a continuous function f but $(f_d)_{d\in D}$ does not $\mathcal{I}$-converge uniformly to f.

Since X is not pseudocompact, there exists a locally finite family $\mathcal{F}$ of nonempty open sets which is not finite. Let $\leq$ be a well-order in $\mathcal{F}$ and let α be the order type of $(\mathcal{F}, \leq)$. By D we denote the directed set of all ordinal numbers less than α. Hence, the family $\mathcal{F}$ can be presented as $\{U_d : d \in D\}$. For each $d \in D$ we select a point $x_d \in U_d$. Since X is completely regular, there exists a continuous function $f_d : X \to [0, 1]$ such that $f_d(x_d) = 0$ and $f_d(X \setminus U_d) = \{1\}$. Consider the function $f : X \to [0, 1]$ defined by $f(t) = 1$, for every $t \in X$.

Let $\mathcal{I}$ be an admissible non-trivial ideal on D and $\mathcal{U}$ be the usual uniformity for $[0,1]$. We shall prove that $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$. Let $x \in X$ and $U \in \mathcal{U}$. Since the family $\mathcal{F}$ is locally finite, there exist an open neighbourhood O_x of x and a finite subset $\{d_1, \dots, d_n\}$ of D such that

$$O_x \cap U_{d_i} \neq \emptyset,\ i = 1, \dots, n$$

and

$$O_x \cap U_d = \emptyset, \text{ for every } d \in D \setminus \{d_1, \dots, d_n\}.$$

We set $A = \{d_1, \dots, d_n\}$. Since $\mathcal{I}$ is admissible, $A \in \mathcal{I}$. For every $d \notin A$ and for every $t \in O_x$ we have $f_d(t) = f(t) = 1$ and, therefore, $(f(t), f_d(t)) \in U$.

Now, we prove that $(f_d)_{d\in D}$ does not $\mathcal{I}$-converge uniformly to f. Indeed, let

$$U = \{(y_1, y_2) \in [0,1] \times [0,1] : |y_1 - y_2| < \frac{1}{2}\}$$

and A be an arbitrary element of $\mathcal{I}$. Since $\mathcal{I}$ is non-trivial, there exists $d \notin A$ such that $x_d \in U_d$. Hence, $|f(x_d) - f_d(x_d)| = 1$ and, therefore, $(f(x_d), f_d(x_d)) \notin U$.

Example 6.4. Let X be a completely regular space which is not locally compact. We construct a net $(f_d)_{d\in D}$ of functions from X into the real unit interval $[0,1]$ endowed with the usual uniformity such that $(f_d)_{d\in D}$ $\mathcal{I}_D$-converges uniformly to a continuous function f on compact sets but $(f_d)_{d\in D}$ does not $\mathcal{I}_D$-converge almost uniformly to f.

Since X is not locally compact, there exists $x \in X$ such that for each open neighbourhood O of x and for each compact set C we have $O \nsubseteq C$. Let $\mathcal{O}(x)$ be the family of all open neighbourhoods of x and let $\mathcal{C}$ be the family of all nonempty compact subsets of X. We consider the directed set $(D, \leqslant)$, where $D = \mathcal{O}(x) \times \mathcal{C}$ and

$$(O_1, C_1) \leqslant (O_2, C_2) \text{ if and only if } O_2 \subseteq O_1 \text{ and } C_1 \subseteq C_2.$$

For each $d = (O, C) \in D$ we select a point $x_d \in O \setminus C$. Since X is completely regular, there exists a continuous function $f_d : X \to [0,1]$ such that $f_d(x_d) = 0$ and $f_d((X \setminus O) \cup C) = \{1\}$ (C is closed because it is a compact subset of the Hausdorff space X). We consider the function $f : X \to [0,1]$ defined by $f(t) = 1$, for every $t \in X$.

Let $\mathcal{U}$ be the usual uniformity for $[0,1]$. We shall prove that $(f_d)_{d\in D} \xrightarrow{\mathcal{I}_D\text{-}u} f$ on every compact subset of X. Let K be a compact subset of X and $U \in \mathcal{U}$. Let O_x be an arbitrary open neighbourhood of x. We set $d_0 = (O_x, K)$ and $A = \{d \in D : d \ngeqslant d_0\}$. By the definition of $\mathcal{I}_D$ we have $A \in \mathcal{I}_D$. For every $t \in K$ and for every $d = (U, C) \notin A$ we have

$$f_d(t) \in f_d(K) \subseteq f_d(C) = \{1\}.$$

Therefore, $(f(t), f_d(t)) \in U$.

Now, we prove that $(f_d)_{d\in D}$ does not $\mathcal{I}_D$-converge almost uniformly to f. Indeed, let $x \in X$ and

$$U = \{(y_1, y_2) \in [0,1] \times [0,1] : |y_1 - y_2| < \frac{1}{2}\}.$$

It suffices to prove that for every $A \in \mathcal{I}_D$ and for every open neighbourhood O_x of x there exist $d \notin A$ and $t \in O_x$ such that $(f(t), f_d(t)) \notin U$. Let $A \in \mathcal{I}_D$ and O_x be an open neighbourhood of x. By the definition of $\mathcal{I}_D$ there exists $d_0 = (O_0, C_0) \in D$ such that $A \subseteq \{d \in D : d \ngeqslant d_0\}$. We set $d = (O_0 \cap O_x, C_0)$. Since $O_0 \cap O_x \subseteq O_0$, we have $d \geqslant d_0$. Hence, $d \notin A$. Moreover, $x_d \in O_x$ and $f_d(x_d) = 0$. We conclude that $|f(x_d) - f_d(x_d)| = 1$. Thus, $(f(x_d), f_d(x_d)) \notin U$.

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