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On Product of Spaces of Quasicomponents

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Abstract. We use a characterization of quasicomponents by continuous functions to obtain the well known theorem which states that product of quasicomponents Q_x, Q_y of topological spaces X, Y, respectively, gives quasicomponent in the product space $X \times Y$. If spaces X, Y are locally-compact, paracompact and Haussdorf, then we prove that the space of quasicomponents of the product $Q(X \times Y)$ is homeomorphic with the product space $Q(X) \times Q(Y)$, so these two spaces have the same topological properties.

1. Introduction

First, we repeat some basic definitions and well known facts about quasicomponents and space of quasicomponents.

The set O is *clopen* in the topological space X if it is open and closed subset of X.

The *quasicomponent* Q_x of a point x in a space X is the intersection of all clopen subsets of X which contain the point x.

Quasicomponents are closed subsets of X. The quasicomponents of two distinct points of a topological space X either coincide or are disjoint, so all quasicomponents constitute a decomposition of the space X into pairwise disjoint closed subsets. The component C_x of a point x in a topological space X is contained in the quasicomponent Q_x of the point x ([6], page 356).

For compact Hausdorff spaces, components and quasicomponents coincide ([6], Theorem 6.1.23.). Also, if the space is locally connected then components and quasicomponents coincide ([3] Prop. 2.4). Every open quasicomponent is a component ([3] Prop. 1.3).

Let QX be the set of all quasicomponents of X.

The *quasicomponent space* (or *space of quasicomponents*) of X is the space QX whose points are the quasicomponents of X and whose topology has a base consisting of sets of the form $QF = \{A \,|\, A \in QX,\ A \subseteq F\}$, where F is clopen subset of X. The space QX has a base of clopen sets (i.e., QX is 0-dimensional) and hence is regular and totally disconnected (see [1]).

For more details about quasicomponents, quasicompactification, space of quasicomponents, see [1, 3–5, 7–9].

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2. Product of Quasicomponents

In [9] quasicomponents are defined in terms of continuous functions.

Let X be a topological space and let $\{0, 1\}$ be two element space with discrete topology.

Definition 2.1. Two points $x, y \in X$ are *continuously separated* if there exists a continuous function $f : X \to \{0, 1\}$ such that $f(x) = 0$ and $f(y) = 1$.

Lemma 2.2. *The quasicomponent Q_x of a point x in a space X is the set of all points of X that could not be continuously separated from x.*

Proof. Let $a \in X$ and F_a be the set of all points of X that cannot be functionally separated from a.

We will show that $F_a = Q_a$, where Q_a is the quasicomponent of the point a.

Let $b \in Q_a$ and let suppose the contrary, that there exists a continuous function $f : X \to \{0, 1\}$ such that $f(a) = 0$, $f(b) = 1$. We have $f^{-1}(\{0\})$ is clopen subset of X that contains the point a and doesn't contain b. This is contradiction with $b \in Q_a$, so we have $b \in F_a$.

For the opposite, let $b \in F_a$. If we suppose that $b \notin Q_a$, then there exists clopen subset O of X such that $a \in O$ and $b \notin O$. If we define $g : X \to \{0, 1\}$ by $g(x) = \begin{cases} 0, & x \in O \\ 1, & x \notin O \end{cases}$, then f is continuous and it separates a from b. The last argument contradicts with $b \in F_a$, so Q_a must contain the point b.

We proved that $F_a = Q_a$. □

This definition of quasicomponents is used in [9] for proving Borsuk's theorem about mapping between spaces of quasicomponents, induced from shape morphism between topological spaces.

In [7] (Ch.V, Theorem 2) it is shown that by taking product of quasicomponents we obtain quasicomponent of product space. In this section we prove the same property using characterization by continuous functions.

Theorem 2.3. *Let X and Y be topological spaces and $x \in X$, $y \in Y$. If Q_x, Q_y are the quasicomponents of x, y, respectively, and $Q_{(x,y)}$ the quasicomponent of (x, y), then*

$$Q_{(x,y)} = Q_x \times Q_y.$$

Proof. 1) First we will prove the inclusion $Q_{(x,y)} \subseteq Q_x \times Q_y$:

Let $(a, b) \in Q_{(x,y)}$ be arbitrary. There is no continuous function from $X \times Y$ to $\{0, 1\}$ which separates the points $(a, b), (x, y)$.

Suppose that $(a, b) \notin Q_x \times Q_y$. Let $a \notin Q_x$. There exists continuous function $f : X \to \{0, 1\}$ such that $f(a) = 0$, $f(x) = 1$.

Then, the function $F : X \times Y \to \{0, 1\}$ defined by $F = f \circ p_X$ is continuous, where $p_X : X \times Y \to X$ is the projection on X.

We have $F(a, b) = f(p_X(a, b)) = f(a) = 0$ and $F(x, y) = f(p_X(x, y)) = f(x) = 1$, but this is not possible since $(a, b) \in Q_{(x,y)}$.

It follows that $(a, b) \in Q_x \times Q_y$.

In a similar way we prove the case when $b \notin Q_y$.

2) $Q_{(x,y)} \supseteq Q_x \times Q_y$:

Let $(c, d) \in Q_x \times Q_y$ i.e., $c \in Q_x$ and $d \in Q_y$. Suppose to the contrary, $(c, d) \notin Q_{(x,y)}$. There exists a continuous function $H : X \times Y \to \{0, 1\}$ such that $H(c, d) = 0$ and $H(x, y) = 1$. The space $\underline{Y} = \{x\} \times Y$ is subspace of $X \times Y$ and (x, y), $(x, d) \in \underline{Y}$. If we take the projection $p_Y : X \times Y \to Y$, then the restriction $p_Y\big|_{\underline{Y}}$ is homeomorphism from $\underline{Y}$ to Y.

We define $h : Y \to \{0, 1\}$ by $h = H\big|_{\underline{Y}} \circ \left(p_Y\big|_{\underline{Y}}\right)^{-1}$. The function h is continuous and $h(y) = H\big|_{\underline{Y}} \left(\left(p_Y\big|_{\underline{Y}}\right)^{-1}(y)\right) = H\big|_{\underline{Y}}(x, y) = H(x, y) = 1$. If we suppose that $h(d) = 1$, we obtain $H(x, d) = 1$.

The function $\alpha = H\big|_{X\times\{d\}} \circ \left(p_X\big|_{X\times\{d\}}\right)^{-1} : X \to \{0, 1\}$ is continuous and $\alpha(x) = 1$, $\alpha(c) = H(c, d) = 0$ which is not possible. So we have $h(d) = 0$, but this is a contradiction with the fact that $d \in Q_y$.

We proved that $(c, d) \in Q_{(x,y)}$, so $Q_x \times Q_y \subseteq Q_{(x,y)}$. □

3. Product of Spaces of Quasicomponents

In this section we prove that for locally compact, Hausdorff and paracompact X and Y the spaces $Q(X \times Y)$ and $QX \times QY$ are homeomorphic. At the end we show that paracompactness and locally-compactness of spaces in our theorem are important.

Definition 3.1. A *clopen box* in a space $X \times Y$ is a clopen subset of the form $U \times V$, where U and V are clopen subsets of X and Y, respectively.

We use the following theorem ([2], Theorem 3) for our proof:

Theorem 3.2 (Keneth Kunen). *Suppose X and Y are both locally compact, Hausdorff and paracompact. Then any clopen subset of $X \times Y$ is a union of clopen boxes.*

Proposition 3.3. *Let F be clopen subset of X and G is clopen subset of Y. Then $F \times G$ is clopen subset of $X \times Y$.*

Proof. It is obvious that $F \times G$ is open. The complement of the set $F \times G$ in the space $X \times Y$ is $(X \times G^C) \cup (F^C \times Y)$. F^C is open in X and G^C is open in Y so $(X \times G^C) \cup (F^C \times Y)$ is open in $X \times Y$. Hence $F \times G$ is closed. □

We can easily prove the following proposition.

Proposition 3.4. *Let $A_i, i \in I$ and $\bigcup_{i \in I} A_i$ be clopen subsets of X and $x \in \bigcup_{i \in I} A_i$. Then $Q_x \in Q\left(\bigcup_{i \in I} A_i\right)$ if and only if there exists $i \in I$ such that $Q_x \in Q(A_i)$.*

Proof. Let the requirements of the proposition be fulfilled and let $Q_x \in Q\left(\bigcup_{i \in I} A_i\right)$. Then $Q_x \in QX$ and $Q_x \subseteq \bigcup_{i \in I} A_i$. From the last inclusion there exists a $i_0 \in I$ such that $x \in A_{i_0}$. The set A_{i_0} is clopen subset of X and it contains the point x, so $Q_x \subseteq A_{i_0}$. For the opposite, let $j \in I$ and $Q_x \in Q\left(A_j\right)$ where $x \in \bigcup_{i \in I} A_i$. It implies that $Q_x \in QX$ and $Q_x \subseteq A_j$. From the last condition we have $Q_x \subseteq \bigcup_{i \in I} A_i$, so $Q_x \in Q\left(\bigcup_{i \in I} A_i\right)$. □ □

Theorem 3.5. *Suppose X and Y are both locally compact, Hausdorff and paracompact. Then the spaces $Q(X \times Y)$ and $QX \times QY$ are homeomorphic.*

Proof. We will prove that $QX \times QY \cong Q(X \times Y)$.

We define a function $f : Q(X \times Y) \to QX \times QY$ by $f\left(Q_{(x,y)}\right) = \left(Q_x, Q_y\right)$.

1) From Theorem 2.3 we obtain that the function f is well defined.

2) Again, from Theorem 2.3 it follows that f is a bijection.

We will prove the following statements:

3) f is open function.

Let $\underline{C}$ be arbitrary element from the base of $Q(X \times Y)$. Then $\underline{C} = Q\left(\underline{M}\right)$ where $\underline{M}$ is clopen subset of $X \times Y$. From Theorem 3.2 it follows that

$$\underline{M} = \bigcup_{i \in I} U_i \times V_i,$$

where U_i is clopen in X and V_i is clopen in Y for every $i \in I$.

Using Proposition 3.4 we obtain

$$Q\left(\bigcup_{i \in I} U_i \times V_i\right) = \left\{Q_{(x,y)} \,\middle|\, Q_{(x,y)} \in Q(X \times Y),\ Q_{(x,y)} \subseteq \bigcup_{i \in I} U_i \times V_i\right\} = \\ = \bigcup_{i \in I} \left\{Q_{(x,y)} \,\middle|\, Q_{(x,y)} \in Q(X \times Y),\ Q_{(x,y)} \subseteq U_i \times V_i\right\}$$

If we denote by $A_i = \left\{ Q_{(x,y)} \,\middle|\, Q_{(x,y)} \in Q(X \times Y),\ Q_{(x,y)} \subseteq U_i \times V_i \right\}$ we could simplify the previous notation as

$$Q\left(\bigcup_{i\in I} U_i \times V_i\right) = \bigcup_{i\in I} \bigcup_{Q_{(x,y)}\in A_i} \left\{Q_{(x,y)}\right\}$$

and we have

$$f\left(\underline{C}\right) = f\left(Q\left(\underline{M}\right)\right) = f\left(Q\left(\bigcup_{i\in I} U_i \times V_i\right)\right) = f\left(\bigcup_{i\in I} \bigcup_{Q_{(x,y)}\in A_i} \left\{Q_{(x,y)}\right\}\right) =$$
$$= \bigcup_{i\in I} \bigcup_{Q_{(x,y)}\in A_i} f\left(\left\{Q_{(x,y)}\right\}\right) = \bigcup_{i\in I} \bigcup_{Q_{(x,y)}\in A_i} \left\{\left(Q_x, Q_y\right)\right\}.$$

Now, from

$$\bigcup_{Q_{(x,y)}\in A_i} \left\{\left(Q_x, Q_y\right)\right\} = \left\{\left(Q_x, Q_y\right) \middle| Q_{(x,y)} \in A_i\right\} =$$
$$= \left\{\left(Q_x, Q_y\right) \middle| Q_x \in Q(X),\ Q_y \in Q(Y),\ Q_x \subseteq U_i, Q_y \subseteq V_i\right\},$$

and from: $QU_i \times QV_i = \left\{\left(Q_x, Q_y\right) \middle| Q_x \in Q(X),\ Q_y \in Q(Y),\ Q_x \subseteq U_i, Q_y \subseteq V_i\right\}$, we obtain $f\left(\underline{C}\right) = \bigcup_{i\in I} (QU_i \times QV_i)$.

The sets $QU_i \times QV_i$ are open in $QX \times QY$ for every $i \in I$ so $f\left(\underline{C}\right)$ is open in $QX \times QY$.

4) f is continuous.

Let $\underline{D}$ be a element from base of $QX \times QY$.

Then $\underline{D} = \bigcup_{\alpha\in A} QF_\alpha \times \bigcup_{\beta\in B} QG_\beta$, where QF_α is a basis element of QX and QG_β is a basis element of QY.

Hence $QF_\alpha = \{Q_x \,|Q_x \in QX,\ Q_x \subseteq F_\alpha\}$, $QG_\beta = \left\{Q_y \middle| Q_y \in QY,\ Q_y \subseteq G_\beta\right\}$.

Let

$M_\alpha = \{x \,|x \in X,\ Q_x \in QX,\ Q_x \subseteq F_\alpha\}$ and $N_\beta = \left\{y \middle| y \in Y,\ Q_y \in QY,\ Q_y \subseteq G_\beta\right\}$.

Then we have

$$QF_\alpha = \bigcup_{x\in M_\alpha} \{Q_x\} \text{ and } QG_\beta = \bigcup_{y\in N_\beta} \left\{Q_y\right\}.$$

For the inverse image we obtain

$$f^{-1}\left(\underline{D}\right) = f^{-1}\left(\bigcup_{\alpha\in A} \bigcup_{x\in M_\alpha} \{Q_x\} \times \bigcup_{\beta\in B} \bigcup_{y\in N_\beta} \left\{Q_y\right\}\right) =$$
$$= f^{-1}\left(\bigcup_{(\alpha,\beta)\in A\times B} \bigcup_{(x,y)\in M_\alpha\times N_\beta} \left\{\left(Q_x, Q_y\right)\right\}\right) =$$
$$= \bigcup_{(\alpha,\beta)\in A\times B} \bigcup_{(x,y)\in M_\alpha\times N_\beta} f^{-1}\left\{\left(Q_x, Q_y\right)\right\} =$$
$$= \bigcup_{(\alpha,\beta)\in A\times B} \bigcup_{(x,y)\in M_\alpha\times N_\beta} \left\{Q_{(x,y)}\right\}$$

For $\bigcup_{(x,y)\in M_\alpha\times N_\beta} \left\{Q_{(x,y)}\right\}$ we have

$$\bigcup_{(x,y)\in M_\alpha\times N_\beta} \left\{Q_{(x,y)}\right\} = \left\{Q_{(x,y)} \middle| Q_x \subseteq F_\alpha,\ Q_y \subseteq G_\beta\right\} =$$
$$= \left\{Q_{(x,y)} \middle| Q_{(x,y)} \subseteq F_\alpha \times G_\beta\right\} = Q\left(F_\alpha \times G_\beta\right).$$

From Proposition 3.3 it follows that the set $f^{-1}\left(\underline{D}\right) = \bigcup_{(\alpha,\beta)\in A\times B} Q\left(F_\alpha \times G_\beta\right)$ is open in $Q(X \times Y)$. □

Theorem 3.6. *Let $Q(X \times Y) \cong QX \times QY$, then every clopen subset W of the product $X \times Y$ can be represented as a union of clopen boxes.*

Proof. Let W be clopen subset of $X \times Y$. From $Q(X \times Y) \cong QX \times QY$, there exists e homeomorphism $f : Q(X \times Y) \to QX \times QY$ hence $f(QW)$ is open in $Q(X) \times Q(Y)$. Therefore $f(QW) = \bigcup_{\alpha \in I} (U_\alpha \times V_\alpha)$, where

$$U_\alpha = \bigcup_{i \in A_\alpha} QF_{\alpha,i} \text{ and } V_\alpha = \bigcup_{j \in B_\alpha} QG_{\alpha,j}$$

In a similar way as in Theorem 3.5 we prove that

$$QW = f^{-1}\left(\bigcup_{\alpha \in I} (U_\alpha \times V_\alpha)\right) = \bigcup_{\alpha \in I} \bigcup_{(i,j) \in A_\alpha \times B_\alpha} Q\left(F_{\alpha,i} \times G_{\alpha,j}\right).$$

It is easy to show that

$$W = \bigcup_{\alpha \in I} \bigcup_{(i,j) \in A_\alpha \times B_\alpha} \left(F_{\alpha,i} \times G_{\alpha,j}\right).$$

□

Examples 1 and 2 from [2] together with Theorem 3.6 ensures us that paracompactness and local compactness could not be omitted in Theorem 3.5.

Remark 3.7. Let local compactness from Theorem 3.5 be omitted. From Example 1 of [2] it follows that there exist two separable metrizable spaces X and Y whose product contains a clopen subset that cannot be represented as a union of clopen boxes. This is contradiction with Theorem 3.6.

Remark 3.8. If paracompactness from Theorem 3.5 is omitted, then from Example 2 of [2] it follows that there exist two locally compact Hausdorff spaces X and Y whose product contains a clopen subset that cannot be represented as a union of clopen boxes. This is a contradiction with Theorem 3.6.

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