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Analysis of Direct and Inverse Problems for a Fractional Elastoplasticity Model

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Abstract. This study is devoted to a nonlinear time fractional inverse coeficient problem. The unknown coefficient depends on the gradient of the solution and belongs to a set of admissible coefficients. First we prove that the direct problem has a unique solution. Afterwards we show the continuous dependence of the solution of the corresponding direct problem on the coefficient, the existence of a quasi-solution of the inverse problem is obtained in the appropriate class of admissible coefficients.

1. Introduction

According to deformation theory of plasticity, the stress-strain relation between deviators is described by the Hencky correlation

$$\sigma_{ij}^D = 2g(\Gamma^2)\epsilon_{ij}^D, \quad i,j = 1,2,3.$$

Then the following relation holds between the intensities of shift strain $\Gamma := \left(2\epsilon_{ij}^D\epsilon_{ij}^D\right)^{\frac{1}{2}}$ and tangential stress $T := \left(\frac{1}{2}\sigma_{ij}^D\sigma_{ij}^D\right)^{\frac{1}{2}}$

$$T = g\left(\Gamma^2\right)\Gamma, \tag{1}$$

where the function $g\left(\Gamma^2\right)$ describes the elastoplastic properties of the material and is sometimes called the modulus of plasticity. Equation (1) can be formally regarded as a general condition encompassing different phases strain. Thus, putting $g\left(\Gamma^2\right) = \frac{\tau_s}{\Gamma}$, we obtain the Von Mises's criterion $T = \tau_s$; while putting $g\left(\Gamma^2\right) = G$, we obtain the case of Hooke's elastic medium, where $T = G\Gamma$ and $G = E/(2(1+\nu))$ is the modulus of rigidity (shear modulus), $E > 0$ is the Young's modulus, $\nu \in \left(0, \frac{1}{2}\right)$ is the Poisson coefficient. The shear modulus is defined as the ratio of shear stress to the shear strain. It describes an object's tendency to shear when acted upon by opposing forces. Also it is used to determine how elastic or bendable materials evolve if they are

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sheared, which is being pushed parallel from opposite sides.

According to deformation theory of plasticity, the function $g\left(\Gamma^2\right)$ satisfies the following conditions [18]:

$$\begin{cases} c_1 \le g(\Gamma^2) \le c_2, \\ g(\Gamma^2) + 2g'(\Gamma^2)\Gamma^2 \ge c_3 > 0, \forall \Gamma^2 \in \left[\Gamma_*{}^2, \Gamma^{*2}\right], \\ g'(\Gamma^2) \le 0, \\ \exists \Gamma_0{}^2 \in (\Gamma_*{}^2, \Gamma^{*2}): \; g(\Gamma^2) = G, \; \forall \Gamma^2 \in \left[\Gamma_0{}^2, \Gamma^{*2}\right], \end{cases} \tag{2}$$

where $c_i > 0, \; i = 1, 2, 3$ are constants. Thus $g(\Gamma^2)$ is a decreasing function of Γ^2 with $c_1 \le g(\Gamma^2) \le c_2$, there exists an inverse function $\Gamma = f(T^2)T$ such that $g(\Gamma^2)f(T^2) = 1$ and $f(T^2)$ satisfies the following conditions [18]:

$$\begin{cases} c_4 \le f(T^2) \le c_5, \\ c_4 \le f(T^2) + 2f'(T^2)T^2 \le c_6, \forall T^2 \in \left[T_*{}^2, T^{*2}\right], \\ f'(T^2) \ge 0, \\ \exists T_0{}^2 \in (T_*{}^2, T^{*2}): \; f(T^2) = \frac{1}{G}, \; \forall T^2 \in \left[T_0{}^2, T^{*2}\right], \end{cases} \tag{3}$$

where $c_4 = \frac{1}{c_2}$, $c_5 = \frac{1}{c_1}$ and $c_6 = \frac{1}{c_3}$. A set $\mathbb{F}$ satisfying the conditions (3) is called the class of admissible coefficients in optimal control and inverse problems theory. We note that (3) implies the following inequality for the function $f = f(T^2), \forall T^2 \in \left[T_*{}^2, T^{*2}\right]$

$$\left(f(T^2)T - f(\widetilde{T}^2)\widetilde{T}\right)(T - \widetilde{T}) \ge c\left|T - \widetilde{T}\right|^2, \tag{4}$$

where c is a positive constant.

The quasi-static mathematical model of the elastoplastic torsion of a strain hardening bar is given in [15]. In this model, one seeks the solution $u(x), x \in \Omega \subset \mathbb{R}^2$, of the following nonlinear boundary value problem:

$$\begin{cases} -\nabla.\left(f(T^2)\nabla u\right) = 2\varphi, \; x \in \Omega \subset \mathbb{R}^2, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{5}$$

where $\Omega := (0, a) \times (0, b), a, b > 0$ is the cross section of a bar, φ is the angle of twist per unit length, $T^2 := |\nabla u|^2$ is the stress intensity and $u(x)$ is the Prandtl stress function. Now we define the parameters in (3). First, we define $T_0{}^2 := \max_{x\in\Omega} |\nabla u|^2$. In materials science, it corresponds to the yield stress which is the maximum stress or force per unit area within a material that can arise before the onset of permanent deformation. When stresses up to the yield stress are removed, the material resumes its original size and shape. In other words, there is a temporary shape change that is self-reversing after the force is removed, so that the object returns to its original shape. This kind of deformation is called pure elastic deformation. On the other hand, irreversible deformations are permanent even after stresses have been removed. One type of irreversible deformation is pure plastic deformation. For such materials the yield stress marks the end of the elastic behavior and the beginning of the plastic behavior. For any angle $\varphi > 0$, all points of the bar have non-zero stress intensity which means the condition $T_*{}^2 > 0$ in (3) makes sense. It is also known that in order for the equation in (5) to be elliptic, the first two conditions in (3) are necessary. Further, the last condition of (3) means that the elastic deformations precede the plastic ones.

Let $u = u(x, \varphi; f)$ be the solution of the nonlinear boundary value problem (5) for an angle φ and a function f. Then theoretical value of the torque (moment of force) is given by

$$M[f](\varphi) = 2\int_\Omega u(x, \varphi; f)dx, \; \varphi \in [\varphi_*, \varphi^*], \; \varphi_* > 0, \tag{6}$$

i.e., the torque is equal to twice the volume enclosed within the stress surface $u(x)$ [15].

For a given function $f(T^2)$ and the angle φ, the problem (5) is called the direct (forward) problem. The associated inverse problem consists of determining the pair of functions $\left\{u(x), f(T^2)\right\}$ from the following nonlocal nonlinear identification problem:

$$\begin{cases} -\nabla.\Big(f(T^2)\nabla u\Big) = 2\varphi, \ x \in \Omega \subset \mathbb{R}^2, \\ u(x) = 0, \quad x \in \partial\Omega, \\ 2\int\limits_{\Omega} u(x;\varphi_i)dx = \mathcal{M}_i, \quad i = 1, \cdots, N, \end{cases} \tag{7}$$

where $\mathcal{M}_i := \mathcal{M}(\varphi_i)$ are the measured values of the torque, (measured output data) corresponding to the angles φ_i, $i = 1, \cdots, N$ and $N > 1$ is the number of measurements. These discrete values are assumed to be given during the quasi-static process of torsion, given by the angle of twist $\varphi_i \in [\varphi_*, \varphi^*]$. Therefore in the considered physical model the quasi-static process of torsion is simulated by the monotone increasing values $0 < \varphi_* = \varphi_1 < \varphi_2 < \cdots < \varphi_N = \varphi^*$ of the angle $\varphi \in [\varphi_*, \varphi^*]$. Hence the torque, defined by (6), may be considered as a function of the angle $\varphi > 0$. The direct problem (5) and the inverse problem (7) have been well-studied both theoretically and numerically in the mathematical literature (see [11], [12], [13], [33], [34], [35]).

In practice, a quasi-static process must be carried out on a time-scale which is much longer than the relaxation time of the system. Note that the relaxation time is the typical time-scale for the system to return to equilibrium after being suddenly disturbed. However in real applications the torsion process is not quasi-static, it depends on the time. The mathematical model of the real torsion process is given by the following evolutional problem:

$$\begin{cases} u_t - \nabla.\Big(f(T^2)\nabla u\Big) = 2t, \ (x,t) \in \Omega_{\mathcal{T}}, \\ u(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\mathcal{T}), \end{cases} \tag{8}$$

where $\Omega_{\mathcal{T}} := \Omega \times (0,\mathcal{T})$. The associated inverse problem consists of determining the pair of functions $\left\{u(x,t), f(T^2)\right\}$ from the following nonlocal nonlinear identification problem:

$$\begin{cases} u_t - \nabla.\Big(f(T^2)\nabla u\Big) = 2t, \ (x,t) \in \Omega_{\mathcal{T}}, \\ u(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\mathcal{T}), \\ 2\int\limits_{\Omega} u(x;t_i)\, dx = \mathcal{M}(t_i), \quad i = 1, \cdots, N, \end{cases} \tag{9}$$

where $\mathcal{M}_i := \mathcal{M}(t_i)$, $i = 1, \cdots, N$ are the measured values of the torque. Similar to (6), the theoretical value of the torque is defined by

$$M[f](t) = 2\int\limits_{\Omega} u(x,t;f)dx, \ t \in [t_*, t^*], \ t_* > 0, \tag{10}$$

where $u(x,t;f)$ is a solution of (8) for a given function f.

The direct problem (8) and the inverse problem (9) have been well-studied both theoretically and numerically in the mathematical literature [36].

A strong motivation for investigating equations involving fractional time derivatives comes from physics. Fractional diffusion equations describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials). In normal diffusion, described by the heat equation or more general parabolic equations, the mean square displacement of a diffusive particle behaves like constant t for $t \to \infty$. A typical behavior for anomalous diffusion is constant t^{β}, and this was the reason to invoke equations with fractional time derivative, see [6] and some of the references cited therein.

We now motivate the use of fractional time derivative for our model, which is considered in an anomalous medium. We first note that the classical diffusion equation $\partial_t u = \Delta u$ is used to describe a cloud of spreading particles at the macroscopic level. The point source solution is a Gaussian probability density that predicts the relative particle concentration. For microscopic picture, Brownian motion is employed, which describes the path of individual particles. The space-time fractional diffusion equation $\partial_t^{\beta} u = -(-\Delta)^{\alpha/2} u$ with $0 < \beta < 1$ and $0 < \alpha < 2$ is used to model anomalous diffusion [23]. Here $(-\Delta)^{\alpha/2}$ is the fractional Laplacian operator, see [16], [37], [39] and some of the references cited therein to get further background on it. In this equation the fractional derivative in time is used to describe particle sticking and trapping phenomena and the fractional space derivative is used to model long particle jumps. These two effects combined together produces a concentration profile with a sharper peak, and heavier tails.

When a porous medium equation is considered in an anomalous medium, fractional derivative appears in the new model due to the new scaling relations between time and space variables. Such an equation has the following general form in one space dimensions [8]

$$\frac{\partial^{\beta} c}{\partial t^{\beta}} = \frac{\partial}{\partial x}\left(D_f \frac{\partial^{\gamma} c}{\partial x^{\gamma}}\right), \tag{11}$$

where D_f is a constant and called the effective coefficient of diffusion. This equation can be found in many papers related to the diffusion phenomenon in the chaotic migration of the particles, see [4, 25] and some of the references cited therein. It is shown in [7] that this equation can effectively be used for modeling the anomalous contaminant diffusion from a fracture into a porous rock matrix with an alteration zone bordering the fracture. Assuming the porous medium has a comb-like structure of fractal geometry, $\gamma = 1$ case is also considered in the mathematical literature, see [26]-[28].

Fractional derivative is also used in mechanics. For example, in [24], the authors present a one-dimensional elastoplastic model based on fractional calculus, in order to improve the elastoplastic behaviour modeling by taking advantage of the nonlocality property of fractional operators. The general form of the proposed model is defined as a series of fractional derivatives

$$\sum_{q=1}^{q_{max}} a_q D^{\beta_q} \sigma(\epsilon) = 1,$$

where $D^{\beta_q}\sigma(\epsilon)$ represents the β_q order derivative of stress σ with respect to strain ϵ. Another interesting application is related to viscoelastic materials [5]. The behaviors of viscoelastic materials with memory are usually described by Kelvin model, Voigt model, Maxwell model, and so on, in terms of the strain, the stress, and their integer-order derivatives or integrals. Comparing with fractional models, such integer-order models reflect memory effects much less accurately. Early observations show that viscoelastic materials behave between elasticity and viscosity. It is reasonable, hence, to assume that models of viscoelasticity take the form of the following equation

$$D_t^{\beta} \epsilon(t) = \nu \sigma(t),$$

where $D_t^{\beta}\epsilon(t)$ is the fractional derivative which depends on the strain history from 0 to t, and ν is a positive constant. This model covers two extremes: $\beta = 0$ for Hooke's Law of elasticity, and $\beta = 1$ for Newton's Law of viscosity. The authors also find that the order of fractional derivative is an index of memory. This is an answer to the open problem: what is the physical meaning of fractional derivative ? In a very recent paper

[32], the application of fractional continuum mechanics to rate independent plasticity is presented. In the presented model, a small fractional strains assumption is used together with additive decomposition of total fractional strains into elastic and plastic parts. Classical local rate independent plasticity is recovered as a special case.

Motivated by the above mentioned works, in this paper we consider our elastoplastic torsional equation in an environment where anomalous torsion takes place. For this setting, the following fractional elastoplastic problem is considered where the usual time derivative is replaced by a fractional time derivative. The general form of our problem reads

$$\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta}u(x,t) - \nabla.\Big(f(T^2)\nabla u\Big) = 2t, \ (x,t) \in \Omega_{\mathcal{T}}, \\ u(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\mathcal{T}), \end{cases} \tag{12}$$

where $\mathcal{T} > 0$ is a final time, $\beta \in (0,1)$ is the fractional order of the time derivative, $\frac{\partial^\beta}{\partial t^\beta}u(x,t)$ is the Caputo time-fractional derivative of order $0 < \beta < 1$ and is defined by

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} := \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \frac{\partial}{\partial \tau} u(x,\tau) d\tau, \tag{13}$$

where Γ is the Gamma function. There is another kind of fractional derivative used frequently called Riemann-Lioville fractional derivative. We note that these two fractional derivatives agree when the initial condition is zero. So for our case any result for one of these can be used throughout the paper since our initial condition is zero. Kilbas et al [17] and Podlubny [29] can be referred for further properties of the Caputo and Riemann-Lioville fractional derivatives.

For given input $f(T^2)$, the problem (12) is called the direct problem. In this paper, we study the following inverse problem that consists of determining the pair of functions $\Big\{u(x,t), f(T^2)\Big\}$ from the following nonlocal nonlinear identification problem:

$$\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta}u(x,t) - \nabla.\Big(f(T^2)\nabla u\Big) = 2t, \ (x,t) \in \Omega_{\mathcal{T}}, \\ u(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\mathcal{T}), \\ 2\displaystyle\int_\Omega u(x;t_i)\, dx = \mathcal{M}(t_i), \quad i = 1, \cdots, N. \end{cases} \tag{14}$$

Recently, there has been a growing interest in inverse problems with fractional derivatives. Usually, in these works a fractional time derivative is considered and determination of that or a source term under some additional condition(s) is the inverse problem. These problems are physically and practically very important. We list some of the references [3, 14, 19–21, 30, 31, 37–40, 42, 43, 45]. Our paper can be regarded as an addition to these studies.

This paper is organized as follows: In the next section we formulate the inverse problem as a minimization problem and present the quasi-solution approach. Existence and uniqueness of solutions for the direct and inverse problems under consideration are discussed in Section 3.

2. Formulation of the Direct and the Inverse Problems

Throughout the paper while, $\|\cdot\|$ and $\Big\langle , \Big\rangle$ denote the usual $L^2(\Omega)-$norm and inner product respectively, $\|\cdot\|_X$ denotes the norm in a Hilbert space X. Now we define a weak solution of the problem (12).

Definition 2.1. *A weak solution of the problem (12) is a function* $u \in L^2\left(0, \mathcal{T}; H_0^1(\Omega)\right) \cap W_2^\beta\left(0, \mathcal{T}; L^2(\Omega)\right)$ *such that the following integral identity holds for a.e.* $t \in [0, \mathcal{T}]$

$$\int\limits_\Omega \frac{\partial^\beta u}{\partial t^\beta} v\, dx + \int\limits_\Omega f\left(|\nabla u|^2\right) \nabla u \cdot \nabla v\, dx = \int\limits_\Omega 2t\, v\, dx, \tag{15}$$

for each $v \in L^2\left(0, \mathcal{T}; H_0^1(\Omega)\right) \cap W_2^\beta(0, \mathcal{T}; L_2(\Omega))$, *where* $W_2^\beta(0, \mathcal{T}) := \left\{u \in L^2[0, \mathcal{T}] : \frac{\partial^\beta u}{\partial t^\beta} \in L^2[0, \mathcal{T}] \text{ and } u(0) = 0\right\}$ *is the fractional Sobolev space of order* β.

Hereafter, we denote the space $L^2\left(0, \mathcal{T}; H_0^1(\Omega)\right)$ by V. A weak solution of the problem (12) is also defined as a solution of the following abstract operator equation

$$Lu + Au = F, \tag{16}$$

where $Lu := \left\langle \hat{L}u, v\right\rangle$, $\hat{L}u := \frac{\partial^\beta u}{\partial t^\beta}$, $\hat{L} : D(\hat{L}) \subset V \to V^*$ with the domain $D(\hat{L}) = \left\{u \in V : \frac{\partial^\beta u}{\partial t^\beta} \in V^*\right\}$, the nonlinear operator $A : V \to V^\star$ is defined by

$$\left\langle Au, v\right\rangle := \int\limits_\Omega f\left(|\nabla u|^2\right) \nabla u \cdot \nabla v\, dx, \tag{17}$$

and F on V is defined by

$$\left\langle F, v\right\rangle := \int\limits_\Omega 2t\, v\, dx. \tag{18}$$

For each $t \in (0, \mathcal{T})$, the inverse problem can be reformulated as a solution of the following nonlinear functional equation

$$2\int\limits_\Omega u(x, t; f)\, dx = \mathcal{M}(t),\ f \in \mathbb{F}. \tag{19}$$

Evidently, in practice the measured output data $\mathcal{M}(t)$ may only be given with some measurement error, which means that exact fulfillment of the equation (19) is not possible. Hence, one needs to introduce the following auxiliary (cost) functional

$$I(f) = \int_0^{\mathcal{T}} \left| \int\limits_\Omega u(x, t; f) dx - \mathcal{M}(t)\right|^2 dt, \tag{20}$$

and consider the following minimization problem:

$$I(\widetilde{f}) = \min_{f \in \mathbb{F}} I(f). \tag{21}$$

A solution of the minimization problem (21) is called quasi-solution (or approximate solution) of the considered inverse problem [41]. Evidently, the existence of the quasi-solution depends on the compactness of the class of admissible coefficients $\mathbb{F}$ and continuity of the functional (20). In [22], it is proved that the class of admissible coefficients $\mathbb{F}$ is compact in $C[0, T^{*2}]$. We refer the readers to Section 3 for details on the continuity of the functional (20).

3. Analysis of the Direct and Inverse Problems

In this section, we prove that the direct problem (12) has a unique solution and the solution depends continuously on the coefficient. Also we prove existence of the quasi-solution of the inverse problem in the appropriate class of admissible coefficients.

Theorem 3.1. *Let $f \in \mathbb{F}$. Then the direct problem (12) has a unique weak solution $u \in L^2\left(0,\mathcal{T};H_0^1(\Omega)\right) \cap W_2^\beta\left(0,\mathcal{T};L^2(\Omega)\right)$. Moreover, for a.e $t \in [0,\mathcal{T}]$ there exist some constant $c, C > 0$ such that*

$$\frac{\partial^\beta}{\partial t^\beta}\left(\|u\|^2\right) + c\,\|u\|^2_{H_0^1(\Omega)} \le C\mathcal{T}^2.$$

Proof. First, we prove that the direct problem (12) has a unique weak solution. We note that the operator $\hat{L}$ is linear, densely defined and m-accretive, see [2]. By Proposition 31.5 in [44], the operator L is a maximal monotone operator. We also note that the operator A is bounded, coercive, strongly monotone and hemicontinuous, see [9], [10] and some of the references cited therein. Proposition 27.6 with Theorem 31.A in [44] imply the existence and uniqueness of solutions to problem (12).

Let u be a weak solution of (12) and let us take $v = u$ in the weak formulation (15). We then have

$$\int_\Omega \frac{\partial^\beta u}{\partial t^\beta}\, u\, dx + \int_\Omega f\left(|\nabla u|^2\right)|\nabla u|^2\, dx = \int_\Omega 2t\, u\, dx. \tag{22}$$

By the first condition of (3), it follows that

$$c_4 \int_\Omega |\nabla u|^2\, dx \le \int_\Omega f\left(|\nabla u|^2\right)|\nabla u|^2\, dx. \tag{23}$$

Moreover, using Cauchy-Schwarz, Young and Poincaré inequalities we have

$$\int_\Omega 2t\, u\, dx \le \varepsilon \int_\Omega |\nabla u|^2\, dx + C_\varepsilon \mathcal{T}^2. \tag{24}$$

Now, employing the Alikhanov inequality [1] we have

$$\int_\Omega \frac{\partial^\beta u}{\partial t^\beta}\, u\, dx \ge \frac{1}{2}\frac{\partial^\beta \|u\|^2}{\partial t^\beta}. \tag{25}$$

Using (23) - (25) in (22) we get

$$\frac{\partial^\beta \|u\|^2}{\partial t^\beta} + c \int_\Omega |\nabla u|^2\, dx \le C\mathcal{T}^2. \tag{26}$$

This completes the proof. □

Theorem 3.2. *Suppose that a sequence of coefficients $\{f_m\} \subset \mathbb{F}$ converges pointwise in $[0,\infty)$ to a function $f \in \mathbb{F}$. Then the sequence of solutions $u_m := u(x,t;f_m)$ converges to the solution $u := u(x,t;f) \in L^2\left(0,\mathcal{T};H_0^1(\Omega)\right) \cap W_2^\beta\left(0,\mathcal{T};L^2(\Omega)\right)$, where $u := u(x,t;f)$ denotes the solution of the direct problem (12) for a given coefficient $f \in \mathbb{F}$.*

Proof. Since $f, f_m \in \mathbb{F}$, by Theorem (3.1), the solutions u, u_m are well-defined. Then, by (15) we have for a.e $t \in [0, \mathcal{T}]$

$$\int\limits_{\Omega} \frac{\partial^\beta u_m}{\partial t^\beta} v\, dx + \int\limits_{\Omega} f_m\left(|\nabla u_m|^2\right)\nabla u_m \cdot \nabla v\, dx = \int\limits_{\Omega} 2tv\, dx, \tag{27}$$

$$\int\limits_{\Omega} \frac{\partial^\beta u}{\partial t^\beta} v\, dx + \int\limits_{\Omega} f\left(|\nabla u|^2\right)\nabla u \cdot \nabla v\, dx = \int\limits_{\Omega} 2tv\, dx. \tag{28}$$

(27) and (28) imply, by taking $v = u_m - u$, that

$$\begin{aligned} &\int\limits_{\Omega} \frac{\partial^\beta (u_m - u)}{\partial t^\beta} (u_m - u)\ dx \\ &+ \int\limits_{\Omega} \left[f_m\left(|\nabla u_m|^2\right)\nabla u_m - f\left(|\nabla u|^2\right)\nabla u\right] \cdot \nabla (u_m - u)\ dx = 0. \end{aligned} \tag{29}$$

Employing Alikhanov inequality [1] in (29) we get

$$\begin{aligned} &\frac{1}{2}\frac{\partial^\beta \|u_m - u\|^2}{\partial t^\beta}\, dx \\ &+ \int\limits_{\Omega} \left[f_m\left(|\nabla u_m|^2\right)\nabla u_m - f_m\left(|\nabla u|^2\right)\nabla u\right] \cdot \nabla (u_m - u)\ dx \\ &+ \int\limits_{\Omega} \left[f_m\left(|\nabla u|^2\right) - f\left(|\nabla u|^2\right)\right] \nabla u \cdot \nabla (u_m - u)\ dx \le 0. \end{aligned} \tag{30}$$

By (4) we have

$$\begin{aligned} &\frac{1}{2}\frac{\partial^\beta \|u_m - u\|^2}{\partial t^\beta}\, dx + c\int\limits_{\Omega} |\nabla (u_m - u)|^2\, dx \\ &\le \int\limits_{\Omega} \left|\left[f\left(|\nabla u|^2\right) - f_m\left(|\nabla u|^2\right)\right] \nabla u \cdot \nabla (u_m - u)\right| dx. \end{aligned} \tag{31}$$

By using Cauchy-Schwarz inequality, the first condition in (3) and Lebesgue's Dominated Convergence Theorem we obtain

$$\begin{aligned} &\lim_{m\to\infty}\left(\frac{\partial^\beta}{\partial t^\beta}\|u_m - u\|^2\right) = 0, \\ &\lim_{m\to\infty}\left\|\nabla (u_m - u)\right\| = 0. \end{aligned} \tag{32}$$

□

Now we prove the following existence theorem.

Theorem 3.3. *The inverse problem (21) has at least one quasi-solution in the set of admissible coefficients* $\mathbb{F}$.

Proof. Let $\{f_m\} \subset \mathbb{F}$ be a minimizing sequence of the functional I, defined by (20). Since the class of admissible coefficients $\mathbb{F}$ is compact in $C[0, T^{*2}]$, there exists a subsequence, which we still denote by $\{f_m\}$, such that $\|f_m - f\|_C \to 0$ as $m \to \infty$. By Theorem 3.2, the sequence $u_m = u(x, t; f_m)$ converges to $u = u(x, t; f)$ in $L^2\left(0, \mathcal{T}; H_0^1(\Omega)\right)$. Therefore we have

$$\min_{f \in \mathbb{F}} I(f) = \lim_{m \to \infty} I(f_m) = I(f).$$

□

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