



# On the Stability of Solution Mappings Parametric Generalized Vector Quasivariational Inequality Problems of the Minty Type

Lam Quoc Anh<sup>a</sup>, Nguyen Van Hung<sup>b,c</sup>

<sup>a</sup>Department of Mathematics, Teacher College, Cantho University, Cantho, Vietnam

<sup>b</sup>Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam

<sup>c</sup>Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

**Abstract.** In this paper, we study two parametric weak and strong vector quasivariational inequality problems of the Minty type. The stability properties of the exact solution sets and approximate solution sets for these problems such as the upper semicontinuity, the lower semicontinuity, the Hausdorff lower semicontinuity, the continuity and the Hausdorff continuity are obtained. The results presented in the paper improve and extend the main results in the literature.

## 1. Introduction and Preliminaries

A vector variational inequality problem was first introduced and studied by Giannessi [15] in the setting of finite-dimensional Euclidean spaces. Since then, many authors have investigated vector variational inequality problems in abstract spaces, see [11–14, 18, 19, 23–25, 27–31] and the references therein. Semicontinuity of the solution sets for parametric vector variational inequality problems is an important topic in optimization theory and applications. Recently, the semicontinuity, especially the upper semicontinuity, the lower semicontinuity and the Hausdorff lower semicontinuity of the solution sets for parametric optimization problems [22, 33], parametric vector variational inequality problems [12, 13, 18, 21, 23–25, 27, 30, 31] and parametric vector quasiequilibrium problems [1–8, 16, 17, 20, 26, 32] have been established by many authors in different ways.

In 2007, Khanh and Luu [24] established the sufficient conditions for the exact solution sets and approximate solution sets of parametric scalar quasivariational inequality problems with the stability properties such as the upper semicontinuity and lower semicontinuity. Very recently, Lalitha and Bhatia [25] considered a parametric scalar quasivariational inequality problem of the Minty type, and the upper semicontinuity, lower semicontinuity, Hausdorff lower semicontinuity of the exact solution sets and approximate solution sets for this problem were also obtained.

---

2010 *Mathematics Subject Classification.* Primary 90C31; Secondary 49J53, 49J40, 49J45

*Keywords.* Parametric weak and strong generalized vector quasivariational inequality problems of the Minty type; Upper semicontinuity; Lower semicontinuity; Hausdorff lower semicontinuity; Continuity; Hausdorff continuity; Closedness.

Received: 02 January 2015; Accepted: 12 March 2015

Communicated by Predrag Stanimirović

The corresponding author: Nguyen Van Hung

The first author was funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2017.18. The second author was supported by Ministry of Education and Training of Vietnam under grant number B2017.SPĐ01.

*Email addresses:* quocanh@ctu.edu.vn (Lam Quoc Anh), nguyenvanhung2@tdt.edu.vn (Nguyen Van Hung)

Motivated by research works mentioned above, in this paper, we introduce two kinds of parametric weak and strong generalized vector quasivariational inequality problems of the Minty type in Hausdorff topological vector spaces. We also discuss the upper semicontinuity, lower semicontinuity, Hausdorff lower semicontinuity, continuity and Hausdorff continuity of the exact solution sets and approximate solution sets for these problems. Our results are new and extend corresponding results in the literature.

Let  $X, Y$  be two Hausdorff topological vector spaces and  $\Gamma, \Lambda$  be two topological vector spaces. Let  $C \subset Y$  be a closed, convex and pointed cone with  $\text{int}C \neq \emptyset$ . The cone  $C$  induces a partial ordering in  $Y$  defined by

$$\begin{aligned} y \geq x &\Leftrightarrow y - x \in C, \quad \forall x, y \in Y, \\ y \not\geq x &\Leftrightarrow y - x \notin C, \quad \forall x, y \in Y, \\ y < x &\Leftrightarrow y - x \in -\text{int}C, \quad \forall x, y \in Y, \\ y \not< x &\Leftrightarrow y - x \notin -\text{int}C, \quad \forall x, y \in Y, \end{aligned}$$

where  $\text{int}C$  denotes the interior of  $C$ .

Let  $L(X, Y)$  be the space of all linear continuous operators from  $X$  into  $Y$ , and  $A \subset X$  be a nonempty subset. Let  $K_1 : A \times \Gamma \rightarrow 2^A, K_2 : A \times \Gamma \rightarrow 2^A$  and  $T : A \times \Gamma \rightarrow 2^{L(X, Y)}$  be set-valued mappings, and let  $H : L(X, Y) \rightarrow L(X, Y), \psi : A \times A \times \Lambda \rightarrow A$  be continuous single-valued mappings. Denoted by  $\langle z, x \rangle$  the value of a linear operator  $z \in L(X, Y)$  at  $x \in X$ , we always assume that  $\langle \cdot, \cdot \rangle$  is continuous.

Now we adopt the following notations (see [4, 16, 17]). Letters  $w, m$  and  $s$  are used for weak, middle and strong, respectively, kinds of considered problems. For subsets  $M$  and  $N$  under consideration we adopt the notations

$$\begin{aligned} (u, v) \text{ w } M \times N &\text{ means } \forall u \in M, \exists v \in N, \\ (u, v) \text{ m } M \times N &\text{ means } \exists v \in N, \forall u \in M, \\ (u, v) \text{ s } M \times N &\text{ means } \forall u \in M, \forall v \in N \\ (u, v) \bar{w} M \times N &\text{ means } \exists u \in M, \forall v \in N \text{ and similarly for } \bar{m}, \bar{s}. \end{aligned}$$

Let  $\alpha \in \{w, m, s\}, \bar{\alpha} \in \{\bar{w}, \bar{m}, \bar{s}\}$  and for  $\gamma \in \Gamma, \lambda \in \Lambda$ , we consider the following parametric weak and strong generalized vector quasivariational inequality problems of the Minty type (in short, (WMQVIP) and (SMQVIP)), respectively.

**(WMQVIP)** Find  $\bar{x} \in K_1(\bar{x}, \gamma)$  such that  $(y, z) \alpha K_2(\bar{x}, \gamma) \times T(y, \gamma)$  satisfying

$$\langle H(z), \psi(y, \bar{x}, \lambda) \rangle \not\leq 0.$$

**(SMQVIP)** Find  $\bar{x} \in K_1(\bar{x}, \gamma)$  such that  $(y, z) \alpha K_2(\bar{x}, \gamma) \times T(y, \gamma)$  satisfying

$$\langle H(z), \psi(y, \bar{x}, \lambda) \rangle \geq 0.$$

For each  $\gamma \in \Gamma, \lambda \in \Lambda$ , and let  $E(\gamma) := \{x \in A \mid x \in K_1(x, \gamma)\}$ . We denote  $\Psi_\alpha(\gamma, \lambda)$  and  $\Xi_\alpha(\gamma, \lambda)$  are solution sets of (WMQVIP) and (SMQVIP), respectively.

Throughout the article, we assume that  $\Psi_\alpha(\gamma, \lambda) \neq \emptyset$  and  $\Xi_\alpha(\gamma, \lambda) \neq \emptyset$  for each  $(\gamma, \lambda)$  in a neighborhood of  $(\gamma_0, \lambda_0) \in \Gamma \times \Lambda$ .

Next, we recall some basic definitions and their properties.

Let  $X$  and  $Z$  be two topological vector spaces and  $G : X \rightarrow 2^Z$  be a multifunction.

- (i)  $G$  is said to be *lower semicontinuous (lsc)* at  $x_0$  if  $G(x_0) \cap U \neq \emptyset$  for some open set  $U \subseteq Z$  implies the existence of a neighborhood  $V$  of  $x_0$  such that  $G(x) \cap U \neq \emptyset, \forall x \in V$ .
- (ii)  $G$  is said to be *upper semicontinuous (usc)* at  $x_0$  if for each open set  $U \supseteq G(x_0)$ , there is a neighborhood  $V$  of  $x_0$  such that  $U \supseteq G(x), \forall x \in V$ .
- (iii)  $G$  is said to be *Hausdorff upper semicontinuous (H-usc)* at  $x_0$  if for each neighborhood  $U$  of the origin in  $Z$ , there exists a neighborhood  $V$  of  $x_0$  such that,  $G(x) \subseteq G(x_0) + U, \forall x \in V$ .
- (iv)  $G$  is said to be *Hausdorff upper semicontinuous (H-lsc)* at  $x_0$  if for each neighborhood  $U$  of the origin in  $Z$ , there exists a neighborhood  $V$  of  $x_0$  such that  $G(x_0) \subseteq G(x) + U, \forall x \in V$ .

- (v)  $G$  is said to be *continuous* at  $x_0$  if it is both lsc and usc at  $x_0$  and to be *H-continuous* at  $x_0$  if it is both H-lsc and H-usc at  $x_0$ .

**Lemma 1.1.** ([9, 10]). *Let  $X$  and  $Z$  be two topological vector spaces and  $G : X \rightarrow 2^Z$  be a multifunction.*

- (i) *If  $G$  is usc at  $x_0$ , then  $G$  is H-usc at  $x_0$ . Conversely if  $G$  is H-usc at  $x_0$  and if  $G(x_0)$  is compact, then  $G$  is usc at  $x_0$ ;*  
 (ii) *If  $G$  is H-lsc at  $x_0$  then  $G$  is lsc at  $x_0$ . The converse is true if  $G(x_0)$  is compact;*  
 (iii) *If  $Z$  is compact and  $G$  is closed at  $x_0$ , then  $G$  is usc at  $x_0$ ;*  
 (iv) *If  $G$  is usc at  $x_0$  and  $G(x_0)$  is closed, then  $G$  is closed at  $x_0$ ;*  
 (v) *If  $G$  has compact values, then  $G$  is usc at  $x_0$  if and only if, for each net  $\{x_\alpha\} \subseteq X$  which converges to  $x_0$  and for each net  $\{y_\alpha\} \subseteq G(x_\alpha)$ , there are  $y_0 \in G(x_0)$  and a subnet  $\{y_\beta\}$  of  $\{y_\alpha\}$  such that  $y_\beta \rightarrow y_0$ .*

## 2. Upper Semicontinuity of Solution Maps

In this section, we discuss the upper semicontinuity of the exact solution sets and approximate solution sets for the problems (WMQVIP) and (SMQVIP).

**Theorem 2.1.** *Assume for the problem (WMQVIP) that*

- (i)  *$E$  is usc at  $\gamma_0$  and  $E(\gamma_0)$  is compact;*  
 (ii) *in  $K_1(A, \Gamma) \times \{\gamma_0\}$ ,  $K_2$  is lsc;*  
 (iii)  *$T(y, \cdot)$  is usc with compact values at  $\gamma_0$  if  $\alpha = w$  (or  $\alpha = m$ ), and  $T$  is lsc in  $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$  if  $\alpha = s$ .*

*Then,  $\Psi_\alpha$  is usc at  $(\gamma_0, \lambda_0)$ . Moreover,  $\Psi_\alpha(\gamma_0, \lambda_0)$  is compact and  $\Psi_\alpha$  is closed at  $(\gamma_0, \lambda_0)$ .*

**Proof.** Since  $\alpha = \{w, m, s\}$ , we have in fact three cases. Since the proof techniques are similar, we demonstrate only the cases  $\alpha = s$ . We first prove that  $\Psi_s$  is upper semicontinuous at  $(\gamma_0, \lambda_0)$ . Indeed, suppose to the contrary that  $\Psi_\alpha$  is not usc at  $(\gamma_0, \lambda_0)$ . Then there exist an open superset  $U$  of  $\Psi_s(\gamma_0, \lambda_0)$  and a net  $\{(\gamma_n, \lambda_n)\}$  converging to  $\{(\gamma_0, \lambda_0)\}$  such that, for each  $n$ , there is  $x_n \in \Psi_s(\gamma_n, \lambda_n)$ ,  $x_n \notin U$ . Since  $E$  is usc with compact values at  $\gamma_0$ , we can assume that  $x_n$  tends to  $x_0$  for some  $x_0 \in E(\gamma_0)$ . If  $x_0 \notin \Psi_s(\gamma_0, \lambda_0)$ ,  $\exists y_0 \in K_2(x_0, \gamma_0)$ ,  $\exists z_0 \in T(y_0, \gamma_0)$  such that

$$\langle H(z_0), \psi(y_0, x_0, \lambda_0) \rangle < 0.$$

By the lower semicontinuity of  $K_2$  at  $(x_0, \gamma_0)$ , there exists  $y_n \in K_2(x_n, \gamma_n)$  such that  $y_n \rightarrow y_0$ . By the lower semicontinuity of  $T$  at  $(y_0, \gamma_0)$ , there exists  $z_n \in T(y_n, \gamma_n)$  such that  $z_n \rightarrow z_0$ . Since  $x_n \in \Psi_s(\gamma_n, \lambda_n)$ , we have

$$\langle H(z_n), \psi(y_n, x_n, \lambda_n) \rangle \notin 0. \quad (2.1)$$

On the other hand, by the continuity of  $\psi, H$  and  $\langle \cdot, \cdot \rangle$ , it follows from (2.1) that

$$\langle H(z_0), \psi(y_0, x_0, \lambda_0) \rangle \notin 0,$$

it is impossible. Hence,  $x_0$  belongs to  $\Psi_s(\gamma_0, \lambda_0) \subseteq U$ , which is again a contradiction, since  $x_n \notin U$ , for all  $n$ . Therefore,  $\Psi_s$  is usc at  $(\gamma_0, \lambda_0)$ .

Now we prove that  $\Psi_s(\gamma_0, \lambda_0)$  is compact by checking its closedness. Let  $x_n \in \Psi_s(\gamma_0, \lambda_0)$  converge to  $x_0$ . If  $x_0 \notin \Psi_s(\gamma_0, \lambda_0)$ , there exist  $y_0 \in K_2(x_0, \gamma_0)$  and  $z_0 \in T(y_0, \gamma_0)$  such that

$$\langle H(z_0), \psi(y_0, x_0, \lambda_0) \rangle < 0. \quad (2.2)$$

Proceeding similarly as before, we arrive at a contradiction to (2.2). Hence  $x_0 \in \Psi_s(\gamma_0, \lambda_0)$ . Therefore,  $\Psi_s(\gamma_0, \lambda_0)$  is closed. The compactness of  $E(\gamma_0)$  derives that of  $\Psi_s(\gamma_0, \lambda_0)$ . By the condition (iv) of Lemma 1.1, it follows that  $\Psi_s$  is closed at  $(\gamma_0, \lambda_0)$ . And so the proof is completed.  $\square$

The following example shows that the upper semicontinuity and the compactness of  $E$  are essential.

**Example 2.2.** Let  $A = B = X = Y = \mathbb{R}$ ,  $\Gamma = \Lambda = [0, 1]$ ,  $C = \mathbb{R}_+$ ,  $\gamma_0 = 0$ ,  $H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A$ ,  $T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Gamma \rightarrow A$  be defined by

$$\begin{aligned} K_1(x, \gamma) &= (-\gamma - 1, \gamma] \\ \psi(y, x, \gamma) &= \{\gamma^2 + \gamma + 2\}, \\ T(y, \gamma) &= \left\{ \frac{1}{2^{\gamma+2}} \right\}, \\ K_2(x, \gamma) &= [0, e^{\gamma^2+1}], \end{aligned}$$

Then, we have  $E(0) = (-1, 0]$  and  $E(\gamma) = (-\gamma - 1, \gamma]$ ,  $\forall \gamma \in (0, 1]$ . We show that assumptions (ii) and (iii) of Theorem 2.1 are fulfilled. But  $\Psi_\alpha$  is neither usc nor closed at  $(0, 0)$ . The reason is that  $E$  is not usc at 0 and  $E(0)$  is not compact. In fact,

$$\Psi_\alpha(\gamma, \lambda) = \begin{cases} (-1, 0], & \text{if } \gamma = 0, \\ (-\gamma - 1, \gamma], & \text{if } \gamma \in (0, 1]. \end{cases}$$

The following example shows that all assumptions of Theorem 2.1 are satisfied.

**Example 2.3.** Let  $X = Y = \mathbb{R}$ ,  $A = B = [0, 3]$ ,  $\Gamma = \Lambda = [0, 1]$ ,  $C = \mathbb{R}_+$ ,  $\gamma_0 = 0$ ,  $H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A$ ,  $T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Gamma \rightarrow A$  be defined by

$$\begin{aligned} K_1(x, \gamma) &= K_2(x, \gamma) = [0, 1], \\ \psi(y, x, \gamma) &= \{\gamma^2 + \gamma\}, \\ T(y, \gamma) &= \left\{ \frac{1}{e^{\cos^4 \gamma + \sin^2 \gamma} + 2} \right\}. \end{aligned}$$

We see that the all assumptions of Theorem 2.1 are satisfied. So,  $\Psi_\alpha$  is both usc and closed at  $(0, 0)$ . In fact,  $\Psi_\alpha(\gamma, \lambda) = [0, 1]$ ,  $\forall \gamma \in [0, 1]$ .

**Theorem 2.4.** Assume for the problem (SMQVIP) that

- (i)  $E$  is usc at  $\gamma_0$  and  $E(\gamma_0)$  is compact;
- (ii) in  $K_1(A, \Gamma) \times \{\gamma_0\}$ ,  $K_2$  is lsc;
- (iii)  $T(y, \cdot)$  is usc with compact values at  $\gamma_0$  if  $\alpha = w$  (or  $\alpha = m$ ), and  $T$  is lsc in  $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$  if  $\alpha = s$ .

Then,  $\Xi_\alpha$  is usc at  $(\gamma_0, \lambda_0)$ . Moreover,  $\Xi_\alpha(\gamma_0, \lambda_0)$  is compact and  $\Xi_\alpha$  is closed at  $(\gamma_0, \lambda_0)$ .

**Proof.** We omit the proof since the technique is similar as that for Theorem 2.1 with suitable modifications.  $\square$

**Remark 2.5.** In the special case, if  $H$  is the identity map,  $\Lambda = \Gamma$ ,  $C = \mathbb{R}_+$ ,  $\alpha = s$  and  $\psi(y, x, \gamma) = y - x$ ,  $K_1(x, \gamma) = K(x, \gamma) \cap A$ ,  $K_2(x, \gamma) = K(x, \gamma)$  with  $K : A \times \Gamma \rightarrow 2^A$ . Then, the problems (WMQVIP) and (SMQVIP) reduce to the problem (MVI( $\gamma$ )) studied in [25]. In this special case, Theorem 3.1 in [25] is a particular case of Theorems 2.1 and 2.4.

The following example shows a case where the assumed compactness in Theorems 3.1 and 3.2 of [25] is violated but the assumptions of Theorems 2.1 and 2.4 are fulfilled.

**Example 2.6.** Let  $X = Y = \mathbb{R}$ ,  $\Lambda = \Gamma = [0, 1]$ ,  $C = \mathbb{R}_+$ ,  $A = B = [0, 3]$ ,  $\gamma_0 = 0$ ,  $H$  be the identity map,  $K_1 = K_2 = K : A \times \Gamma \rightarrow 2^A$ ,  $T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Gamma \rightarrow A$  be defined by

$$\begin{aligned} K_1(x, \gamma) &= K_2(x, \gamma) = K(x, \gamma) = \left[ \frac{1}{2}, \frac{3}{2} \right], \\ \psi(y, x, \gamma) &= \{x - y\}, \\ T(y, \gamma) &= \{1\}. \end{aligned}$$

It is clear that the assumptions of Theorems 2.1 and 2.4 are fulfilled, and hence  $\Psi_\alpha$  and  $\Xi_\alpha$  are usc and closed at  $(0, 0)$ , although  $A$  is not compact. In fact,  $\Psi_\alpha(\gamma, \lambda) = \Xi_\alpha(\gamma, \lambda) = \{\frac{3}{2}\}$ ,  $\forall \gamma \in [0, 1]$ .

Next, we consider the approximate solution sets for the problems (WMQVIP) and (SMQVIP).

For each  $\gamma \in \Gamma, \lambda \in \Lambda$  and  $\varepsilon \in C$ . We denote the approximate solution sets of (WMQVIP) and (SMQVIP) by  $\widetilde{\Psi}_\alpha(\gamma, \lambda, \varepsilon)$  and  $\widetilde{\Xi}_\alpha(\gamma, \lambda, \varepsilon)$ , respectively, i.e.,

$$\begin{aligned} \widetilde{\Psi}_\alpha(\gamma, \lambda, \varepsilon) &:= \{ \bar{x} \in E(\gamma) \mid (y, z) \in K_2(\bar{x}, \gamma) \times T(y, \gamma), \langle H(z), \psi(y, \bar{x}, \lambda) \rangle + \varepsilon \not\leq 0 \}, \\ \widetilde{\Xi}_\alpha(\gamma, \lambda, \varepsilon) &:= \{ \bar{x} \in E(\gamma) \mid (y, z) \in K_2(\bar{x}, \gamma) \times T(y, \gamma), \langle H(z), \psi(y, \bar{x}, \lambda) \rangle + \varepsilon \geq 0 \}. \end{aligned}$$

**Theorem 2.7.** Assume for the problem (WMQVIP) that all conditions in Theorem 2.1 are satisfied.

Then,  $\widetilde{\Psi}_\alpha$  is usc at  $(\gamma_0, \lambda_0, \varepsilon)$ . Moreover,  $\widetilde{\Psi}_\alpha(\gamma_0, \lambda_0, \varepsilon)$  is compact and  $\widetilde{\Psi}_\alpha$  is closed at  $(\gamma_0, \lambda_0, \varepsilon)$ , for all  $\varepsilon \in C$ .

**Proof.** We consider only the cases  $\alpha = s$ . We first prove that  $\widetilde{\Psi}_s$  is upper semicontinuous at  $(\gamma_0, \lambda_0, \varepsilon)$ . Suppose to the contrary that  $\widetilde{\Psi}_s$  is not usc at  $(\gamma_0, \lambda_0, \varepsilon)$ . Then there are an open superset  $V$  of  $\widetilde{\Psi}_s(\gamma_0, \lambda_0, \varepsilon)$  and a net  $\{(\gamma_n, \lambda_n, \varepsilon_n)\}$  converging to  $\{(\gamma_0, \lambda_0, \varepsilon)\}$  in  $\Gamma \times \Lambda \times C$  such that, for each  $n$ , there is  $x_n \in \widetilde{\Psi}_s(\gamma_n, \lambda_n, \varepsilon_n)$ ,  $x_n \notin V$ . Since  $E$  is usc with compact values at  $\gamma_0$ , we can assume that  $x_n$  tends to  $x_0$  for some  $x_0 \in E(\gamma_0)$ . If  $x_0 \notin \widetilde{\Psi}_s(\gamma_0, \lambda_0, \varepsilon)$ ,  $\exists y_0 \in K_2(x_0, \gamma_0), \exists z_0 \in T(y_0, \gamma_0)$  such that

$$\langle H(z_0), \psi(y_0, x_0, \lambda_0) \rangle + \varepsilon < 0.$$

By the lower semicontinuity of  $K_2, T$  at  $(x_0, \gamma_0)$  and  $(y_0, \gamma_0)$ ,  $\forall y_0 \in K_2(x_0, \gamma_0), \forall z_0 \in T(y_0, \gamma_0)$  there exist  $y_n \in K_2(x_n, \gamma_n), z_n \in T(y_n, \gamma_n)$  such that  $y_n \rightarrow y_0, z_n \rightarrow z_0$ . Since  $x_n \in \widetilde{\Psi}_s(\gamma_n, \lambda_n, \varepsilon_n)$ , we have

$$\langle H(z_n), \psi(y_n, x_n, \lambda_n) \rangle + \varepsilon_n \not\leq 0. \tag{2.3}$$

Let  $\text{id} : C \rightarrow C$  be the identity map. By the continuity of  $\psi, H$  and  $\langle \cdot, \cdot \rangle$ , it follows that  $\langle \cdot, \cdot \rangle + \text{id}$  is continuous. So, (2.3) implies that

$$\langle H(z_0), \psi(y_0, x_0, \lambda_0) \rangle + \varepsilon \not\leq 0,$$

it is impossible. Hence,  $x_0$  belongs to  $\widetilde{\Psi}_s(\gamma_0, \lambda_0, \varepsilon) \subseteq V$ , which is again a contradiction, since  $x_n \notin V$ , for all  $n$ . Therefore,  $\widetilde{\Psi}_s$  is usc at  $(\gamma_0, \lambda_0, \varepsilon)$ .

Now we prove that  $\widetilde{\Psi}_s(\gamma_0, \lambda_0, \varepsilon)$  is compact and  $\widetilde{\Psi}_s$  is closed at  $(\gamma_0, \lambda_0, \varepsilon)$ . By using the argument is the same as in Theorem 2.1, the proof is completed.  $\square$

The following example shows that the lower semicontinuity assumption of  $K_2$  in Theorem 2.7 is essential.

**Example 2.8.** Let  $A = B = X = Y = \mathbb{R}, \Gamma = \Lambda = [0, 1], C = \mathbb{R}_+, \varepsilon \in \mathbb{R}_+, \gamma_0 = 0, H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A, T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Lambda \rightarrow A$  be defined by

$$\begin{aligned} K_2(x, \gamma) &= \begin{cases} \{-5, 0, 5\}, & \text{if } \gamma = 0, \\ \{0, 5\}, & \text{if } \gamma \neq 0, \end{cases} \\ \psi(y, x, \gamma) &= \{x + y + \gamma\}, \\ T(y, \gamma) &= \{1\}, \\ K_1(x, \gamma) &= [0, 5], \end{aligned}$$

Then  $E(\gamma) = [0, 5], \forall \gamma \in [0, 1]$ . Hence,  $E$  is usc at 0 and  $E(0)$  is compact, assumption (iii) is satisfied. For each  $\varepsilon \geq 0$ , we have

$$\widetilde{\Psi}_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} [5 - \varepsilon, 5] \cap [0, 5], & \text{if } \gamma = 0, \\ [0, 5], & \text{if } \gamma \in (0, 1]. \end{cases}$$

Therefore,  $\widetilde{\Psi}_\alpha$  is not usc at  $(0, 0, \varepsilon)$ . The reason is that  $K_2$  is not lsc at  $(x, 0)$ .

By using similar argument as in the proof of Theorem 2.7, we can prove the following result.

**Theorem 2.9.** Assume for the problem (SMQVIP) that all conditions in Theorem 2.4 are satisfied.

Then,  $\widetilde{\Xi}_\alpha$  is usc at  $(\gamma_0, \lambda_0, \varepsilon)$ . Moreover,  $\widetilde{\Xi}_\alpha(\gamma_0, \lambda_0, \varepsilon)$  is compact and  $\widetilde{\Xi}_\alpha$  is closed at  $(\gamma_0, \lambda_0, \varepsilon)$ , for all  $\varepsilon \in C$ .

**Remark 2.10.** Note that, our Theorems 2.7 and 2.9 are different from Theorems 3.4 and 3.5 in [25]. However, if we let  $\varepsilon$  is a fixed non-negative real number together with Remark 2.5 (i). Then, Theorems 3.4 and 3.5 in [25] are particular cases of Theorems 2.7 and 2.9. The following example shows that in this special case, the assumptions of Theorems 2.7 and 2.9 are satisfied, but Theorems 3.4 and 3.5 in [25] cannot be applied.

**Example 2.11.** Let  $X = Y = \mathbb{R}, A = B = [0, 2), \Gamma = \Lambda = [0, 1], C = \mathbb{R}_+, \gamma_0 = 0$ , and let  $\varepsilon \geq 0$  be fixed,  $H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A, T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Gamma \rightarrow A$  be defined by

$$\begin{aligned} K_1(x, \gamma) &= K_2(x, \gamma) = [0, 1], \\ \psi(y, x, \gamma) &= y - x - \gamma, \\ T(y, \gamma) &= \{1\}, \end{aligned}$$

Then  $E(\gamma) = [0, 1], \forall \gamma \in [0, 1]$ . We see that the assumptions of Theorems 2.7 and 2.9 are satisfied. So,  $\widetilde{\Psi}_\alpha$  and  $\widetilde{\Xi}_\alpha$  are both usc and closed at  $(0, 0, \varepsilon)$ . But Theorems 3.4 and 3.5 in [25] cannot be applied. The reason is  $A$  is not compact. In fact, for a fixed  $\varepsilon \geq 0$ , we have

$$\widetilde{\Psi}_\alpha(\gamma, \lambda, \varepsilon) = \widetilde{\Xi}_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} [0, \varepsilon] \cap [0, 1], & \text{if } \gamma = 0, \\ [0, \varepsilon - \gamma] \cap [0, 1], & \text{if } \varepsilon \geq \gamma > 0. \end{cases}$$

### 3. Lower Semicontinuity of Solution Maps

In this section, we discuss the lower semicontinuity and the Hausdorff lower semicontinuity of the exact solution sets and approximate solution sets for the problems (WMQVIP) and (SMQVIP).

**Theorem 3.1.** Assume for the problem (WMQVIP) that

- (i)  $E$  is lsc at  $\gamma_0$ ;
- (ii)  $\forall x_0 \in K_1(x_0, \gamma_0), \forall (x_n, \gamma_n, \lambda_n) \rightarrow (x_0, \gamma_0, \lambda_0)$  and

$$(y, z) \in K_2(x_0, \gamma_0) \times T(y, \gamma_0) \text{ satisfying } \langle H(z), \psi(y, x_0, \lambda_0) \rangle \neq 0$$

implies that there exists a positive integer  $n$ , such that

$$(y, z) \in K_2(x_n, \gamma_n) \times T(y, \gamma_n) \text{ satisfying } \langle H(z), \psi(y, x_n, \lambda_n) \rangle \neq 0.$$

Then  $\Psi_\alpha$  is lsc at  $(\gamma_0, \lambda_0)$ .

**Proof.** Similar arguments can be applied to three cases. We present only the proof for the cases where  $\alpha = s$ . Suppose that  $\Psi_\alpha$  is not lsc at  $(\gamma_0, \lambda_0)$ . Then there exist  $x_0 \in \Psi_s(\gamma_0, \lambda_0)$  and a net  $\{(\gamma_n, \lambda_n)\}$  converging to  $(\gamma_0, \lambda_0)$  such that, for all  $x'_n \in \Psi_s(\gamma_n, \lambda_n)$ , the net  $\{x'_n\}$  cannot converge to  $x_0$ . Since  $E$  is lsc at  $\gamma_0$ , there is  $x_n \in E(\gamma_n), x_n \rightarrow x_0$ . By the above contradiction assumption, without loss of generality, we can assume that  $x_n \notin \Psi_s(\gamma_n, \lambda_n)$ , for all  $n$ , i.e.,  $\exists y_n \in K_2(x_n, \gamma_n), \exists z_n \in T(y_n, \gamma_n)$

$$\langle H(z_n), \psi(y_n, x_n, \lambda_n) \rangle < 0. \tag{3.1}$$

Since  $x_0 \in \Psi_s(\gamma_0, \lambda_0), \forall z \in T(y, \gamma_0), \forall y \in K_2(x_0, \gamma_0)$ , we have

$$\langle H(z), \psi(y, x_0, \lambda_0) \rangle \neq 0.$$

Since  $(x_n, \gamma_n, \lambda_n) \rightarrow (x_0, \gamma_0, \lambda_0)$  and by the condition (ii), there exists  $n$ , such that  $\forall z \in T(y, \gamma_n), \forall y \in K_2(x_n, \gamma_n)$

$$\langle H(z), \psi(y, x_n, \lambda_n) \rangle \not\geq 0.$$

which contradicts (3.1). Therefore,  $\Psi_s$  is lsc at  $(\gamma_0, \lambda_0)$ . □

The following example shows that the lower semicontinuity of  $E$  is essential.

**Example 3.2.** Let  $X = Y = \mathbb{R}, A = B = [0, 1], \Gamma = \Lambda = [0, 1], C = \mathbb{R}_+, \gamma_0 = 0, H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A, T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Lambda \rightarrow A$  be defined by

$$K_1(x, \gamma) = \begin{cases} \{-\frac{1}{2}, 0, \frac{1}{2}\}, & \text{if } \gamma = 0, \\ \{0, \frac{1}{2}\}, & \text{if } \gamma \neq 0, \end{cases}$$

$$\psi(y, x, \lambda) = \gamma + \cos^4(\gamma) + \sin^2(\gamma),$$

$$T(y, \gamma) = \{3\gamma^{2+3}\},$$

$$K_2(x, \gamma) = [0, \frac{1}{2}].$$

Then,  $K_2$  is usc with compact values in  $A \times \{\gamma_0\}$  and the assumptions (ii) and (iii) of Theorem 3.1 are fulfilled. But  $\Psi_\alpha$  is not lsc at  $(0, 0)$ . The reason is that  $E$  is not lsc at 0. In fact,

$$\Psi_\alpha(\gamma, \lambda) = \begin{cases} \{0, \frac{1}{2}\}, & \text{if } \gamma \in (0, 1], \\ \{-\frac{1}{2}, 0, \frac{1}{2}\}, & \text{if } \gamma = 0. \end{cases}$$

The following example shows that all assumptions of Theorem 3.1 are satisfied.

**Example 3.3.** Let  $A = B = X = Y = \mathbb{R}, \Gamma = \Lambda = [0, 1], C = \mathbb{R}_+, \gamma_0 = 0, H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A, T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Lambda \rightarrow A$  be defined by

$$K_1(x, \gamma) = \begin{cases} [0, 1], & \text{if } \gamma = 0, \\ [-1, 2], & \text{if } \gamma \neq 0, \end{cases}$$

$$\psi(y, x, \gamma) = \{\gamma + \sin^4(\gamma) + \cos^2(\gamma)\},$$

$$T(y, \gamma) = \{\frac{1}{2\gamma^{2+2}}\},$$

$$K_2(x, \gamma) = [0, 1].$$

We have  $E(\gamma) = [-1, 2]$  for all  $\gamma \in (0, 1]$  and  $E(0) = [0, 1]$ . It is not hard to see that (i)-(iii) in Theorem 3.9 are satisfied and, according to Theorem 3.1,  $\Psi_\alpha$  is lsc at  $(0, 0)$ . In fact,  $\Psi_\alpha(\gamma, \lambda) = [-1, 2]$  for all  $\gamma \in (0, 1]$  and  $\Psi_\alpha(0, 0) = [0, 1]$ .

**Theorem 3.4.** Assume for the problem (SMQVIP) that

- (i)  $E$  is lsc at  $\gamma_0$ ;
- (ii)  $\forall x_0 \in K_1(x_0, \gamma_0), \forall (x_n, \gamma_n, \lambda_n) \rightarrow (x_0, \gamma_0, \lambda_0)$  and

$$(y, z) \in \alpha K_2(x_0, \gamma_0) \times T(y, \gamma_0) \text{ satisfying } \langle H(z), \psi(y, x_0, \lambda_0) \rangle \geq 0$$

implies that there exists a positive integer  $n$ , such that

$$(y, z) \in \alpha K_2(x_n, \gamma_n) \times T(y, \gamma_n) \text{ satisfying } \langle H(z), \psi(y, x_n, \lambda_n) \rangle \geq 0.$$

Then  $\Xi_\alpha$  is lsc at  $(\gamma_0, \lambda_0)$ .

**Proof.** We omit the proof since the technique is similar as that for Theorem 3.1 with suitable modifications.  $\square$

Next, we study the Hausdorff lower semicontinuity of the exact solution sets for the problems (WMQVIP) and (SMQVIP).

**Theorem 3.5.** *Impose the assumptions of Theorem 3.1 and the following additional conditions:*

- (iii)  $K_2$  is lsc in  $K_1(A, \Gamma) \times \{\gamma_0\}$  and  $E(\gamma_0)$  is compact;
- (iv)  $T(y, \cdot)$  is usc with compact values at  $\gamma_0$  if  $\alpha = w$  (or  $\alpha = m$ ), and  $T$  is lsc in  $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$  if  $\alpha = s$ .

Then  $\Psi_\alpha$  is H-lsc at  $(\gamma_0, \lambda_0)$ .

**Proof.** Similar arguments can be applied to three cases. We consider only the cases  $\alpha = s$ . Using the similar argument as in the last part of proof of Theorem 2.1, we have  $\Psi_s(\gamma_0, \lambda_0)$  is compact. Applying Theorem 3.1, we obtain the lower semicontinuity property of  $\Psi_s$ . The Hausdorff lower semicontinuity of  $\Psi_s$  is derived directly from condition (ii) of Lemma 1.1.  $\square$

The following shows that the compactness of  $E$  in Theorem 3.5 is essential.

**Example 3.6.** Let  $A = B = X = \mathbb{R}^2, Y = \mathbb{R}, \Gamma = \Lambda = [0, 1], C = \mathbb{R}_+, \gamma_0 = 0, H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A, T : A \times \Gamma \rightarrow 2^{L(X, Y)}$  and  $\psi : A \times A \times \Lambda \rightarrow A$  be defined by

$$\begin{aligned} K_1(x, \gamma) &= K_2(x, \gamma) = \{(x_1, \lambda x_1^4)\}, x = (x_1, x_2) \in \mathbb{R}^2, \\ \psi(y, x, \gamma) &= \{2\gamma^4 + \sin^2(\gamma)\}, \\ T(y, \gamma) &= \{2^{1+\gamma^4+\cos^4(\gamma)}\}. \end{aligned}$$

We have  $E(0) = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$  and  $E(\gamma) = \{x \in \mathbb{R}^2 \mid x_2 = \gamma x_1^4\}, \forall \gamma \in (0, 1]$ . It is easy to see that all assumptions of Theorem 3.5 are satisfied, but the compactness of  $E(0)$  is not satisfied. Direct computations give  $\Psi_\alpha(0, 0) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\}$  and  $\Psi_\alpha(\gamma, \lambda) = \{x \in \mathbb{R}^2 \mid x_2 = \gamma x_1^4\}, \forall \gamma \in (0, 1]$  is not Hausdorff lower semicontinuous at  $(0, 0)$ .

Using the similar argument as in the proof of Theorem 3.5, we obtain the following result.

**Theorem 3.7.** *Impose the assumptions of Theorem 3.4 and the following additional conditions:*

- (iii)  $K_2$  is lsc in  $K_1(A, \Gamma) \times \{\gamma_0\}$  and  $E(\gamma_0)$  is compact;
- (iv)  $T(y, \cdot)$  is usc with compact values at  $\gamma_0$  if  $\alpha = w$  (or  $\alpha = m$ ), and  $T$  is lsc in  $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$  if  $\alpha = s$ .

Then  $\Xi_\alpha$  is H-lsc at  $(\gamma_0, \lambda_0)$ .

**Remark 3.8.** In the special case studied in Remark 2.5, Theorems 3.1 and 3.4 extend Theorem 4.1 in [25]. Theorems 3.5 and 3.7 extend Corollary 4.1 in [25].

**Theorem 3.9.** *Suppose that all conditions in Theorems 2.1 and 3.1 (Theorem 3.5, respectively) are satisfied. Then,  $\Psi_\alpha$  is both continuous (H-continuous, respectively) and closed at  $(\gamma_0, \lambda_0)$ .*

**Theorem 3.10.** *Suppose that all conditions in Theorems 2.4 and 3.4 (Theorem 3.7, respectively) are satisfied. Then,  $\Xi_\alpha$  is both continuous (H-continuous, respectively) and closed at  $(\gamma_0, \lambda_0)$ .*

Now, we study the semicontinuity of the approximate solution sets for the problems (WMQVIP) and (SMQVIP).

**Theorem 3.11.** *Assume for the problem (WMQVIP) that*

- (i)  $E$  is lsc at  $\gamma_0$ ;



(ii)  $\forall x_0 \in K_1(x_0, \gamma_0), \forall (x_n, \gamma_n, \lambda_n, \varepsilon_n) \rightarrow (x_0, \gamma_0, \lambda_0, \varepsilon)$  and

$$(y, z) \alpha K_2(x_0, \gamma_0) \times T(y, \gamma_0) \text{ satisfying } \langle H(z), \psi(y, x_0, \lambda_0) \rangle + \varepsilon \not\leq 0$$

implies that there exists a positive integer  $n$ , such that

$$(y, z) \alpha K_2(x_n, \gamma_n) \times T(y, \gamma_n) \text{ satisfying } \langle H(z), \psi(y, x_n, \lambda_n) \rangle + \varepsilon_n \not\leq 0.$$

Then  $\widetilde{\Psi}_\alpha$  is lsc at  $(\gamma_0, \lambda_0, \varepsilon)$ , for all  $\varepsilon \in \text{int}C$ .

**Proof.** We present only the proof for the cases where  $\alpha = s$ . Suppose that  $\widetilde{\Psi}_\alpha$  is not lsc at  $(\gamma_0, \lambda_0, \varepsilon)$ . Then, there exist  $x_0 \in \widetilde{\Psi}_s(\gamma_0, \lambda_0, \varepsilon)$  and a net  $\{(\gamma_n, \lambda_n, \varepsilon_n)\}$  converging to  $(\gamma_0, \lambda_0, \varepsilon)$  such that, for all  $x'_n \in \widetilde{\Psi}_s(\gamma_n, \lambda_n, \varepsilon_n)$ , the net  $\{x'_n\}$  does not converge to  $x_0$ . Since  $E$  is lsc at  $\gamma_0$ , there is  $x_n \in E(\gamma_n), x_n \rightarrow x_0$ . By the above contradiction assumption, we conclude that  $x_n \notin \widetilde{\Psi}_s(\gamma_n, \lambda_n, \varepsilon_n)$ , for all  $n$ , i.e.,  $\exists y_n \in K_2(x_n, \gamma_n), \exists z_n \in T(y_n, \gamma_n)$

$$\langle H(z_n), \psi(y_n, x_n, \lambda_n) \rangle + \varepsilon_n < 0. \tag{3.2}$$

Since  $x_0 \in \widetilde{\Psi}_s(\gamma_0, \lambda_0, \varepsilon), \forall z \in T(y, \gamma_0), \forall y \in K_2(x_0, \gamma_0)$ , we have

$$\langle H(z), \psi(y, x_0, \lambda_0) \rangle + \varepsilon \not\leq 0.$$

Since  $(x_n, \gamma_n, \lambda_n, \varepsilon_n) \rightarrow (x_0, \gamma_0, \lambda_0, \varepsilon)$  and by the condition (ii), there exists  $n$ , such that  $\forall z \in T(y, \gamma_n), \forall y \in K_2(x_n, \gamma_n)$

$$\langle H(z), \psi(y, x_n, \lambda_n) \rangle + \varepsilon_n \not\leq 0.$$

which contradicts (3.2). Therefore,  $\widetilde{\Psi}_s$  is lsc at  $(\gamma_0, \lambda_0, \varepsilon)$ . □

The following example shows that all assumptions of Theorem 3.11 are satisfied.

**Example 3.12.** Let  $A = B = X = Y = [0, 1], \Gamma = \Lambda = [0, 1], C = \mathbb{R}_+, \varepsilon \in \text{int}\mathbb{R}_+, \gamma_0 = 0, H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A, T : A \times \Gamma \rightarrow 2^{L(X,Y)}$  and  $\psi : A \times A \times \Lambda \rightarrow A$  be defined by

$$\begin{aligned} K_1(x, \gamma) &= K_2(x, \gamma) = [0, 1], \\ \psi(y, x, \gamma) &= \{2^{2\gamma - \varepsilon^2 - \cos^2(\gamma)}\}, \\ T(y, \gamma) &= \left\{ \frac{1}{e^{\gamma+1}} \right\}. \end{aligned}$$

We have  $E(\gamma) = [0, 1]$  for all  $\gamma \in [0, 1]$ . We see that the assumptions of Theorem 3.11 are satisfied. So,  $\widetilde{\Psi}_\alpha$  is lsc at  $(0, 0, \varepsilon)$ . In fact,  $\widetilde{\Psi}_\alpha(\gamma, \lambda, \varepsilon) = [0, 1]$  for all  $\gamma \in [0, 1]$ .

By using similar argument as in the proof of Theorem 3.11, we can prove the following result.

**Theorem 3.13.** Assume for the problem (SMQVIP) that

- (i)  $E$  is lsc at  $\gamma_0$ ;
- (ii)  $\forall x_0 \in K_1(x_0, \gamma_0), \forall (x_n, \gamma_n, \lambda_n, \varepsilon_n) \rightarrow (x_0, \gamma_0, \lambda_0, \varepsilon)$  and

$$(y, z) \alpha K_2(x_0, \gamma_0) \times T(y, \gamma_0) \text{ satisfying } \langle H(z), \psi(y, x_0, \lambda_0) \rangle + \varepsilon \geq 0$$

implies that there exists a positive integer  $n$ , such that

$$(y, z) \alpha K_2(x_n, \gamma_n) \times T(y, \gamma_n) \text{ satisfying } \langle H(z), \psi(y, x_n, \lambda_n) \rangle + \varepsilon_n \geq 0.$$

Then  $\widetilde{\Xi}_\alpha$  is lsc at  $(\gamma_0, \lambda_0, \varepsilon)$ , for all  $\varepsilon \in \text{int}C$ .

**Theorem 3.14.** *Impose the assumptions of Theorem 3.11 and the following additional conditions:*

- (iii)  $K_2$  is lsc in  $K_1(A, \Gamma) \times \{\gamma_0\}$  and  $E(\gamma_0)$  is compact;
- (iv)  $T(y, \cdot)$  is usc with compact values at  $\gamma_0$  if  $\alpha = w$  (or  $\alpha = m$ ), and  $T$  is lsc in  $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$  if  $\alpha = s$ .

Then  $\widetilde{\Psi}_\alpha$  is H-lsc at  $(\gamma_0, \lambda_0, \varepsilon)$ , for all  $\varepsilon \in \text{int}C$ .

**Proof.** Similar arguments can be applied to three cases. We consider only the cases  $\alpha = s$ . The similar argument as in the last part of proof of Theorem 2.7, we also have  $\widetilde{\Psi}_s(\gamma_0, \lambda_0)$  is compact. From the Theorem 3.11 implies the lower semicontinuity of  $\widetilde{\Psi}_s$ . The Hausdorff lower semicontinuity of  $\widetilde{\Psi}_s$  is direct from condition (ii) of Lemma 1.1.  $\square$

The following example shows that all assumptions of Theorem 3.14 are satisfied.

**Example 3.15.** Let  $A = B = X = Y = \mathbb{R}$ ,  $\Gamma = \Lambda = [0, 1]$ ,  $C = \mathbb{R}_+$ ,  $\varepsilon \in \text{int}\mathbb{R}_+$ ,  $\gamma_0 = 0$ ,  $H$  be the identity map,  $K_1, K_2 : A \times \Gamma \rightarrow 2^A$ ,  $T : A \times \Gamma \rightarrow 2^{L(X, Y)}$  and  $\psi : A \times \Lambda \rightarrow A$  be defined by

$$K_1(x, \gamma) = \begin{cases} [\frac{1}{2}, 1], & \text{if } \gamma = 0, \\ [0, 2], & \text{if } \gamma \neq 0, \end{cases}$$

$$\psi(y, x, \gamma) = \{\gamma^2 + 2\gamma + 1 + \varepsilon^2 - \varepsilon + \cos^2(\gamma)\},$$

$$T(y, \gamma) = \{1\},$$

$$K_2(x, \gamma) = [0, 1].$$

We have  $E(0) = [\frac{1}{2}, 1]$  and  $E(\gamma) = [0, 2]$  for all  $\gamma \in (0, 1]$ . It is easy to see that the assumptions of Theorem 3.14 are satisfied. So,  $\widetilde{\Psi}_\alpha$  is Hausdorff lower semicontinuous at  $(0, 0, \varepsilon)$ . In fact,  $\widetilde{\Psi}_\alpha(0, 0, \varepsilon) = [\frac{1}{2}, 1]$  and  $\widetilde{\Psi}_\alpha(\gamma, \lambda, \varepsilon) = [0, 2]$  for all  $\gamma \in (0, 1]$ .

Similarly, we have the following result.

**Theorem 3.16.** *Impose the assumptions of Theorem 3.13 and the following additional conditions:*

- (iii)  $K_2$  is lsc in  $K_1(A, \Gamma) \times \{\gamma_0\}$  and  $E(\gamma_0)$  is compact;
- (iv)  $T(y, \cdot)$  is usc with compact values at  $\gamma_0$  if  $\alpha = w$  (or  $\alpha = m$ ), and  $T$  is lsc in  $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$  if  $\alpha = s$ .

Then  $\widetilde{\Xi}_\alpha$  is H-lsc at  $(\gamma_0, \lambda_0, \varepsilon)$ , for all  $\varepsilon \in \text{int}C$ .

**Remark 3.17.** In the special case considered in Remark 2.5, Theorems 3.11 and 3.13 are different from Theorem 4.5 in [25]. Moreover, even for this special case, Theorems 3.9, 3.10, 3.14 and 3.16 are new.

## Acknowledgment

The authors would like to thank the referees for their valuable remarks and suggestions, which helped to improve the paper.

## References

- [1] L. Q. Anh, P. Q. Khanh, Semicontinuity of the solution sets of parametric multivalued vector quasiequilibrium problems, *J. Math. Anal. Appl.* 294 (2004) 699–711.
- [2] L. Q. Anh, P. Q. Khanh, On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems, *J. Math. Anal. Appl.* 321 (2006) 308–315.
- [3] L. Q. Anh, P. Q. Khanh, Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces, *J. Glob. Optim.* 37 (2007) 449–465.

- [4] L. Q. Anh, P. Q. Khanh, On the stability of the solution sets of general multivalued vector quasiequilibrium problems, *J. Optim. Theory Appl.* 135 (2007) 271–284.
- [5] L. Q. Anh, P. Q. Khanh, Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions, *J. Glob. Optim.* 42 (2008) 515–531.
- [6] L. Q. Anh, P. Q. Khanh, Hölder continuity of the unique solution to quasiequilibrium problems in metric spaces, *J. Optim. Theory Appl.* 141 (2009) 37–54.
- [7] L. Q. Anh, P. Q. Khanh, Continuity of solution maps of parametric quasiequilibrium problems, *J. Glob. Optim.* 46 (2010) 247–259.
- [8] L. Q. Anh, N. V. Hung, The existence and stability of solutions for symmetric generalized quasi-variational inclusion problems, *Filomat* 29 (2015) 2147–2165.
- [9] J. P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, John Wiley and Sons, New York, 1984.
- [10] C. Berge, *Topological Spaces*, Oliver and Boyd, London, 1963.
- [11] J. W. Chen, Z. P. Wan, Semicontinuity for parametric Minty vector quasivariational inequalities in Hausdorff topological vector spaces, *Comput. Appl. Math.* 33 (2014) 111–129.
- [12] G. Y. Chen, S. J. Li, K. L. Teo, Solution semicontinuity of parametric generalized vector equilibrium problems, *J. Glob. Optim.* 45 (2009) 309–318.
- [13] C. R. Chen, S. J. Li, Z. M. Fang, On the solution semicontinuity to a parametric generalized vector quasivariational inequality, *Comput. Math. Appl.* 60 (2010) 2417–2425.
- [14] X. P. Ding, Salahuddin, Generalized vector mixed general quasi-variational-like inequalities in Hausdorff topological vector spaces, *Optim. Lett.* 7 (2012) 893–902.
- [15] F. Giannessi, Theorems of alternative, quadratic programmes and complementarity problems, in: R. W. Cottle, F. Giannessi, J. L. Lions (Eds.), *Variational Inequalities and Complementarity Problems*, Wiley, Chichester, (1980) 151–186.
- [16] N. V. Hung, Continuity of solutions for parametric generalized quasivariational relation problems, *Fixed Point Theory Appl.* 102 (2012) 1–10.
- [17] N. V. Hung, Sensitivity analysis for generalized quasi-variational relation problems in locally  $G$ -convex spaces, *Fixed Point Theory Appl.* 158 (2012) 1–13.
- [18] N. V. Hung, Stability of solution set for parametric generalized vector mixed quasivariational inequality problem, *J. Inequal. Appl.* 176 (2013) 1–17.
- [19] N. V. Hung, Well-posedness for parametric generalized vector quasivariational inequality problems of the Minty type, *J. Inequal. Appl.* 178 (2014) 1–16.
- [20] N. V. Hung, P. T. Kieu, On the existence and essential components of solution sets for systems of generalized quasi-variational relation problems, *J. Inequal. Appl.* 250 (2014) 1–12.
- [21] N. V. Hung, On the lower semicontinuity of the solution sets for parametric generalized vector mixed quasivariational inequality problems, *Bull. Korean Math. Soc.* 52 (2015) 1777–1795.
- [22] B. T. Kien, On the lower semicontinuity of optimal solution sets, *Optimization* 54 (2005) 123–130.
- [23] P. Q. Khanh, L. M. Luu, Upper semicontinuity of the solution set of parametric multivalued vector quasivariational inequalities and applications, *J. Glob. Optim.* 32 (2005) 551–568.
- [24] P. Q. Khanh, L. M. Luu, Lower and upper semicontinuity of the solution sets and approximate solution sets to parametric multivalued quasivariational inequalities, *J. Optim. Theory Appl.* 133 (2007) 329–339.
- [25] C. S. Lalitha, G. Bhatia, Stability of parametric quasivariational inequality of the Minty type, *J. Optim. Theory Appl.* 148 (2011) 281–300.
- [26] X. B. Li, S. J. Li, Continuity of approximate solution mappings for parametric equilibrium problems, *J. Glob. Optim.* 51 (2011) 541–548.
- [27] S. J. Li, C. R. Chen, Stability of weak vector variational inequality problems, *Nonlinear Anal. TMA* 70 (2009) 1528–1535.
- [28] S. J. Li, G. Y. Chen, K. L. Teo, On the stability of generalized vector quasivariational inequality problems, *J. Optim. Theory Appl.* 113 (2002) 283–295.
- [29] X. Q. Yang, Vector variational inequality and its duality, *Nonlinear Anal. TMA* 95 (1993) 729–734.
- [30] R. Y. Zhong, N. J. Huang, Stability analysis for minty mixed variational inequality in reflexive Banach spaces, *J. Optim. Theory Appl.* 147 (2010) 454–472.
- [31] R. Y. Zhong, N. J. Huang, Lower semicontinuity for parametric weak vector variational inequalities in reflexive Banach spaces, *J. Optim. Theory Appl.* 150 (2011) 2417–2425.
- [32] R. Y. Zhong, N. J. Huang, Strict feasibility for generalized mixed variational inequality in reflexive Banach spaces, *J. Optim. Theory Appl.* 152 (2012) 696–709.
- [33] J. Zhao, The lower semicontinuity of optimal solution sets, *J. Math. Anal. Appl.* 207 (1997) 240–254.