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Relative Entropy and L^2 Stability to a Shock for Scalar Balance Laws

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Abstract. We consider a scalar balance law with a strict convex flux. In this paper, we study L^2 stability to a shock for a scalar balance law up to a shift function, which is based on the relative entropy. This result generalizes Leger's works [18] and provides more a simple proof than Leger's proof.

1. Introduction

We consider the following balance law in one dimensional spaces $\mathbb{R}$,

$$\partial_t U + \partial_x A(U) = g(U) \tag{1}$$

where the flux $A''(v) := a'(v) \geq c$ for some constant $c > 0$. The existence of global unique weak solutions of (1) have been studied by Kruzkov. In this paper, we are interested in getting L^2 stability for a balance law up to a shift function.

Let us consider the shock solutions of the scalar conservation laws with the given source term (1) with the initial data

$$S_0(x) = \begin{cases} C_L \text{ if } x < 0, \\ C_R \text{ if } x \geq 0. \end{cases} \tag{2}$$

with two constants $C_L > C_R$ where the source term g is defined as follows:

$$g \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad g(C_L) = g(C_R) = 0. \tag{3}$$

Then, the Rankine-Hugoniot condition ensures that the function

$$S_0(x - \sigma t) \qquad \text{with} \qquad \sigma := \frac{A(C_L) - A(C_R)}{C_L - C_R}, \tag{4}$$

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is a solution to the equation (1). Notice that the condition $C_L > C_R$ implies that they verify the entropy conditions, that is:

$$\partial_t \eta(U) + \partial_x G(U) - \eta'(U) g(U) \leq 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{5}$$

for any convex functions η, and $G' = \eta' A'$. An easy dimensional analysis shows that, because of those layers, we may have in general

$$\|U(t) - S(\cdot - \sigma t)\|_{L^2}^2 \geq C\varepsilon,$$

for some $\epsilon > 0$ which means that the L^2 stability for two solutions U, S does not hold. We are interested in deriving the extremal L^2 stability up to a shift function. The main result is as follows.

Theorem 1.1. *Let $C_L > C_R$ and $T > 0$ be any number. Suppose that U is a solution of the equation (1) and (5). Then there exists a Lipschitz curve $X \in L^\infty(0, T)$ and $C := C(\|\eta''\|_{L^\infty}, \|g'\|_{L^\infty}, T)$ such that $X(0) = 0$ and for any $0 < t < T$:*

$$\|U(t) - S(t)\|_{L^2(\mathbb{R})} \leq C\|U_0 - S_0\|_{L^2(\mathbb{R})} \tag{6}$$

where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (2). Moreover, this curve satisfies

$$|\dot{X}(t)| \leq C \quad \text{and } |X(t) - \sigma t| \leq C t^{\frac{1}{4}} \|U_0 - S_0\|_{L^2(\mathbb{R})}. \tag{7}$$

This is L^2 stability result to a shock for balance laws up to a shift function. The main point is how to construct a shift function $X(t)$ such that the time derivative of the relative entropy is non-positive. Our method is based on the method developed in Leger and Vasseur [18, 19] together with using the relative entropy idea and the result cannot be true without shift (see [18]).

The relative entropy method introduced by Dafermos [9] and Diperna [11] provides an efficient tool to study the stability and asymptotic limits among thermomechanical theories, which is related to the second law of thermodynamics. They showed, in particular, that if $\overline{U}$ is a Lipschitzian solution of a suitable conservation law on a lapse of time $[0, T]$, then for any bounded weak entropic solution U it holds:

$$\int_{\mathbb{R}} |U(t) - \overline{U}(t)|^2\, dx \leq C \int_{\mathbb{R}} |U(0) - \overline{U}(0)|^2\, dx, \tag{8}$$

for a constant C depending on $\overline{U}$ and T. Since Dafermos [9, 10] and Diperna [11]'s works, there have been many recent progress as applications of the relative entropy method. Chen et al. [6] have applied the relative entropy method to obtain the stability estimates to shocks for gas dynamics which deive the time asymptotic stability of Riemann solutions with large oscilaation for the 3×3 system of Euler equations. For incompressible limits, see Bardos, Golse, Levermore [2, 3], Lions and Masmoudi [20], Saint Raymond et al. [14, 21, 23, 24] have studied incompressible limit problems. There are also many recent results of the weak-uniqueness for the compressible Navier Stokes equations together with using relative entropy by Germain [13], Feireisl, Novotny [12]. For the relaxation there is an application for compressible models by Lattanzio, Tzavaras [17, 26] and we can also see Berthelin, Tzavaras, Vasseur [4, 5] as some applications of hydrodynamical limit problems. However, in all those cases, the method works as long as the limit solution has a good regularity such that the solution is Lipschitz. This is due to the fact that strong stability as (8) is not true when $\overline{U}$ has a discontinuity. It has been proven in [18, 19], however, that some shocks are strongly stable up to a shift. Choi and Vasseur [7] have recently used this stability property to study sharp estimates for the inviscid limit of viscous scalar conservation laws to a shock. With the same idea, Kwon and Vasseur [16] develop sharp estimates of hydrodynamical limits to shocks for BGK models. The L^2 contraction for extremal shocks of systems of conservation laws is recently developed by Vasseur [27]

$$\int_{\mathbb{R}} |U(t) - S(t)|^2\, \mathrm{dx} \leq C \int_{\mathbb{R}} |U_0 - S_0|^2\, \mathrm{dx}, \tag{9}$$

and he has introduced a pseudo-distance based on the notion of relative entropy to deal with the extremal shocks for systems of conservation laws. For this result, we need the strong trace notion of solution of conservation laws and it was proved by Kwon, Vasseur [16, 28] for only scalar conservation laws. For this paper, we also develop L^2 contraction for shocks of balance laws up to a shift function $X(t)$. Thus it generalizes Leger's work [18] and provides more a simple proof than the proof of Leger's work [18]. The outline of this article is as follows: In Section 2 we introduce the relative entropy and construct a shift function. In section 3, we give the proof of Theorem 1.1 together with using Lemma 2.4.

2. Relative Entropy and some Properties

In this section we introduce a special drift function $X(t), t \in (0, T)$, defined in Leger [18] and the relative entropy. To begin with we need some notations and properties provided in Leger [18]. Fix any strictly convex function $\eta \in C^2$, we first define the normalized relative entropy flux $g(\cdot, \cdot)$ by

$$f(x, y) := \frac{F(x, y)}{\eta(x|y)}$$

where the associated relative entropy functional $\eta(\cdot|\cdot)$ is given by

$$\eta(x|y) := \eta(x) - \eta(y) - \eta'(y)(x - y)$$

and the flux of the relative entropy $F(\cdot, \cdot)$ is defined by

$$F(x, y) := G(x) - G(y) - \eta'(y)(A(x) - A(y)). \tag{10}$$

Note that for any fixed y and any weak entropic solution u of (1), we have

$$\partial_t \eta(u|y) + \partial_x F(u, y) \le (\eta'(u) - \eta'(y))g(u).$$

Hence, f can be seen as a typical velocity associated to the relative entropy $\eta(\cdot, y)$.

Using the strict convexity of the function η, Leger showed in [18] the following lemma.

Lemma 2.1. *Let $x, y \in [-L, L]$ for any $L > 0$. There exists a constant $\Lambda > 0$, such that we have*

- $\frac{1}{\Lambda} \le \eta''(x) \le \Lambda$,
- $\frac{1}{2\Lambda}(x - y)^2 \le \eta(x|y) \le \frac{1}{2}\Lambda(x - y)^2$,
- $|F(x, y)| \le \Lambda(x - y)^2$,
- $0 \le (\partial_x f)(x, y) \le \Lambda$,
- $\frac{1}{\Lambda} \le (\partial_y f)(x, y)$.

In the spirit of Leger [18], we consider the solution of the following differential equation in order to define the shift function X

$$\begin{cases} \dot{X}(t) = f\Big(\frac{U(t,X(t)+)+U(t,X(t)-)}{2}, \frac{C_L+C_R}{2}\Big) \\ X(0) = 0 \end{cases}. \tag{11}$$

where the the existence of strong trace of solution $U(t, X(t))$ for scalar balance laws is provided in Coclite, Karsen, and Kwon [8]. Note that the existence and uniqueness of X comes from the Cauchy-Lipschitz theorem.

First, X is Lipschitz, since we have from Lemma 2.1

$$|\dot{X}(t)| \le \frac{\left|F\left(\frac{U(t,X(t)+)+U(t,X(t)-)}{2}, \frac{C_L+C_R}{2}\right)\right|}{\eta\left(\frac{U(t,X(t)+)+U(t,X(t)-)}{2}\middle|\frac{C_L+C_R}{2}\right)} \le 2\Lambda^2 \tag{12}$$

where we used the fact $\|U(t)\|_{L^\infty} \le L$ for $t > 0$ and it is guaranteed from the initial data(see Lemma 2.3.)

Let us consider

$$\partial_t u + \partial_x A(u) = \epsilon\partial_{xx}^2 u + g(u). \tag{13}$$

Lemma 2.2. *Let u_ϵ be a solution of the equation (13). Then, for every $t \in (0,T)$, we have*

$$\|u_\epsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})}t. \tag{14}$$

proof From the equation in(13), we get

$$\partial_t u_\epsilon + \partial_x A(u_\epsilon) - \epsilon\partial_{xx}^2 u_\epsilon \le \|g\|_{L^\infty(\mathbb{R})}. \tag{15}$$

Since $\|u_0\|_{L^\infty(\mathbb{R})} + t\|g\|_{L^\infty(\mathbb{R})}$ satisfies (15) and $|u_0(x)| \le \|u_0\|_{L^\infty(\mathbb{R})}$ for all $x \in \mathbb{R}$, the comparison principle for parabolic equations provide

$$u_\epsilon(t,x) \le \|u_0\|_{L^\infty(\mathbb{R})} + t\|g\|_{L^\infty(\mathbb{R})}$$

for all $(t,x) \in (0,T) \times \mathbb{R}$. In the same method, we also get

$$u_\epsilon(t,x) \ge -(\|u_0\|_{L^\infty(\mathbb{R})} + t\|g\|_{L^\infty(\mathbb{R})})$$

for all $(t,x) \in (0,T) \times \mathbb{R}$.□ From the Lemma 2.2, we get

Lemma 2.3. *Let u_ϵ be a solution of the equation (13). Then, for every $t \in (0,T)$, there exists a subsequence $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\epsilon\}$ and a function $U \in L^\infty((0,T) \times \mathbb{R})$ such that*

$$\{u_{\epsilon_k}\} \to U \text{ a.e.}$$

proof Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and let $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q'(u) = \eta'(u)A'(u)$. By multiplying the first equation in (13) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t\eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon\partial_{xx}^2\eta(u_\varepsilon)}_{=:\mathcal{L}_\varepsilon^1} \underbrace{-\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 + \eta'(u_\varepsilon)g(u_\varepsilon)}_{=:\mathcal{L}_\varepsilon^2}, \tag{16}$$

where $\mathcal{L}_\varepsilon^1, \mathcal{L}_\varepsilon^2$ are distributions. Taking $\eta(s) = s^2$, integrating (16) for space variable x, and using Gronwall's inequality imply

$$\sqrt{\epsilon}\partial_x u_\epsilon \in L^2((0,T) \times \mathbb{R}). \tag{17}$$

By Lemma 2.2 and (17),

$$\begin{aligned} &\mathcal{L}_\varepsilon^1 \to 0 \text{ in } H_{\mathrm{loc}}^{-1}((0,T) \times \mathbb{R}), \\ &\mathcal{L}_\varepsilon^2 \text{ is uniformly bounded in } L_{\mathrm{loc}}^1((0,T) \times \mathbb{R}). \end{aligned} \tag{18}$$

Therefore, Murat's lemma [22] implies that

$$\{\partial_t\eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H_{\mathrm{loc}}^{-1}((0,T) \times \mathbb{R}). \tag{19}$$

Thus, the Tartar's compensated compactness method [25] give the existence of a subsequence $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $u \in L^\infty((0,T) \times \mathbb{R})$.□

The idea of the proof is to find a Lipschiz surve $X(t)$ with initial $X(0) = 0$ verifying

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t) := \frac{\mathrm{d}}{\mathrm{dt}}\Big(\int_{-\infty}^{X(t)} \eta(U(t,x)|C_L)\mathrm{dx} + \int_{X(t)}^{\infty} \eta(U(t,x)|C_R)\mathrm{dx}\Big) \le C\mathcal{E}(t). \tag{20}$$

Remark 1. *Let U be an entropy solution of (1) with initial data $U_0 \in L^\infty(\mathbb{R})$. Then, Oleinik's estimate provides that the one side limits $U(t, X(t)-)$ and $U(t, X(t)+)$ exist and satisfies $U(t, X(t)-) \geq U(t, X(t)+)$. Furthermore, $U(t, X(t)-)$ and $U(t, X(t)+)$ is also well-defined according to Coclite, Karsen, and Kwon [8].*

Thus, the derivative of $\mathcal{E}(t)$ implies the following lemma

Lemma 2.4. *Let $X(t)$ be a Lipschitzian curve and U be an entropic solution of balance laws (1). Then, for almost every $t > 0$, we have the following on $(0, T)$*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t) &\leq -F(U(t, X(t)-), C_L) + X'(t)\eta(U(t, X(t)-)|C_L)\\ &+F(U(t, X(t)+), C_R) + X'(t)\eta(U(t, X(t)+)|C_R) + C\mathcal{E}(t)\\ &:= I + J + C\mathcal{E}(t)\end{aligned} \tag{21}$$

in the sense of distribution.

proof Following Leger and Vasseur [19], it is easily seen to see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t) &\leq -F(U(t, X(t)-), C_L) + x'(t)\eta(U(t, X(t)-)|C_L)\\ &+F(U(t, X(t)+), C_R) + X'(t)\eta(U(t, X(t)+)|C_R)\\ &+ \int_{-\infty}^{X(t)} (\eta'(U(t, x)) - \eta'(C_L))g(U(t, x))\mathrm{dx}\\ &+ \int_{X(t)}^{\infty} (\eta'(U(t, x)) - \eta'(C_R))g(U(t, x))\mathrm{dx}\end{aligned} \tag{22}$$

in the sense of distribution.

Let us first compute the third line of (22) and we can do the fourth line in the same method. Using the assumption of source term g and the Mean Value Theorem for entropy function η' and g give

$$\begin{aligned}&\int_{-\infty}^{X(t)} (\eta'(U(t, x)) - \eta'(C_L))g(U(t, x))\mathrm{dx}\\ &\qquad= \int_{-\infty}^{X(t)} (\eta'(U(t, x)) - \eta'(C_L))(g(U(t, x)) - g(C_L))\mathrm{dx}\\ &\qquad\leq C\int_{-\infty}^{X(t)} (U(t, x) - C_L)^2\mathrm{dx}\\ &\qquad\leq C\int_{-\infty}^{X(t)} \eta(U(t, x)|C_L)\mathrm{dx} \leq C\mathcal{E}(t)\end{aligned}$$

where C depends on $\|\eta''\|_{L^\infty}$, $\|g'\|_{L^\infty}$ and we have used Lemma 2.1 on the fourth line.□

3. Proof of Theorem 1.1

In this section we prove the Theorem 1.1 together with Lemma 2.4. Let us first show that I is non-negative. From now on, we denote $U-$, $U+$ for convenient presentation by

$$U- := U(t, X(t)-) \text{ and } U+ := U(t, X(t)+).$$

In virtue of the definition of shift function defined in (11) and Lemma 2.4, we find

$$
\begin{aligned}
I &= -F(U-, C_L) + X'(t)\eta(U - |C_L) \\
&= \eta(U - |C_L)\Big(f\Big(\frac{(U+)+(U-)}{2}, \frac{C_L + C_R}{2}\Big) - f(U-, C_L)\Big) \\
&\leq \eta(U - |C_L)\Big(f\Big(U-, \frac{C_L + C_R}{2}\Big) - f(U-, C_L)\Big) \\
&\leq \eta(U - |C_L)(f(U-, C_L) - f(U-, C_L)) = 0
\end{aligned}
$$

where we have here used Remark 1. In the same spirit, we obtain the following result

$$
\begin{aligned}
J &= F(U+, C_R) - X'(t)\eta(U + |C_R) \\
&= \eta(U + |C_R)\Big(f(U+, C_R) - f\Big(\frac{(U+)+(U-)}{2}, \frac{C_L + C_R}{2}\Big)\Big) \\
&\leq \eta(U + |C_R)\Big(f\Big(\frac{(U+)+(U-)}{2}, C_R\Big) - f\Big(\frac{(U+)+(U-)}{2}, \frac{C_L + C_R}{2}\Big)\Big) \\
&\leq \eta(U + |C_R)\Big(f\Big(\frac{(U+)+(U-)}{2}, \frac{C_L + C_R}{2}\Big) - f\Big(\frac{(U+)+(U-)}{2}, \frac{C_L + C_R}{2}\Big)\Big) \\
&= 0.
\end{aligned}
$$

Consequently, from Lemma 2.4, we obtain that there exists $C := C(\|\eta''\|_{L^\infty}, \|g'\|_{L^\infty}, T)$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t) \leq C\mathcal{E}(t),$$

which implies

$$\int_{\mathbb{R}} \eta(U(t,x)|S(t,x))\mathrm{dx} \leq C \int_{\mathbb{R}} \eta(U_0(x)|S_0(x))\mathrm{dx}$$

where we have here used the Gronwall's inequality and thus it provides (6) in the main Theorem by taking $\eta(s) = s^2$.

To end the proof, we next show (7) and define the function ψ by

$$\psi(x) := \begin{cases} 0 \text{ if } |x| > 2, \\ 1 \text{ if } |x| \leq 1 \\ 2 - |x| \text{ if } 1 < |x| \leq 2. \end{cases}$$

Let $s \in (0, t)$ and $R > 0$. Multiplying $\Psi_R(s,x) := \psi(\frac{x - X(s)}{R})$ to the equation (1) and integrating in x, we get

$$
\begin{aligned}
0 &= -\frac{d}{ds}\int \Psi_R \cdot U dx + \int \partial_x(\Psi_R)A(U)dx + \int \partial_t(\Psi_R)U\mathrm{dx} + \int \Psi_R\, g(U)\mathrm{dx} \\
&= \underbrace{-\frac{d}{ds}\int \psi\Big(\frac{x - X(s)}{R}\Big) \cdot U(s,x)dx}_{(I)} + \underbrace{\frac{1}{R}\int \psi'\Big(\frac{x - X(s)}{R}\Big) \cdot \Big(A(U(s,x)) - \dot{X}(s)U(s,x)\Big)dx}_{(II)} \\
&\quad + \underbrace{\int \psi\Big(\frac{x - X(s)}{R}\Big)g(U)\mathrm{dx}}_{(III)}.
\end{aligned}
$$

By using the above observation, we have

$$\begin{aligned}(\sigma - \dot{X}(s)) &= \frac{1}{C_L - C_R}\Big(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s)\Big)\\ &= \frac{1}{C_L - C_R}\Big(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II) + (I) + (III)\Big).\end{aligned}$$

Then we integrate the above equation in time on $[0,t]$ to get:

$$\begin{aligned}|\sigma t - X(t)| \le C\Big(&t \cdot \max_{s\in(0,t)} \underbrace{\Big|A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II)\Big|}_{(II')}\\ &+ \Big|\int_0^t (I)ds\Big| + t\cdot \max_{s\in(0,t)}\Big|(III)\Big|\Big).\end{aligned} \tag{23}$$

From the result of Choi and Vasseur [7], we already know the following results:

$$(II')^2 \le \frac{C}{R}\cdot \int_{\mathbb{R}} \eta(U(s)|S(s))dx. \tag{24}$$

and

$$\Big|\int_0^t (I)ds\Big|^2 \le CR\Big(\int_{\mathbb{R}} \eta(U(t)|S(t))dx + \int_{\mathbb{R}} \eta(U_0|S_0)dx\Big). \tag{25}$$

We now estimate (III). This directly follows from the definition of source term (3) and Holder's inequality:

$$\begin{aligned}|(III)| &\le \int_{-2R+X(s)}^{X(s)} |g(U) - g(C_L)|\mathrm{dx} + \int_{X(s)}^{2R+X(s)} |g(U) - g(C_R)|\mathrm{dx}\\ &\le C\sqrt{R}\|U - S\|_{L^2(\mathbb{R})}\\ &\le C\sqrt{R}\|U_0 - S_0\|_{L^2(\mathbb{R})}.\end{aligned} \tag{26}$$

Finally, by using (26), we combine (24) and (25) together with (23) to get, for any $t \in (0,T)$,

$$|\sigma t - X(t)|^2 \le C\Big(\frac{t^2}{R^2} + R + t^2 R\Big)\cdot\Big(\int_{\mathbb{R}} |U_0 - S_0|^2 dx\Big) \tag{27}$$

Consequently, taking $R = t^{\frac{1}{2}}$ provides the estimate (7).

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